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Abstract

This dissertation comprises three essays in distinct areas of economic theory, yet all are related to experimentation and learning.

In the first chapter, I study how organizations assign tasks to identify the best candidate to promote among a pool of workers. Task allocation and workers' motivation interact through the organization's promotion decisions. The organization designs the workers' careers to both screen and develop talent. When only non-routine tasks are informative about a worker's type and non-routine tasks are scarce, the organization's preferred promotion system is an *index contest*. Each worker is assigned a number that depends only on his *own type*. The principal delegates the non-routine task to the worker whose current index is the highest and promotes the first worker whose type exceeds a threshold. Each worker's threshold is independent of the other workers' types. Competition is mediated by the allocation of tasks: who gets the opportunity to prove themselves is a determinant factor in promotions. Finally, features of the optimal promotion contest rationalize the prevalence of fast-track promotion, the role of seniority, or when a group of workers is systemically advantaged.

The second chapter is co-authored with Matteo Camboni. We formulate a general optimal stopping problem that can accommodate various non-stationary environments, such as situations where the decision maker's patience, time pressure, and learning speed can change gradually and abruptly over time. We show that, under mild regularity conditions, this problem has a well-defined solution. Furthermore, we characterize the shape of the stopping region in a large class of *monotone environments* and obtain comparative statics on the timing and quality of decisions for many sequential sampling problems à la Wald. For example, we show that accuracy increases (decreases) over time when, over time, (i) the learning speed increases (decreases), or (ii) the discount rate decreases (increases) (i.e., the decision maker values the future more (less) over time), or (iii) the time pressure decreases

(increases). Since our main comparative static results hold locally, we can also capture non-monotone relations between time and accuracy that consistently arise in perceptual and cognitive testing.

The final chapter, co-authored with Udayan Vaidya and Boli Xu, concerns robustness in mechanism design, with a particular emphasis on dynamic problems of learning and experimentation. We develop a new approach to identify the class of mechanisms that contains a robust optimum. Notably, our approach avoids the issues associated with explicitly solving for the worst-case scenario, allowing us to consider new applications of robustness to dynamic problems. In particular, we use our tools to characterize the robustly optimal dynamic mechanism in a repeated seller-buyer model and the robustly optimal mechanism in a principal-agent model in which the agent can search à la Weitzman (1979). In the first case, the seller offers a sequence of statically optimal random posted prices, while in the second, a debt contract is optimal.

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Chapter 1

Prove yourself: Dynamic delegation in promotion contests

1.1 Introduction

Matching tasks to the right workers is crucial to an organization's success. First, productive efficiency requires that more talented workers perform more complex, non-routine tasks. Second, workers' success in their current tasks is informative: Organizations also allocate tasks to learn about the workers and improve future matches. Third, a worker's assignment determines what he learns on the job. Assigning the right worker to the right task is then especially important when the organization seeks to identify and develop talented workers.[1] Non-routine tasks are opportunities for workers to prove themselves. Workers understand that their career trajectories in the organization depend on the opportunities to showcase their talents. Task allocation and workers' motivation then interact through the organization's promotion decisions. Designing a good promotion system is thus both challenging and

[1]Former Xerox CEO Anne Mulcahy insists in Mulcahy (2010) that it is crucial to identify candidates for promotion early and give them "developmental responsibilities" to develop strong workers and test their abilities.

essential for the organization's success.[2]

A sound promotion system must include the task allocation process and the promotion rule. It must also balance exploitation (delegating non-routine tasks to a worker known to be good and eventually promoting him) and exploration (giving responsibilities to new workers). Ignoring exploration, one misses an essential part of the story: who gets the opportunity to prove themselves is a determining factor in promotions. In this paper, I ask how the organization optimally designs the promotion system to motivate workers and tackle the exploration-exploitation trade-off. I also address the following questions: How does incentive provision affect the allocation of tasks and the promotion decision? Can the allocation of opportunities exacerbate initial differences to induce significant disparities over time? What characteristics of a worker increase his chance of being promoted?

I explore these questions in a *centralized dynamic contest model*. A principal (she) has one prize to award, the promotion. She decides how to allocate a non-routine task sequentially to N workers (he/they) and when to give the prize. Each worker has a type represented by a stochastic process, and the processes are independent. When the principal allocates the non-routine task to a worker and the worker exerts effort, the principal gets a reward that depends on the worker's type. The worker's type also evolves, and the evolution of types could reflect the principal learning about the worker's or the worker acquiring new skills on the job. The other workers are assigned uninformative routine tasks, and their types remain frozen. Finally, the principal can only use the promise of future promotion to motivate the workers. In particular, in the baseline model, I rule out transfers to focus on the interaction

[2]For example, Rosen (1982) insists on the importance of selecting the right person to lead an organization as they set its course, and their decisions are "magnifies" many times. Rohman et al. (2018) note that when employees believe that promotion decisions are efficient, they are more likely to exert effort and follow the organization's leaders' directions and recommendations. The same authors also point out that "stock returns are nearly three times the market average, voluntary turnover is half that of industry peers, and metrics for innovation, productivity, and growth consistently outperform competitors" at companies that manage promotion effectively.

between the two classical and conflicting[3] purposes of promotions: to "assign people to the roles where they can contribute the most to the organization's performance" and to provide "incentives and rewards" (Roberts and Milgrom (1992)).

My model builds on the canonical multi-armed bandit model but departs in one critical aspect: the arms are workers who exert effort. Hence the arms are strategic. In the classic bandit model, when the decision maker pulls an arm, she gets a reward drawn stochastically from some fixed distribution. In some contexts, this assumption fits the behavior of the problem inputs: In clinical trials, it is natural to think that the patients will comply. However, in my problem, each arm corresponds to a worker whose incentives differ from the principal's. So, the arms are *strategic*. When the principal allocates the non-routine task, her reward and the flow of information are controlled by the workers' choices of effort, and the workers exert effort only when compensated for it by the promise of future promotion. To incentivize effort, the principal must eventually promote a worker, stopping exploration.

In this setting, I characterize the principal-optimal promotion contest. Solving the principal's problem is challenging for three reasons. As in bandit models, the first one is the problem's dimensionality. The set of feasible promotion contests is large. Second, the principal's promotion decision can depend on all workers' types and their effort histories. So, it introduces a degree of dependence among the workers that complicates the problem.[4] Finally, the workers are strategic. Not all delegation and promotion rules incentivize effort. The set of "implementable" promotion contests is complex.

Nevertheless, the optimal contest is simple. I prove that, as in the canonical model, *indexability* holds: The optimal promotion contest takes the form of an *index contest*. Each worker is assigned a number (his *index*) that depends only on his type and the cost of

[3]As famously illustrated by the Peter's principle described in Peter et al. (1969).

[4]The stoppable bandit models studied in the literature were so under the (exogenous) restriction that the decision to freeze an arm can only depend on the state of this arm. See, for example, Glazebrook (1979) or Fershtman and Pavan (2022b).

incentive provision. The principal sequentially delegates the non-routine task to the worker whose *index* is the highest. Eventually, she promotes the *first* worker whose type exceeds a threshold. Both the worker's index and promotion threshold are *independent* of the successes and failures of the other workers. Finally, I also show that it is optimal to promote one worker only when the principal can design the prize-sharing rule, i.e., decide to promote multiple workers. The optimal contest is a winner-take-all contest.

In the index contest, the delegation rule mediates the competition for promotion between the workers. To understand the determinants of promotions, it is crucial to consider the factors that affect the allocation and timing of opportunities. This has two significant consequences: (i) for the contractual arrangements between the principal and the workers and (ii) for the effect of initial differences on promotion decisions (especially when thinking about discrimination in promotion practices).

First, no mention of competition needs to appear in the contractual arrangement between each worker and the principal. One interpretation of the index contest is the following. The principal successively offers short-term individual trial contracts to one worker at a time. Each trial contract specifies a target and a (potentially stochastic) deadline. The worker gets the promotion if he achieves the target before the deadline. Otherwise, the manager offers a new trial contract to one of the other workers until a worker eventually succeeds. The trial contracts do not rely on relative performance measures: they are independent of what the other workers do [5] The principal uses contracts that incentivize workers separately: the promotion thresholds do not condition on indicators of relative performance (on the

[5]This appears consistent with some evidence that contracts and promotion guidelines rarely mention relative performance explicitly. For example, both Bretz Jr et al. (1989) and Bretz Jr et al. (1992) find that less than a third of organization uses rankings explicitly. Even among organizations that use ranking measures, they generally "supplement other performance appraisal methods" such as the management by objectives approach (that relies on absolute performance, see Drucker (1954)). More recently, Campbell (2008) provides evidence that promotion decisions are made on an absolute measure of performance in the fast food industry. Finally, anecdotal evidence suggests that organizations that used explicit ranking appear to be abandoning it, see O'Connor (2021). Unfortunately, more evidence and data on contracts and promotion guidelines are seldom available.

other workers' types). This may be surprising: why would not the principal use relative performance to compare the candidates and select the best of them? However, it should not be. Who gets the opportunity already summarizes the relevant ranking information. In the *index contest*, other workers' efforts and successes only affect the likelihood that the worker will be delegated the non-routine task and, hence, get the chance to prove themselves. It is irrelevant to the promotion decision once the worker gets the opportunity.

Second, the principal delegates the non-routine task sequentially to the workers in the index contest. This generates significant path dependence in promotion decisions: a worker who is not given a chance initially may never get the opportunity to prove himself, hence, will not be promoted. In particular, early successes have an outsized impact on the probability of promotion. They increase both the likelihood of being promoted before any of the other workers gets the opportunity to showcase their talent and the likelihood that the worker is allocated the non-routine task again later. One should therefore be careful where to look to identify discrimination in promotion practices. In particular, a firm may always promote the most qualified candidate and yet discriminates. That is because the principal may also discriminate through the allocation of opportunities. In the context of my model, the index delegation rule may treat different groups very differently. For example, minor differences among workers in learning speed or the cost of effort may lead the principal to delegate to one first. In *reinforcing environments*,[6] this dramatically affects their promotion chance and expected time to promotion. There, the early assignment of the non-routine task largely determines the promotion decision. If the principal delegation rule is biased toward one group, workers from the other group will never get an opportunity to be promoted. Moreover, at the time of promotion, they will also appear less qualified than the promoted worker. Their type will be lower than that of the promoted worker, and they will not have

[6] This includes bad news Poisson learning or realistic representations of on-the-job learning. See Definition 8 in Section 1.4.

worked on non-routine tasks as much. This is an instance of what has been described outside of economics as systemic discrimination: discrimination based on systemic group differences in observable characteristics or treatment (see Bohren et al. (2022) for a treatment of systemic discrimination within economics).

I also obtain further predictions for organizational design from my characterization. A notable feature of the index contest is that the promotion thresholds decrease over time. So, a worker's type, when promoted, decreases with time. A first consequence is that fast tracks (i.e., that a quickly promoted worker often gets another promotion soon after, see Baker et al. (1994) and Ariga et al. (1999)) should then not be surprising. When a worker is promoted quickly, his type upon promotion is higher. Hence, he starts from a better place when entering a potential new promotion contest at the next level. So, his expected time to promotion decreases: the worker is expected to be promoted again soon. Second, the type of a worker at the time of promotion decreases the longer he stays in his current position. So, faster-promoted workers should perform better upon promotion than more slowly-promoted workers. Third, in the *index contest*, seniority is not explicit but still confers an advantage. It becomes easier to be promoted for a worker as time passes. The *index contest* backloads incentives, implicitly putting weight on seniority.[7]

Finally, I study multiple extensions: I relax some of the assumptions made in the model. I show that the winner-take-all index context is optimal when the principal can design the prize. I consider the possibility of transfers, and I study different information structures. In my setting, if transfers are unrestricted, the manager can incentivize effort at no cost, and the first best is achieved. However, if wages only depend on the workers' current types and the workers are protected by limited liability, the principal cannot freely punish a worker who decides to shirk. So, intertemporal distortions like the ones absent transfers are reintroduced,

[7]Seniority has been widely used as an explicit promotion criterion (especially in public administrations), and, although it has fallen from favor since the 1980s, it is still seen as an important determinant of promotions, see Dobson (1988) or Phelan and Lin (2001).

and the *index contest* (with adapted indices and promotion thresholds) is optimal. Secondly, in the baseline model, information is symmetric. All the players observe the types of all workers. If only the workers observe their types and can credibly communicate them to the principal, the *index contest* is still implementable (and optimal). In particular, workers do not need to observe who has been in charge of the project and how successful other workers were.

Besides promotion decisions, my results apply to various problems in which a principal owns an asset and wants to allocate it to the best agent among a pool of candidates of unknown ability. This includes outsourcing and procurement decisions by firms, a venture capitalist's investment decision between multiple start-ups, or the CERN research board deciding which team of researchers can use the colliders and when an experiment should be abandoned, for example. When the principal earns rewards and learns by delegating the asset, the optimal mechanism is an *index contest*.

The rest of the article is organized as follows. The related literature is discussed in the remainder of the introduction. In Section 1.2, I formally describe the environment. In Section 1.3, I introduce the *index contest* and presents its properties. In Section 1.4, I study the implications of my findings on discrimination. In Section 1.5, I present an outline of the proof of my main result: the optimality of the *index contest*. In Section 1.6, I consider extensions. I conclude in Section 1.7 with a brief discussion of the results and lines of future research. All proofs not in the main text are in the Appendices A.1, A.2, and supplemental Appendix A.3.

Literature: This paper studies a dynamic contest for experimentation and characterizes the optimal promotion contest when the principal can control the learning process. I then use this model to study a big under-studied topic in personnel economics: how the allocation of tasks affects promotions and worker careers. So, my paper builds on several streams of literature.

First, as mentioned above, I build on the canonical bandit problem solved in Gittins and Jones (1974). Gittins et al. (2011) offer a textbook treatment. Bergemann and Valimaki (2006) is a good survey on bandit problems (in economics). Here, the authors solve for the optimal delegation rule when the arms are passive. Other papers in this literature consider learning about multiple alternatives before making an (irreversible) decision. Examples includes Austen-Smith and Martinelli (2018), Fudenberg et al. (2018), Ke and Villas-Boas (2019), Ke et al. (2016), and Che and Mierendorff (2019). More closely related is the study of stoppable bandit models in Glazebrook (1979). Glazebrook considers a multi-armed bandit model in which the decision maker decides which arm to pull every period but can also choose to freeze an arm and play it forever. He gives conditions under which indexability is preserved. Again, all these papers are concerned with a decision problem: the arms or alternatives are passive, and they study the optimal way to allocate attention before making a decision absent incentive consideration. This is fundamentally different from my model. I am interested in how organizations allocate tasks when the arms are *strategic*. To solve this problem, I identify a new condition under which indexability holds in bandit superprocess problems. I then use this condition to show that the problem's separability is preserved. Despite the strategic nature of the problem, the optimal delegation rule is an index rule.

A few other papers also look at bandit problems with strategic arms. The key distinction between these papers and mine is that, in my paper, the principal is constrained in her ability to provide incentives. In particular, there are no transfers (or only limited transfers), and promotions are scarce. So the agents compete for the prize, which creates a strategic dependence between arms. This strategic dependence is largely absent in the other papers in this literature. In Bergemann and Välimäki (1996) and Felli and Harris (1996, 2018), a principal allocates an asset between two strategic agents over time. The values from allocating to each agent for the principal are initially unknown but can be learned over time. Their models, like mine, can be understood as bandit models with strategic arms. However, in Bergemann and Välimäki (1996); Felli and Harris (1996, 2018), transfers are unrestricted and only affect the "cost of utilization" of each arm, not the information the players get. This implies that all (Markov Perfect) equilibria are efficient (or pairwise efficient in Felli and Harris (2018)). So, the allocation policy is undistorted in any (Markov Perfect) equilibrium.[8] Since the classic Gittins index policy maximizes total surplus, the principal always chooses the agent whose associated Gittins index is the highest. The question is then how surplus is allocated between players. On the other hand, in my framework, the conflict of interest between the principal and the workers prevents allocative efficiency. This is also the case in Kakade et al. (2013), Pavan et al. (2014), Bardhi et al. (2020), or Fershtman and Pavan (2022a), who study strategic bandit models in which the experimentation outcomes are privately observed; or in Guo (2016) and McClellan (2017), who study versions of a $1\frac{1}{2}$-arm strategic bandit model when the principal has limited instruments. For example, in Kakade et al. (2013), Pavan et al. (2014), Bardhi et al. (2020), or Fershtman and Pavan (2022a), to incentivize disclosure, the principal needs to pay rents to the agents. The latter creates dissonance between the principal's and the agents' value for experimentation and, hence, changes the relative value of pulling one arm rather than another. In these papers, the indices are, therefore, also distorted. Contrary to my paper, however, there is no strategic dependency between the arms in the above papers. The presence of transfers allows them to abstract from any linkage of incentive problems across employees. The allocation maximizes the total virtual value and therefore follows from the standard Gittins characterization applied to the "virtual value processes". In my setting, this linkage of incentives is central as promotions, hence incentives, are scarce. The principal has to promise an eventual promotion for which the workers compete. Promise-keeping then distorts the future delegation process. In particular, the set of implementable delegation rules in the continuation game depends on the history. The classical approach to indexability therefore fails. Nevertheless, I show that indexability

[8]Although, in Felli and Harris (2018), the players' investment decisions may be distorted.

still holds. The indices reflect the strategic nature of the problem and the constraints it places on both learning and exploitation. I then focus on how incentives provision distorts the delegation and promotion rules.

My paper is related to a last stream of works on multi-agent experimentation. See, for instance, Bolton and Harris (1999), Keller et al. (2005), Bonatti and Hörner (2011), and Halac et al. (2017). However, the fundamental trade-offs are different. In these papers, the agents experiment on a common bandit machine and therefore have incentives to free-ride on each others' costly experimentation. Free-riding is absent from my model, as each arm is a separate agent and the agent's types are independent. So, there is no positive externality across workers. The central trade-off in my paper is between retaining the option value of experimentation and motivating workers. Two other papers on multi-agent experimentation are related. De Clippel et al. (2021) and Deb et al. (2022) also study how to select the best agent to execute a task when the agents only care about being selected. They focus on different trade-offs than I do. Deb et al. (2022) look at the trade-off between retaining option value via competition and harnessing gains from collaboration, while De Clippel et al. (2021) are interested in mechanisms guaranteeing that the agents willingly display their private information, ensuring efficiency. My paper is complementary to theirs. It illustrates how the principal-optimal allocation rule responds to a different environment and trade-off.

In particular, I show that the optimal allocation rule is an *index contest*. So, my paper also contributes to the growing literature on the design of dynamic contests pioneered by Taylor (1995) and extended by Halac et al. (2017), Benkert and Letina (2020), or Ely et al. (2021). The critical difference between my paper and the rest of the dynamic contest literature is that, in my model, the contest is centralized; i.e., the principal controls the assignment of tasks. Therefore, the set of participants at every point is endogenous and chosen by the principal. This is crucial for my application. Organizations control the allocation of tasks. Therefore, the results I derive are qualitatively different from those in the rest of the dynamic contest literature, where the set of participants is exogeneous. Moreover, the optimal contest in my model is a winner-take-all contest and not a prize-sharing contest as in Halac et al. (2017) or Ely et al. (2021), for example. Comparing my results to these papers can also help us understand when a more meritocratic (winner-take-all) system or a more equal (prize-sharing) system helps the principal.

Finally, my paper contributes to the extensive literature on personnel economics that studies careers in organizations (see Prendergast (1999) for a survey). I consider an environment where learning about workers shapes their career trajectories and hence generates career-concern incentives (Harris and Holmstrom (1982); Holmström (1999)). MacDonald (1982a,b), or Gibbons and Waldman (1999) also emphasize the importance of learning and task assignments in shaping career dynamics, which Pastorino (2019) empirically documents. In these papers, tasks are equally informative. So the players' choices do not affect learning. Instead, I focus on a setting where tasks vary in the information they generate, as in Antonovics and Golan (2012), Canidio and Patrick (2019), or Madsen et al. (2022). These papers, however, focus on the distortionary effects of promotions and career concerns on risk-taking when the workers control their occupational choices. In contrast, in my paper, the principal controls the allocation of tasks. So my paper complements their findings. In particular, I study a trade-off that arises in task allocation problems when the principal is primarily concerned with alleviating the time inconsistency problem of promotions, as in Waldman (2003), which is absent in their papers. Since promotions reward past effort and sort workers, a sound promotion system should do both. Moreover, the optimal way to incentivize effort may be suboptimal for selection. Indeed, as mentioned above, the optimal index contest vastly differs from the optimal dynamic contest to incentivize effort in Ely et al. (2021). I show how the principal-optimal task allocation balances incentives provision and selection. This trade-off is also absent from other papers that look at how firms assign tasks and learn, such as Pastorino et al. (2004) or Bar-Isaac and Lévy (2022), in which the principal can incentivize each worker's effort separately. Finally, in all the previous papers that study learning through task allocation, the employer faces no constraints on learning. She can assign all workers to non-routine tasks simultaneously. On the contrary, I assume that non-routine tasks are scarce, reflecting that not all workers can simultaneously lead a team, for example. So, my paper complements their works by studying how firms design careers to screen and develop workers when learning opportunities are limited. In particular, I show that the delegation process is sequential, meritocratic, and creates a significant path dependence in promotion decisions: The principal first delegates opportunities to the best workers as measured by their *index* and immediately promotes them in case of success. So, my paper also relates to the analysis of turnover in a leadership position. This question has been studied by, among others, Mortensen and Pissarides (1994), Atkeson and Cole (2005), and Garrett and Pavan (2012). As in these papers, seniority matters for promotion decisions, and I extend such finding to a multi-agent setting. Here the main difference is my focus on the dynamic process of experimentation that leads to the promotion decision.

1.2 Model

Let $(\Omega, \bar{\mathcal{F}}, \mathbb{P})$ be a probability space rich enough to accommodate all the objects defined below. A principal (she) and N workers (he/they) interact in an infinite-horizon continuous-time stochastic game. All players discount the future at a common discount rate $r > 0$.[9] The principal has to decide how to delegate one non-routine task and many routine tasks among the workers to maximize her continuation value. When the non-routine task is delegated to one of the workers, the principal gets a flow rewards that depends on the current type of the worker and whether the worker exerts effort. If he does, his type also evolves (stochastically). To motivate the workers to exert effort when delegated the non-routine task, the principal

[9]This assumption can be relaxed: the analysis can also accommodate for random discount factors for example.

can allocate an indivisible prize that the wokers value; i.e., she can decide to promote one of them.

1.2.1 Actions

Heuristically, at each time t, the principal and the workers play in the "stage game" depicted in figure 1.1. Within each period $[t, t+dt)$, the principal chooses who to delegate the non-routine task to. The other workers are allocated routine tasks. When she delegates the non-routine task to worker $i \in \{1, \ldots, N\}$, worker i then decides whether to exert effort to complete the task. If worker i exerts effort, the principal learns about worker i, gets a reward $\pi^i(x^i)$ that depends on worker i's current type x^i, and worker i's type evolves (stochastically). If he does not, the principal gets no reward and worker i's type stay the same. The principal then decides whether to (i) continue to experiment before allocating the prize, (ii) promote one of the workers, or (iii) allocate the prize to an external worker (i.e. take her outside option W). If she chooses to continue to experiment, the next "period", $[t+dt, t+2dt)$ the players play the same "stage game". If she chose to allocate the prize, her only decision in the continuation game is who to delegate the non-routine task to. The workers then decides whether to exert effort to complete the task. I assume that the principal can commit at time zero to an history contingent sequence of plays, while the workers cannot.

Formally, at time $t = 0$, the principal commits to a history-dependent promotion contest comprising of (i) a promotion time τ specifying when the promotion is allocated; (ii) a promotion decision d specifying which of the workers is promoted; and (iii) a delegation rule α that assigns at every instant the non-routine task to some worker.

The promotion time, τ, is a $\bar{\mathcal{F}}$-measurable mapping from Ω to $\mathbb{R}$. The promotion decision is a (stochastic) process $d = \left(d^0 = \{d_t^0\}_{t\geq 0}, \ldots, d^N = \left\{d_t^N\right\}_{t\geq 0}\right)$, which takes value in $\{0,1\}^{N+1} \cap \Delta^{N+1}$, where Δ^{N+1} is the $N+1$-dimensional simplex. If $d_\tau^i = 1$, worker i is

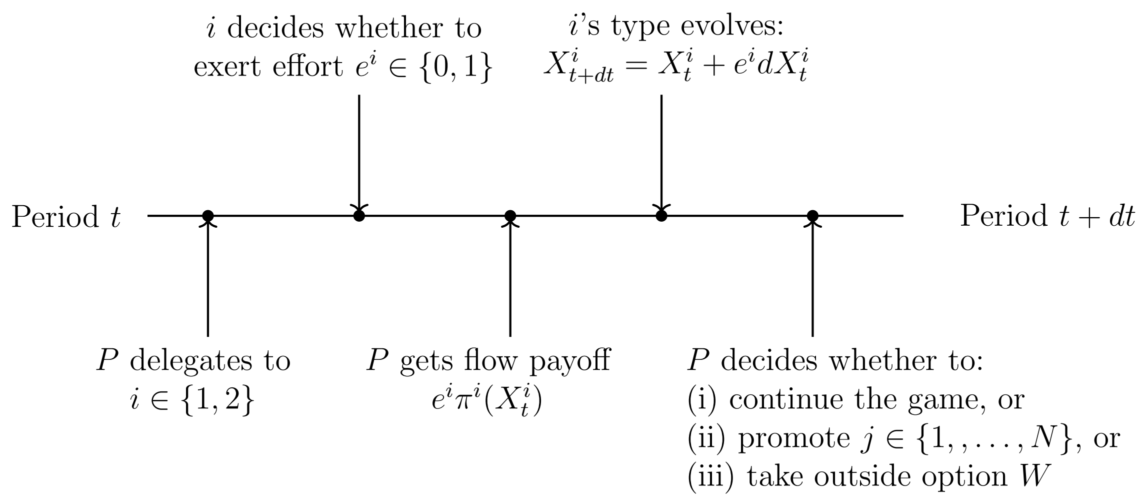

Figure 1.1: Heuristic "stage game"

promoted at time τ. $d^0_\tau = 1$ stands for the principal's decision to take her outside option. Finally, the delegation rule $\alpha = \left(\alpha^1 = \{\alpha^1_t\}_{t \geq 0}, \ldots, \alpha^N = \left\{\alpha^N_t\right\}_{t \geq 0}\right)$ is a (stochastic) process which takes value in the N-dimensional simplex, Δ^N. α^i_t is the share of the non-routine task worker i is responsible for at each instant $t \geq 0$. The process $t \to \alpha_t$ is (at least) Borel measurable $\mathbb{P}$-a.s..[10]

Workers cannot commit. Each instant, they decide whether to exert effort when delegated (a positive share of) the non-routine task. $a^i_t \in \{0, 1\}$ denotes the effort decision of worker i at time $t \geq 0$. The effort process generated by the decisions of worker i is $a^i = \{a^i_t\}_{t \geq 0}$. $t \to a^i_t$ is required to be Borel measurable $\mathbb{P}$-a.s..[11]

1.2.2 Workers' types

Together, the choices of effort and the delegation rule determine the evolution of the workers' types. To describe the state dynamics, I follow the multi-parameter approach pioneered by Mandelbaum in discrete time in Mandelbaum (1986) and in continuous time in Mandelbaum

[10]The filtration to which α, τ, and d are adapted to is not defined yet. This is deferred to Section 1.2.3, after the dynamics of the workers' types and the information structure are introduced.

[11]As above, a more precise measurability requirement is postponed until Section 1.2.3.

(1987).

For all $i \in \{1, \ldots, N\}$, let $\mathcal{F}^i := \{\mathcal{F}_t^i\}_{t\geq 0}$ be a filtration in $\bar{\mathcal{F}}$ and $X^i = \{X_t^i\}_{t\geq 0}$ be a $\mathcal{F}^i$-adapted process with values in the interval $\mathcal{X}^i \subseteq \mathbb{R}$. For simplicity, assume that either X^i does not reach the boundary of the set $\mathcal{X}^i$ or the boundary is absorbing. Define

$$T^i(t) := \int_0^t a_s^i \alpha_s^i ds, \, \forall \, 0 \leq t < \infty. \tag{1.1}$$

$T^i(t)$ is the amount of time worker i worked on the non-routine task. At time t, the type of worker i is $X^i_{T^i(t)}$. So worker i's type evolves (stochastically) only when he exerts effort. When he does not, his type is frozen. Intuitively, one can think of the evolution of the type as follows: Nature first draw a path $X^i = \{X_s^i\}_{s\geq 0}$ for worker i. The delegation rule and the worker's choices of effort then jointly control "the passage of time", $T^i(t)$, i.e., the speed at which the worker's type moves along the path X^i.

Define the delegation process $T = \left(T^1 = \{T^1(t)\}_{t\geq 0}, \ldots, T^N = \left\{T^N(t)\right\}_{t\geq 0}\right)$. The state of the game at time t is

$$X_{T(t)} = \left(X^1_{T^1(t)}, \ldots, X^N_{T^N(t)}\right).$$

$\left\{X_{T(t)}\right\}_{t\geq 0}$ is a multi-parameter process adapted to the multi-parameter filtration

$$\mathcal{F} := \left\{\mathcal{F}_{\bar{t}} := \bigvee_{i=1}^N \mathcal{F}^i_{t^i}, \quad \bar{t} = (t^1, \ldots, t^N) \in [0, \infty)^N\right\}$$

defined on the orthant $[0, \infty)^N$.

I make the following assumptions on the types' processes.

Assumption 1 *The filtrations $\mathcal{F}^i$, $i \in \{1, \ldots N\}$, are mutually independent and they satisfy* *the usual hypothesis of right-continuity and completeness.*[12]

Assumption 1 implies that the manager does not learn anything about the type of one worker by observing the type of another.

Assumption 2 *The processes* $(X^i, \mathcal{F}^i)$, $i = 1, \ldots, N$, *are Feller.*[13]

Assumption 2 is made to guarantee that the type process has the strong Markovian property: the distribution of future realizations only depends on the current value of the process. The Feller property is however stronger: it also guarantees that the expectation operator conditional on the value of the type process is continuous. This second property is not needed, but simplifies the analysis.

Assumption 3 *For all* $i \in \{1, \ldots, N\}$, *if* $X_0^i = x^i \geq \tilde{X}_0^i = \tilde{x}^i$, *then, for all* $s \geq 0$, $X_s^i \geq \tilde{X}_s^i$ $\mathbb{P}$-*a.s.*.

Assumption 3 states that if a worker's initial type increases, so does his type at any instant $t \geq 0$. Because Feller processes are time-homogeneous, it also implies that, if a worker's type is higher at time t along one path than along another, so is his type at any instant $s \geq t$.

Assumption 4 *For each* $i \in \{1, \ldots, N\}$, *if* $t \to X_t^i$ *is not continuous, then either (i)* $X_{t^-}^i - X_t^i > 0$ *at all discontinuity points* $t \in \mathbb{R}$; *or (ii)* $X_{t^-}^i - X_t^i < 0$ *at all discontinuity points* $t \in \mathbb{R}$.

Assumption 4 is a restriction on the jump of the processes X^i.The jumps must be"one-sided"; i.e., if the process X^i jumps up, it cannot jump down, and conversely. In particular, if X^i is a continuous process, Assumption 4 holds trivially.

Assumption 5 *For all* $i \in \{1, \ldots, N\}$ *and for all* $x \in \mathcal{X}^i$, $\mathbb{P}_x\left(\{\tau^i_{(x,\infty)} = 0\}\right) = 1$, *where* $\tau^i_{(x,\infty)} := \inf\{t \geq 0 : X_t^i \in (x, \infty)\}$. *Moreover, if* X^i *jumps down, for all* $\kappa, \epsilon > 0$, *there exists* $\delta > 0$ *such that, for all* $x \in \mathcal{X}^i$, $\mathbb{E}_x\left[\tau_{(x-\delta,x+\delta)}\right] < \epsilon$.

[12] See, e.g., Protter (2005).

[13] Recall that any Feller process admits a càdlàg modification. So I always assume that X^i is càdlàg.

Assumption 5 states that any worker can always become more productive. It simplifies the arguments and guarantees the existence of a solution to the principal's problem. The second part of Assumption 5 strengthens the first part for the case in which X^i jumps down. In particular, it guarantees that the expected time X^i stays in any small interval is small. I relax this assumption in Section 1.6.1.

In Appendix A.1.1, I show that my framework accommodates all jump-diffusion processes that satisfy mild regularity and monotonicity conditions. In particular, it includes the commonly studied cases in which workers can be either good or bad and the principal learns whether the worker is good or bad according to a Brownian signal or a bad news Poisson signal. In these examples, worker i's type at time t, $X^i_{T^i(t)}$, is the belief that worker i is good after he has worked for $T^i(t)$ unit of time on the project.

1.2.3 Information and Strategies

Information: The principal and the workers perfectly observe the delegation rule chosen by the principal, and the effort decisions and types of all the workers. Information is symmetric, but incomplete.[14]

Workers' strategies: It is well-known that perfect monitoring in continuous-time games can come with complications.[15] To avoid the issue, I take a reduced form approach.

Definition 1 *A **dynamic delegation process** is a process*

$$T = \left\{T(t) = \left(T^1(t), \ldots, T^N(t)\right), \, t \geq 0\right\}$$

taking values in $[0, \infty)^N$ *such that, for all* $i \in \{1, \ldots, N\}$,

[14] Alternative information structures are discussed in Section 1.6.

[15] Continuous time is not well-ordered, and, therefore, seemingly well-defined promotion contests and effort strategies can fail to uniquely determine the outcome of the game. For a more detailed discussion, see Simon and Stinchcombe (1989), Bergin and MacLeod (1993), or Park and Xiong (2020) for deterministic games, and Durandard (2022b) for stochastic games.

1. $\{T(t) \leq \bar{t}\} = \bigcap_{i=1}^{N} \{T^i(t) \leq t^i\} \in \mathcal{F}_{\bar{t}}$ *for all* $\bar{t} = (t^1, \ldots, t^N) \in [0, \infty)^N$, $t \geq 0$,

2. $T^i(\cdot)$ *is nondecreasing, right-continuous, with* $T^i(0) = 0$,

3. $\sum_{i=1}^{N} (T^i(t) - T^i(u)) \leq t - u, \quad \forall t \geq u \geq 0.$

Denote by $\mathcal{D}$ the set of all dynamic delegation processes.[16]

Condition 1. in Definition 1 ensures that delegation processes are adapted to the multi-parameter filtration $\mathcal{F}$. So they are non-anticipative: they do not depend on future events.

Given a dynamic delegation process $T \in \mathcal{D}$, define the one parameter filtration $\mathcal{G}^T = \left\{\mathcal{G}_t^T\right\}_{t \geq 0}$ as follows. Let $\nu : \Omega \to [0, \infty)^N$. ν is a stopping point of $\mathcal{F}$ if $\{\nu \leq \bar{t}\} \in \mathcal{F}_{\bar{t}}$ for all $\bar{t} \in [0, \infty)^N$. For any stopping point ν, define the sigma-field

$$\mathcal{F}(\nu) \coloneqq \left\{A \in \bar{\mathcal{F}} \, : \, A \cap \{\nu \leq \bar{t}\} \in \mathcal{F}_{\bar{t}}, \, \forall \bar{t} \in [0, \infty)^N \right\}.$$

Then, for all $0 \leq t < \infty$, let $\mathcal{G}_t^T \coloneqq \mathcal{F}(T(t))$.

In the remaining of the paper, with a small abuse of notation, I will redefine promotion contests as:

Definition 2 *A promotion contest (T, τ, d) consists of a dynamic delegation process T, a $\mathcal{G}^T$-stopping time τ, and a $\mathcal{G}^T$-optional promotion decision rule d, such that $\mathbb{P}$-a.s.*

$$T^i(t) \coloneqq \int_0^t a_s^i \alpha_s^i ds,$$

for all $0 \leq t \leq \tau$ and all $i = 1, \ldots, N$.

Denote by $\mathcal{P}$ the set of all promotion contests.

[16] In the theory of multi-parameter processes, $T(t)$ is a stopping point in $[0, \infty)^N$ and a delegation rule T is called an optional increasing path (Walsh 1981, Walsh (1981)). It can be thought of as a multi-parameter time change.

Finally, a strategy profile is **admissible** if it uniquely defines a promotion contest after all histories $\mathbb{P}$-a.s.. I will require that the space of strategies is such that (i) any strategy profile in which all workers change their effort decision at most once and the principal can adjust the contest upon observing such changes is included, and (ii) if a strategy profile belongs to the strategy space, then any h_t-"truncated" strategy profile does too. The h_t-"truncated" strategy profile is the strategy profile that coincides with the original profile for any history that does not contain h_t and such that all players play a Markov continuation strategy after history h_t. Both conditions (i) and (ii) are richness conditions on the strategy space. They are satisfied for example by the space of semi-Markov strategies or the strategy space defined in Durandard (2022b).

In particular, condition (i) guarantees that any promotion contest can be obtained as the outcome of an **admissible** strategy profile. By definition of admissibility, the set of continuation values at any instant $t \geq 0$ that can be generated in the game coincides with the set of values generated by the set of promotion contests.

1.2.4 Payoffs and objective

At time $t \geq 0$, when worker i is delegated a share α_t^i of the project and exerts effort a_t^i, the principal gets a flow reward $\alpha_t^i a_t^i \pi^i \left(X^i_{T^i(t)} \right)$. Worker i incurs a flow cost $\alpha_t^i a_t^i c^i \left(X^i_{T^i(t)} \right)$, proportional to the fraction of the task he is responsible for. Upon promotion (at time τ), worker i gets a payoff, $g^i > 0$, and is now compensated for working on the non-routine task: he gets a flow payoff $\alpha_t^i a_t^i c^i \left(X^i_{T^i(t)} \right)$. When the principal takes the outside option, i.e., allocate the promotion to an external worker, she gets $W > 0$.

I make the following assumption on the principal's flow rewards and the workers' flow costs.

Assumption 6 *(i)* $\pi^i : \mathcal{X}^i \to \mathbb{R}$ *is upper semicontinuous, nondecreasing, nonnegative, and* *such that*

$$\mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i(X_t^i)dt \mid X_0^i = x\right] < \infty$$

for all $x \in \mathcal{X}$*. (ii)* $c^i : \mathcal{X}^i \to \mathbb{R}$ *is lower semicontinuous, nonincreasing, and nonnegative.*

So, given a promotion contest (T, τ, d), the principal's expected payoff is

$$\Pi^M\left(T,\tau,d;W\right) := \mathbb{E}\left[\sum_{i=1}^N \int_0^\tau e^{-rt}\pi^i(X^i_{T^i(t)})dT^i(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)}, d_\tau\right)\right],$$

The workers' expected payoffs are

$$U^i\left(T,\tau,d\right) := \mathbb{E}\left[e^{-r\tau}gd^i_\tau - \int_0^\infty e^{-rt}(1-d^i_\tau\mathbb{1}_{\{t\geq\tau\}})c^i\left(X^i_{T^i(t)}\right)dT^i(t)\right].$$

Define also the workers' continuations payoff at time $t \geq 0$ as

$$U^i_t\left(T,\tau,d\right) := \mathbb{E}\left[e^{-r(\tau-t)}g^i d^i_\tau \mathbb{1}_{t\leq\tau} - \int_t^\infty e^{-r(s-t)}(1-d^i_\tau\mathbb{1}_{\{s\geq\tau\}})c^i\left(X^i_{T^i(s)}\right)dT^i(s) \mid \mathcal{G}^T_t\right].$$

Definition 3 *A promotion contest* (T, τ, d) *is* ***implementable*** *if there exists a promotion contest* (α, τ, d) *such that (i) there exists a (weak) Perfect Bayesian Nash equilibrium with effort processes* a *in the game defined by* (α, τ, d) *played by the workers, and (ii) such that, for all* $i \in \{1, \dots, N\}$*,*

$$T^i(t) = \int_0^t \alpha^i_s a^i_s ds, \quad 0 \leq t \leq \tau,\ \mathbb{P}\text{-a.s..}$$

Denote by $\mathcal{P}^I$ *be the set of all implementable promotion contests.*

The principal designs the promotion contest to maximize her total expected payoff among all implementable promotion contests:

$$\Pi^M := \sup_{(T,\tau,d)\in\mathcal{P}^I} \mathbb{E}\left[\sum_{i=1}^{N}\int_0^\tau e^{-rt}\pi^i(X^i_{T^i(t)})dT^i(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)}, d_\tau\right)\right]. \tag{Obj}$$

Finally, I make the following assumption.

Assumption 7 *For all $i \in \{1,\dots,N\}$, there exists $(T,\tau,d) \in \mathcal{P}^I$, with $T^i(t) = t$ for all $t \geq 0$, such that*

$$\mathbb{E}\left[\int_0^\tau e^{-rt}\pi^i\left(X^i_t\right)dt + e^{-r\tau}\left((1-d^0_\tau)\int_\tau^\infty e^{-r(t-\tau)}\pi^i\left(X^i_t\right)dt + d^0_\tau W\right)\right]$$
$$> \mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i\left(X^i_t\right)dt\right].$$

Assumption 7 guarantees that the principal's problem when worker i is the only candidate is not trivial, i.e., she can do better than promote worker i immediately. It is not needed, but it simplifies some of the arguments by restricting the number of cases to consider.

1.2.5 Discussion of the model

Before moving to the analysis, I comment on several features of the model.

A constrained multi-armed bandit model: As mentioned in the introduction, the model is a bandit problem with strategic arms. At each instant, the principal chooses which arm to pull (which worker to delegate) or takes her outside option. As in bandit models, the workers' types only evolve when they work on the project. For example, the principal learns about a worker's fixed but unknown potential. Implicit here is that the principal allocates her attention only to the worker delegated the non-routine task or that performing other jobs is not informative for the promotion. So learning is conditional on delegation. Another example corresponds to the acquisition of new skills and on-the-job learning: the workers' skills improve when responsible for the non-routine task.

Moreover, I assume that the workers' types are independent. The performance of one of the workers when delegated the task is uninformative about the potential of another worker. In particular, the workers do not learn from one another. I also focus on environments in which the workers do not need to cooperate: in my model, there is no payoff externality. The workers' efforts are substitutes, and the reward the principal obtains only depends on who is in charge of the non-routine task (and not on the types of other workers).

These assumptions are crucial. As in classic bandit models, very little can be said when the workers' types are correlated or when a worker's type evolves when the principal does not delegate the project to him.[17]

Multi-parameter formulation: To describe the types' dynamics, I adopt the multi-parameter approach pioneered by Mandelbaum in Mandelbaum (1986) for the multi-armed bandit model. This is critical to guarantee that the types' processes can be defined on a fixed (exogenous) probability space. It also simplifies the analysis. It also allows capturing many dynamic contracting environments with one unified approach. The alternative method would be to assume that the type of each worker is defined as the solution of a stochastic differential equation with drift, diffusion, and jump coefficients equal to zero when the workers do not exert effort. However, such stochastic differential equations would be unlikely to admit strong solutions.[18] By taking the multi-parameter approach, I do not need to work with multiple (endogenous) probability measures.

Only value of promotions is strategic: Another assumption of the model is the absence of direct value in promoting someone for the principal. The flow payoff from the non-routine task is the same whether the worker completing it has been promoted. One can

[17]One could relax the last assumption (the absence of payoff externalities), following the analysis in Nash (1980) or Eliaz et al. (2021). They prove that indexability (with Nash indices) holds in multi-armed bandit problems in which the reward from pulling an arm also depends on the states of the other arms. Nash (1980) consider the case when arms are complements, while Eliaz et al. (2021) consider both the cases when arms are substitutes and complements.

[18]See Karatzas (1984), and the discussion in Mandelbaum (1987).

think that a given non-routine task is associated with an opening position in the organization, for example, bringing a new product to the market. The principal allocates this same task whether or not she has already promoted a worker. So, the promotion has *no direct payoff effect*. It has, however, a strategic role. Workers value promotion. Hence, the principal uses the promises of a future promotion to motivate the workers. In particular, upon promotion, the workers get an *exogenous* prize and are compensated for their effort when working on the non-routine task.[19] This is for simplicity. It reflects the idea that the organization designs the position so that the promoted worker willingly exerts effort and obtains a strictly positive rent. It guarantees that the model remains tractable and allows me to focus on the interaction between the allocation of tasks and the promotion decision.

1.3 Main Result

Lemma 1 below characterizes the set of implementable promotion contests $\mathcal{P}^I$. In particular, it shows that it is nonempty and, hence, that the value of the principal is finite (by Assumption 6 (i)).

Lemma 1 *A promotion contest is implementable if and only if the continuation value of each worker is nonnegative after any possible history: For all $i \in \{1, \ldots, N\}$ and all $t \geq 0$, $U_t^i \geq 0$.*

Its proof is in Appendix A.1.2. It follows from Lemma 1 that, in any implementable promotion contest, the non-routine task is allocated to the promoted worker forever once the promotion decision is made. The principal already spent her incentive capital. The best thing she can do is then to delegate the task to the promoted worker. So, with a small abuse

[19] In Section 1.6.2, I relax this assumption and allow the principal to design the prize.

of notation, redefine the continuation value the principal obtains upon promotion as

$$\bar{\pi}\left(X_{T(\tau)}, d\right) := d_\tau^0 W + \sum_{i=1}^{N} d_\tau^i \int_\tau^\infty e^{-rt}\pi^i\left(X^i_{T^i(\tau)+t}\right) dt.$$

The principal's problem (Obj) is then equivalent to:

$$\Pi^M := \sup_{(T,\tau,d)\in\mathcal{P}} \mathbb{E}\left[\sum_{i=1}^{N}\int_0^\tau e^{-rt}\pi^i(X^i_{T^i(t)})dT^i(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)}, d_\tau\right)\right],$$

subject to the dynamic participation constraints: for all i and all possible histories h_t with $t \leq \tau$,

$$\mathbb{E}\left[e^{-r(\tau-t)}g^i d_\tau^i - \int_0^\tau e^{-r(s-t)}c^i\left(X^i_{T^i(s)}\right) dT^i(s) \mid h_t\right] \geq 0.$$

1.3.1 Benchmark

A natural benchmark is when the principal does not need to incentivize the workers to exert effort (which corresponds to $c^i(\cdot) = 0$ for all i). The problem then reduces to the standard multi-armed bandit problem (with passive arms):

$$\sup_{(T,\tau)\in\mathcal{D}\times\mathcal{T}} \mathbb{E}\left[\sum_{i=1}^{N}\int_0^\tau e^{-rt}\pi^i\left(X^i_{T^i(t)}\right) dT^i(t) + e^{-r\tau}W\right]. \tag{Bm}$$

Hence, when $c^i(\cdot) = 0$, all promotion contests give a nonnegative continuation payoff to the worker i after any possible history. Since the flow rewards the principal obtains when worker i performs the non-routine task are the same before and after promotion, promoting worker i has no direct value. It also has no strategic value when $c^i(\cdot) = 0$. However, it has a cost: it restricts the principal's future options. So the principal always wants to postpone the promotion. When the workers do not need to be incentivized, any rationale for promotion disappears, and it is never optimal to promote a worker.

The solution of this problem is well-known. It is the index rule associated with the the (classic) Gittins' index. Both index rules and the Gittins' indices are defined now.

Definition 4 *A delegation process T is called an* ***index rule*** *if, for all $i \in \{1, \ldots, N\}$, there exists a $\mathcal{F}^i$-adapted processes $\Gamma^i := \{\Gamma^i_t\}_{t\geq 0}$ such that $T^i(t)$ is flat off the set*

$$\left\{ t \geq 0 \,:\, \underline{\Gamma}^i_{T^i(t)} = \bigvee_{j=1}^{N} \underline{\Gamma}^j_{T^j(t)} \right\} \mathbb{P}\text{-}a.s.,$$

where $\underline{\Gamma}^i_t = \inf_{0\leq s\leq t} \Gamma^i_s$.

The process Γ^i is ***worker i's index****.*

In continuous time, the existence of index rules is not obvious. It is proved (by construction) in Mandelbaum (1987), El Karoui and Karatzas (1994), or El Karoui and Karatzas (1997). For completeness, in Appendix A.2.1, I reproduce the construction in El Karoui and Karatzas (1997) to obtain an index delegation rule associated with (arbitrary) indices $\left(\Gamma^1, \ldots, \Gamma^N\right)$, as I will need properties specific to this construction.

Definition 5 *The (classic) Gittins' index $\Gamma^{g,i} := \left\{\Gamma^{g,i}_t\right\}_{t\geq 0}$ associated with worker (arm) i is defined by, for all $t \geq 0$,*

$$r\Gamma^{g,i}_t := \sup_{\tau > t} \frac{\mathbb{E}\left[\int_t^\tau e^{-rs}\pi^i(X^i_s)ds \mid \mathcal{F}^i_t\right]}{\mathbb{E}\left[\int_t^\tau e^{-rs}ds \mid \mathcal{F}^i_t\right]}, \tag{GI}$$

with the convention that $\frac{0}{0} = -\infty$

$\Gamma^{g,i}_t$ is the maximal constant price the principal is willing to pay to include worker i in the pool of candidates up to time $t + \tau^*$; where τ^* is the optimal stopping time in (GI). Γ^i_t captures both the payoff from exploiting arm i up to time $t + \tau^*$ and the value of the information the principal obtains.

Proposition 1 *The index rule associated with the Gittin's indices is optimal in the multi-armed bandit problem (with passive arms).*

Proposition 1 restates the well-known optimality of the Gittins' index rule for the multi-armed bandit problem. It is obtained as a special case of the main Theorem 1 below. Its proof is in Appendix A.1.4.

Proposition 1 formally establishes that giving the prize to any of the candidates is never optimal when they do not need to be incentivized. Hence, it confirms that the value of the promotion is purely strategic in my model. When the workers do not need to be incentivized, the principal never promotes them. However, she still takes her outside option (i.e., hire externally) when she becomes too pessimistic any of the workers is good.

As a result, the principal only has to balance exploration and exploitation: delegating to a new worker to learn about him or to a worker known to be good to enjoy the higher reward obtained from the non-routine task. The Gittins' index rule addresses this trade-off. To see this, suppose that every time the principal delegates to worker i, she has to pay $\underline{\Gamma}^i_{T^i(t)} := \inf_{0 \leq s \leq t} \Gamma^i_{T^i(s)}$. By definition, it is the maximal flow price the principal would pay to delegate to worker i from time t to $t + \tau^*$. So the principal is indifferent between allocating the task to worker i and stopping the game: her value from delegating to worker i is zero. Following the Gittins index rule guarantees that her continuation value at all times is zero. If she, however, were to choose a different strategy, her value would be negative. So, given such prices, the index rule is optimal: it maximizes the profit collected by the bandit machine as it moves up the use of the more costly arms and postpones the use of the less costly ones. Since the prices are set to be the greatest possible to ensure the principal participation, they maximize the profit of the bandit machine among all possible prices. The index rule, therefore, maximizes total surplus and hence is optimal. This intuition was developed by Weber in his proof of indexability in Weber (1992).

However, because the Gittins' rule never promotes any of the workers, it is not implementable: the continuation value of each worker when delegated the non-routine task is strictly negative. Hence the need to incentivize the workers to exert effort constrains the principal ability to learn. So the Gittins' "prices" are too high in the index contest: the principal would not delegate to the workers at these prices. In the next Section, I solve the multi-armed bandit problem with strategic arms.

1.3.2 The index contest

The strategic index rule will play a crucial role. To define it formally, I need to introduce the promotion thresholds, $P^i(\cdot)$'s, and promotion times, $\tau^{s,i}$'s, first. Define $\tau^i_{(\underline{x},\bar{x})} := \inf\{t \geq 0 \,:\, , X^i_t \notin (\underline{x}, \bar{x})\}$. For all $i \in \{1, \ldots, N\}$, for all $\underline{x} \leq x \leq \bar{x} \in \mathcal{X}^i$, let

$$U^i(x, \underline{x}, \bar{x}) := \mathbb{E}\left[e^{-r\tau^i_{(\underline{x},\bar{x})}} g^i \mathbb{1}_{\{X^i_\tau \geq \bar{x}\}} - \int_0^{\tau^i_{(\underline{x},\bar{x})}} e^{-rt} c^i(X^i_t) dt \mid X^i_0 = x\right].$$

$U^i(x, \underline{x}, \bar{x})$ is i's continuation value when his current type is x and he exerts effort until his type exceeds $\bar{x}$ and he is promoted or his type falls below $\underline{x}$ and he "quits" and gets payoff 0. Define then i's promotion threshold as:

$$\bar{P}^i(\underline{x}) :- \sup\left\{\bar{x} \geq \underline{x} \,:\, \lim_{x \to \underline{x}} U^i\left(x, \underline{x}, \bar{x}\right) \geq 0\right\}.$$

$\bar{P}^i(\underline{x})$ is the largest promotion threshold for which worker i is willing to stay in the game as long as his type does not fall below $\underline{x}$. Moreover $\bar{P}^i(\cdot)$ is *increasing*.

Define also the stopping time $\tau^{s,i} := \inf\left\{t \geq 0 \,:\, X^i_t \geq \bar{P}^i\left(\underline{X}^i_t\right)\right\}$, where $\underline{X}^i_t := \inf_{0 \leq s \leq t} X^i_s$ is the running minimum of X^i. Theorem 2 in Section 1.5.2 shows that $\tau^{s,i}$ is the optimal promotion time when worker i is the only worker. Next define the $\mathcal{F}^i$-adapted process h^i as

$$h_t^{s,i} := \pi^i \left(X_t^i\right) \mathbb{1}_{\{t<\tau^{s,i}\}} + \bar{\pi}^i \left(X_{\tau^{s,i}}^i\right) \mathbb{1}_{\{t\geq\tau^{s,i}\}}, \quad t \geq 0,$$

where

$$\bar{\pi}^i (x) := r\mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i \left(X_t^i\right) dt \mid X_0^i = x\right].$$

The **strategic index** of worker i is defined by

$$\Gamma_t^{s,i} := \inf\left\{W \geq 0 \,:\, V^i(t;W) \leq W\right\},$$

where

$$V^i(t;W) := \sup_{\tau\geq t} \mathbb{E}\left[\int_0^\tau e^{-r(s-t)}h_t^{s,i}dt + e^{-r(\tau-t)}W \mid \mathcal{F}_t^i\right].$$

Worker i's index is the "equitable surrender value", i.e., the smallest W such that the principal prefers to take the outside option immediately rather than to delegate to worker i for some time before making a decision (when worker i is the only worker). Moreover, observe that, by assumption 2, the strategic index is a function of X_t^i and $\underline{X}_t^i$ only: $\Gamma_t^{s,i} = \Gamma^{s,i}\left(X_t^i, \underline{X}_t^i\right)$.

As in the classical bandit problem, it can be shown to be equal to

$$r\Gamma_t^{s,i} = \sup_{\tau>t} \frac{\mathbb{E}\left[\int_t^\tau e^{-rs}h_s^{s,i}ds \mid \mathcal{F}_t^i\right]}{\mathbb{E}\left[\int_t^\tau e^{-rs}ds \mid \mathcal{F}_t^i\right]}, \tag{1.2}$$

with the convention that $\frac{0}{0} = -\infty$. The strategic index coincides with the classical Gittins index for the modified payoff stream $\{h_s^i\}_{s\geq0}$. In particular, $\Gamma_t^{s,i}$ is the maximum price the principal is willing to pay for the possibility of including the worker in the pool of candidates. Moreover, the second expression also makes it clear that the worker's *strategic index* is similar to the classic Gittins' index (GI). The difference resides in what information is optimally *acquired.*

Here, the workers control both the rewards and the flow of information. Moreover, their incentives differ from the principal's. Worker i's strategic index then takes into account the incentive provision problem. Since worker i only exerts effort if it increases his chance of promotion, the principal has to motivate him to work by promising he will eventually get the prize. However, upon promotion, collecting information has no value for the principal, as she cannot incentivize other workers to work anymore. That's captured in the definition of the process $h^{s,i}$: after the promotion time, the flow reward is $\mathbb{E}\left[\pi^i(X_t^i) \mid \mathcal{F}_\tau^i\right]$. It is as if no new information is obtained. Contrary to Gittins' index, the strategic index ignores the information generated after the promotion decision when assessing the value of delegating to a worker. Interestingly, when the cost of providing incentives goes to zero (when $c^i \to 0$), the strategic index process converges to the Gittins' index from below pointwise $\mathbb{P}$-a.s..

The index delegation rule associated with the strategic index processes $\left(\Gamma^{s,1}, \ldots, \Gamma^{s,N}\right)$ is called the **strategic index rule**.

Definition 6 *The **index contest** (i) follows the **strategic index rule**, (ii) promotes the **first** worker i whose type reaches his **promotion threshold** $\bar{P}^i(\underline{X}_t^i)$, and (iii) takes the outside option at time* $\tau^0 = \inf\left\{t \geq 0 \,:\, \bigvee_{i=1}^N \underline{\Gamma}^i_{T^{s,i}(t)} \leq W\right\}$ *if no worker was promoted before.*

Figure 1.2 illustrates the **index contest** with two workers. Each can be good or bad. Their types are the posterior beliefs that they are good, and the principal learns about them according to the Poisson arrival of bad news. Initially, worker 1 is better, so the principal first delegates to worker 1. However, too much bad news arrives. Therefore, she switches to worker 2. Worker 2 performs well and eventually gets the promotion.

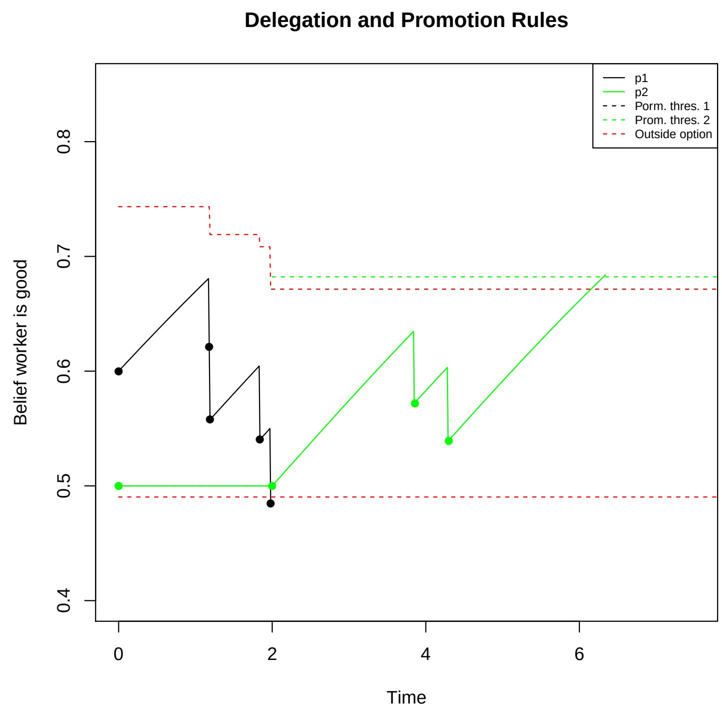

Figure 1.2: Index contest

Proposition 2 *The index contest can be* ***implemented*** *in a (weak) Perfect Bayesian equilibrium without commitment. The workers' strategies only depends on their own type $X^i_{T^i(t)}$, the current running minimum of their type $\underline{X}^i_{T^i(t)}$, and whether the principal promoted a worker. The principal's payoff is*

$$\mathbb{E}\left[\int_0^\infty re^{-rt}\bigvee_{i=1}^N \underline{\Gamma}^{s,i}_{T^{s,i}(t)}dt\right]. \tag{Π^M}$$

The proof of Proposition 2 is in Appendix A.1.3.

1.3.3 Optimality

The main result is the optimality of the *index contest*: despite the agency frictions, indexability is preserved. When deciding who to delegate the non-routine task to and who to promote, the principal considers each worker separately. She delegates to and eventually promotes the best worker, as measured by the value of his associated strategic index.

Theorem 1 *The* ***index contest*** *maximizes the principal's payoff among all implementable promotion contests.*

Theorem 1 is proved in Section 1.5. Its proof requires to address two main challenges.

First, implementable promotion contests have to balance the incentives of all workers. So, there is no reason a priori that it treats them separately. For example, if the principal promotes worker 1 when his type exceeds that of worker 2 by $\frac{1}{2}$, it creates a strategic dependence between the arms. The optimal delegation rule may not be an index rule, and the index of both workers 1 and 2 does not remain frozen when the other worker is in charge of the non-routine task.

To overcome this problem, I show that the principal can treat the workers separately (when it comes to incentive provision), i.e., she chooses N different promotion rules, each incentivizing one worker to exert effort. To do so, I study a relaxed problem in which participation constraints only hold in expectations (conditional on the workers' type). The relaxed problem coincides with the setting where each worker can only see the evolution of his own type. It pulls together many information sets. In that relaxed problem, workers have to be informed when promoted to maximize the length of the experimentation phase. Otherwise, their continuation payoff would be strictly less than the value they associate with the promotion. Therefore the principal could delegate the project a little longer before making her decision (which benefits the principal). So, the promotion time associated with each worker has to be measurable with respect to their own type. As a result, it is without loss of optimality for the principal to choose a delegation rule and N individual promotion contracts (i.e., N individual promotion time and promotion decision that depend only on the type of the worker).

The solution, however, needs not be a solution to the original problem, as the delegation rule and individual promotion contracts may not be jointly implementable. Hence the principal may be unable to keep the independent promises she made to distinct workers. I will come back to this after describing the relaxed problem more carefully.

Second, even if each worker's promotion time and decision depend only on his own type process, the problem is still not a standard bandit problem. To use the techniques developed in the bandit literature, I rewrite the flow payoff the principal gets upon promotion as the expected payoff from delegating to the worker, conditional on the information available at the time of promotion. Each arm is then a superprocess: each arm comprises multiple possible reward processes, one for each individual promotion contract. So, the principal chooses both which arm to pull and which contract to offer. In particular, when the principal pulls an arm, the flow payoff and the information depends on the "promotion contract".

There is no guarantee that indexability holds for superprocesses. However, in the Markovian setting, there exists a condition for which it does, sometimes known as the Whittle condition or condition W (Whittle (1980), Glazebrook (1982)). It says that the optimal action chosen in each state in the single-armed retirement problem is independent of the outside option W. If for some value of the outside option, it is optimal to choose an action rule, then it is also optimal to choose the same action rule for any other value of the outside option (as long as the arm is not retired). If the Whittle condition holds, the bandit problem with superprocesses is indexable.

In my setting, I show that a version of condition W for general (non-Markovian) superprocesses holds in the single-worker promotion problem. At time 0, the optimal promotion contract in the single-worker problem is independent of the outside option (before the principal takes her outside option). That is, provided that the principal has not taken her outside option yet, if the worker is promoted after some history, he is also promoted after this same history when the value of the outside option is smaller. I then show that this condition is sufficient for indexability to hold.

The index contest is the solution to this problem. To build some intuition for this result, consider the case of two workers. Suppose that the strategic index of worker 1 is initially higher than that of worker 2. Suppose also that the value of the principal's outside option equals worker 2's index. If worker 2 were the only worker, the principal would take the outside option immediately. The principal's problem then reduces to the problem in which she can only delegate to worker 1, promote worker 1, or take the outside option. The *index contest* guarantees that the principal offers the optimal single agent promotion mechanism to worker 1 (as Theorem 2 in Section 1.5.2 below shows). Eventually, either worker 1 is promoted, and the game ends, or his strategic index falls below the strategic index of worker 2, hence, below the value of the outside option. In that case, the principal should take her outside option. Instead, imagine that, when it happens, the value of the outside option also falls to the level of worker 1's strategic index. In the new continuation problem, the principal never delegates to worker 1. However, she is willing to delegate to worker 2. In particular, she offers the single-player optimum promotion contract to worker 2. The *index contest* repeatedly plays the single-player optimal promotion mechanism for the best current worker (as measured by the strategic indices) until one worker is promoted or the principal takes her outside option. Promotion happens the first time a worker's type reaches $\bar{P}^i(\underline{X}_t^i)$. The principal takes the outside option when there is no benefit from experimentation anymore, i.e., when $\Gamma^{s,i} \leq W$ for all i. So, at any point in time, when one worker is delegated the project, his promotion threshold is equal to the optimal threshold in the single-agent problem.

As mentioned above, the index contest needs not be implementable. Fortunately, it is, as Proposition 2 shows. Intuitively, when only one worker is allocated the task, the only promise-keeping constraint that matters is the one for the worker currently assigned the task. All other constraints are redundant and can be ignored.

The above intuition suggests the following interpretation of the *index contest*. The principal approaches the workers successively. The indices' ranking determines the order in which the workers are approached. When the principal selects one worker, she offers him an *individual trial contract*. It consists of a target: the promotion threshold, and a (potentially stochastic) deadline. If the worker achieves the target before the deadline, the principal promotes him. He is then in charge of the non-routine task forever. Otherwise, the principal approaches another worker until one succeeds, or the principal becomes too pessimistic and takes her outside option. Interestingly, if the principal could reoptimize at the end of each short-term contract (when she starts delegating to a new worker in the index contest), she would choose the same continuation promotion contest. Every time a new worker gets an opportunity to prove himself, the continuation mechanism is optimal for the principal.

The above interpretation of the index contest is reminiscent of promotion practices described in the strategic management literature. For example, Stumpf and London (1981) propose to evaluate the workers sequentially until the principal identifies a satisfactory one.

More generally, the *index contest* is also related to absolute merit-based promotion systems,[20] in which the first worker who meets a minimum performance target gets the promotion. My results suggest that one should expect organizations to use absolute merit-based promotion systems when it is important to fill the position with the right worker. On the other hand, when motivating effort is more important, other promotion systems, such as the classic winner-take-all promotion contest of Lazear and Rosen (1981) or the cyclical egalitarian contest proposed by Ely et al. (2021) may be better, and, hence, more common. Intuitively, these promotion systems are very good at incentivizing effort but less so at ensuring that the promoted worker's potential is high. The *index contest* guarantees that the non-routine task runs smoothly after the promotion decision is made. It balances incentives provision and selection.

1.3.4 Features of the index contest

No commitment: Often, one may want to assume little commitment within an organization: most of the day-to-day activities are not governed by formal contracts, it is unlikely that the performance of a worker is verifiable... In my setting, the principal does not need any commitment power, as Proposition 2 shows. The *index contest* is implementable even if the principal cannot commit to the delegation rule, delegation time, or promotion decision. Maybe even more interestingly, it does not require sophisticated coordinated punishments. It is implementable in a (weak) Perfect Bayesian equilibrium by "grim trigger" strategies. Moreover, each worker's strategy only depends on his current type, the running minimum of his type process, and whether the principal has promoted a worker yet.

Fast track: In the **index contest**, the promotion thresholds are decreasing over time (as increasing functions of the running minimums of the workers' types). So a worker's potential upon promotion decreases with time:

[20] See Phelan and Lin (2001).

Proposition 3 (Speed and accuracy) *If $\pi^i(\cdot) = \pi(\cdot)$ for all $i \in \{1, \ldots, N\}$ and the processes X^i's have the same law, then the promoted worker's type and the principal's continuation value upon promotion are nonincreasing over time $\mathbb{P}$-a.s..*

Proposition 3 follows from the fact that the promotion threshold is $\mathbb{P}$-a.s. nonincreasing over time. The proof is omitted.

Pushing the interpretation beyond the model, the above proposition suggests that fast tracks[21] should not be surprising. When a worker is promoted quickly, his type upon promotion is high. So, when entering a potential new contest for further promotion at the next level of the organization, he starts from a better position. In turn, it implies that his expected time to promotion is shorter and that the worker's chances to be promoted again soon are high.

Seniority: Finally, the decrease over time of the promotion thresholds also has an interesting implication for seniority. As time passes, it becomes easier for each worker to be promoted (conditional on his type). Proposition 4 formalizes this statements.

Proposition 4 *In the index contest, worker i's promotion probability, $\mathbb{E}\left[d^i \mid \mathcal{G}_t^{T^s}\right]$ and continuation value, U_t^i, are non nondecreasing over time conditional on $X_{T^i(t)} = x$. His expected time to a promotion is nonincreasing in t, conditional on $X^i_{T^i(t)} = x$.*

Proposition 4 also follows immediately from the promotion threshold being nonincreasing over time $\mathbb{P}$-a.s.. The proof is omitted.

Convex compensation structure: Learning is essential when the cost of promoting the wrong worker is high. Hence, the principal always benefits from a larger prize, as illustrated by the following proposition.

Proposition 5 (Value of the project) *The principal's value increases with the value of the promotion $g = \left(g^1, \ldots, g^N\right)$.*

[21] I.e., that a quickly promoted worker often gets another promotion soon after. See Baker et al. (1994) and Ariga et al. (1999).

Proposition 2 is immediate: Let $\bar{g} \geq \underline{g}$. Then any promotion contest feasible for the value vector $\bar{g}$ is also feasible for $\underline{g}$.

But, she should benefit from a larger prize especially when learning is paramount. That's because it makes incentivizing experimentation easier and helps the principal make a better decision. Proposition 6 confirms this point and shows that the principal acquires more information about the promoted worker as g^i increases.

Proposition 6 *As g^i increases, the principal learns more about worker i.*

Proof of Proposition 6. Let $\bar{g}^i \geq \underline{g}^i$. Observe first that the index of worker i is greater when the prize is $\bar{g}^i$; therefore, the principal acquires information about worker i sooner. Moreover, in the index contest with reward $g^i \in \left\{\bar{g}^i, \underline{g}^i\right\}$, worker i is promoted after being responsible for the non-routine task for the time $\tau^i(g^i) = \inf\left\{t \geq 0 : X_t^i \geq \bar{P}^i(\underline{X}_t^i; g^i)\right\}$. Note that $\bar{P}^i(\cdot, \bar{g}^i) \geq \bar{P}^i(\cdot, \underline{g}^i)$, and therefore $\tau^i\left(\bar{g}^i\right) \geq \tau^i\left(\underline{g}^i\right)$. Putting these two observations together concludes the proof. ■

Intuitively, when workers value the promotion more (i.e., the prize is bigger), they are willing to exert effort for an extended time. So the principal can acquire more information and make a better promotion decision.

This can help understand why many organizations have a convex compensation structure (i.e., the bonuses paid upon promotion and the wage spread between positions increase when moving up in the hierarchy).[22] At the top of the organization, the cost of promoting the wrong worker is potentially high. Extending the exploration phase is, therefore, valuable. A convex compensation structure achieves this.

However, how to measure the value of information here is not obvious. I propose to use the following definition:

[22] See DeVaro (2006) for example.

Definition 7 *The value of information in the promotion problem with $\tilde{\pi}^i(\cdot)$ is higher than the value of information in the promotion problem with $\pi^i(\cdot)$ if, for all $t \geq 0$,*

$$\frac{\partial \Gamma_t^{s,i}(\tilde{\pi}^i)}{\partial \tau^{s,i}} \geq \frac{\partial \Gamma_t^{s,i}(\pi^i)}{\partial \tau^{s,i}},$$

$\mathbb{P} - a.s.$, where $\Gamma_t^{s,i}(\pi)$ is the strategic index of worker i when the flow payoff the principal gets when worker i with type x^i leads the project is $\pi(x^i)$.

Intuitively, the above definition says that the benefit from waiting for one more instant before promoting worker i is larger for $\tilde{\pi}^i$ than for π^i, i.e., there is more to gain from acquiring information as the cost of mistakes increases. It captures the extent to which marginal information is actionable: whether it helps the principal to make a better decision.

Proposition 7 *Let $\bar{g} \geq \underline{g}$ and the value of information associated with $\tilde{\pi}^i$ be higher than the value of information associated with π^i, for all $i \in \{1, \dots, N\}$. Then*

$$\Pi^M(\bar{g}, \tilde{\pi}) - \Pi^M(\bar{g}, \pi) \geq \Pi^M(\underline{g}, \tilde{\pi}) - \Pi^M(\underline{g}, \pi). \tag{1.3}$$

Proof of Proposition 7. To prove (1.3), it is enough to show that, for all $i \in \{1, \dots, N\}$,

$$\frac{\partial \Pi^M(g, \pi)}{\partial g^i} \text{ is increasing in } \pi \text{ for the order of Defintion 7.}$$

since Π^M is nondecreasing and locally Lipschitz (by Assumption 5 and the definition of $\tau^{s,i}$ as $\pi^i(\cdot)$ is locally bounded) in g^i, hence, differentiable almost everywhere.

Consider $i \in \{1, \dots, N\}$. By the envelope theorem for dynamic optimization (e.g. Theorem 1 in LaFrance and Barney (1991) and the discussion above),

$$\frac{\partial \Pi^M(g, c)}{\partial g^i} = \mathbb{E}\left[\int_0^\infty e^{-rt} \frac{\partial \underline{\Gamma}^{s,i}_{T^{s,i}(t)}}{\partial g^i} dT^{s,i}(t)\right].$$

This follows immediately from my definition of an increase in the value of information, since

$$\frac{\partial \underline{\Gamma}^{s,i}_{T^{s,i}(t)}}{\partial g^i} = \frac{\partial \tau^{s,i}}{\partial g} \frac{\partial \underline{\Gamma}^{s,i}_{T^{s,i}(t)}}{\partial \tau^{s,i}},$$

$\tau^{s,i}$ is independent of π^i, and $\frac{\partial \tau^{s,i}}{\partial g} \geq 0$ by Proposition 6. ∎

Traditionally, contest theory has suggested that the convexity of the compensation structure in organizations results from the higher return of effort at higher positions in the hierarchy. My results offer a complimentary story: when the returns of selecting the right worker are high, larger bonuses let the principal experiment longer and promote a better worker.

1.4 Strategic amplification

One of the initial questions I asked was whether the allocation of opportunities could exacerbate initial differences to produce significant disparities over time. Because in the *index contest*, the principal delegates the project *sequentially* and promotes the *first* worker whose type reaches his promotion threshold, being delegated first is an advantage. This is especially true if, at every step of the index contest, i.e., during every trial contract, the probability that the worker leading the project reaches his target and hence gets the promotion is large.

In this section, I define a class of environments that I call *reinforcing environments*, in which initial differences compound. In these environments, being delegated the project leads to a significant chance of promotion. This has two main implications: First, the timing of the first opportunity matters. A worker in charge of the non-routine task earlier is much more likely to be promoted. So, what determines the assignment of non-routine tasks early on is crucial to understanding who has a chance to be promoted. Secondly, initial differences lead to substantial differences during the exploration phase. To identify discrimination, conditioning on the potential of the workers upon promotion or their history of responsibilities in the organization may be a bad idea. Both depend on the endogenous delegation path. If discrimination occurs in the allocation of opportunities, it will remain undetected.

The following example illustrates the logic. Two symmetric workers compete for the promotion. Their types' processes X^i keep track of their instantaneous (nonnegative) productivity. When they work on the project, their productivity drifts up at a constant speed μ. This could reflect on-the-job learning. However, they can reach a dead end. Dead ends arrive according to a Poisson process with parameter λ. When a dead end comes, the worker needs to devise a new strategy and restart from scratch: his type jumps to zero. So, the type of each worker evolves according to the differential equation $dX_t^i = \mu dt$ if he does not reach a dead end and jumps down to zero if he does. The principal gets a flow payoff of X_t^i when he delegates the project to worker $i \in \{1, 2\}$. I assume that the workers' costs of effort are constant and equal to $c > 0$ and that both associated value $g > 0$ to the promotion. Finally, I assume the principal's outside option is small and, therefore, never taken.

Let $\bar{t}$ be the unique solution of

$$\lambda c \int_0^{\bar{t}} e^{-(r+\lambda)t} dt = g.$$

The workers' promotion thresholds are given by $\bar{P}^i(\underline{X}_t^i) = \underline{X}_t^i + \mu \bar{t}$. The workers' indices can be taken to be the worker's types.[23]

In the index contest, the first worker is promoted before the second worker even has a chance to lead the project with probability $(1 - e^{-\lambda \bar{t}})$. Moreover, if the principal (lexicographically) prefers worker 1, i.e., when indifferent, she delegates to worker 1, then the probability that worker 2 is promoted in this environment is $(1 - e^{-\lambda \bar{t}})e^{-\lambda \bar{t}}$. That is, worker

[23] For $i \in \{1, 2\}$, $\Gamma_t^{s,i} = \Gamma^s(X_t^i, \underline{X}_t^i)$ and the function $\Gamma^s(\cdot, \cdot)$ is increasing both variables. So the ranking of the indices at any instant is the same as the ranking of types when the principal plays the associated index delegation rule.

2 is promoted if and only if worker 1 does not succeed initially and worker 2 does not hit a dead end the first (and only) time he works on the non-routine task. So, when $\bar{t}$ is either small or large, worker 2's promotion probability is close to zero. On the other hand, worker 1's promotion probability is close to one.

The above example is simple and clearly illustrates that the sequential nature of delegation exacerbates small differences in environments in which the workers' types (and, hence, their indices) tend to go up when they work. Under the condition below, the logic of the above example easily extends.

Definition 8 *An environment* $(X^i, c^i(\cdot), g^i, \pi^i(\cdot))_{i=1}^N$ *is* ***reinforcing*** *if, there exists* $\delta > 0$ *such that, for all* $i \in \underset{j\in\{1,\dots,N\}}{\arg\max}\, \Gamma_0^{i,s}$,

$$\mathbb{P}\left(\tau^i \leq \tau_-^i(X_0^i)\right) > \delta, \tag{RC}$$

where $\tau^i = \inf\left\{t \geq 0 \,:\, X_t^i \geq \bar{P}^i(X_0^i)\right\}$ *and* $\tau_-^i(X_0^i) = \inf\left\{t \geq 0 \,:\, X_t^i < \bar{P}^i(X_0^i)\right\}$.

Proposition 8 *In a reinforcing environment, a worker* $i \notin \underset{j\in\{1,\dots,N\}}{\arg\max}\, \Gamma_0^{i,s}$*'s probability to be promoted is bounded above by*

$$(1-\delta)^K.$$

where K *is the cardinality of* $\underset{j\in\{1,\dots,N\}}{\arg\max}\, \Gamma_0^{i,s}$.

Proof. In the index contest, every worker $k \in \underset{j\in\{1,\dots,N\}}{\arg\max}\, \Gamma_0^{i,s}$ will be delegated the project before worker i. The probability that each of the worker $k \in \underset{j\in\{1,\dots,N\}}{\arg\max}\, \Gamma_0^{i,s}$ succeeds upon being delegated the project is greater than δ. The result then follows. ∎

A direct consequence of Proposition 8 is that if δ is large, then the first worker gets the promotion with a considerable probability, and the other workers will not. Moreover, in large promotion contests with two different groups, each composed of initially identical workers, workers from the disadvantaged group face long odds when it comes to promotions. When the pool of candidates for promotion is large, any slight initial disadvantage is disqualifying. The logic here is reminiscent of Cornell and Welch (1996).

My findings can help understand some of the mechanisms behind the "promotion gaps" documented in the literature (see, for example, Bronson and Thoursie (2021), Benson et al. (2021) and Hospido et al. (2022)). This is especially important as wage growth is known to be closely related to job mobility, especially within firms (see Baker and Holmstrom (1995), Lazear and Shaw (2007), or Waldman (2013) and the references therein). The main point is that understanding and addressing the roots and causes of the different allocations of opportunities is crucial.

1.5 Proof of Theorem 1

The proof of Theorem 1 is divided into the five following steps.

- In Section 1.5.1, I relax the problem: in particular, each worker's dynamic participation constraint must only hold on expectation (conditional on the worker's own type), but not necessarily after all possible histories.

- Section 1.5.2 solves the problem with only one worker. Its solution is given in Theorem 2. The argument adapts the logic of the proof of Theorem 1 in McClellan (2017) to our setting.[24]

- In Section 1.5.3, I show that it is without loss of optimality to focus on promotion contests such that at most one worker is promoted and such that the promotion time of worker i is a $\mathcal{F}^i$-stopping time.

[24] See also Harris and Holmstrom (1982), Thomas and Worrall (1988), or Grochulski and Zhang (2011) for similar ideas.

- Next, in Section 1.5.4, I derive an upper bound on the payoff the principal can get in any promotion contest that gives a nonnegative continuation value to all workers at all times, using the results from the three previous steps. Proposition 11 establishes that the principal's payoff in any implementable promotion contest is at most (Π^M).

- Section 1.5.5 verifies that the **index contest** achieves the upper bound, hence proving Theorem 1. This follows from Proposition 2.

1.5.1 The Relaxed Program

The principal solves the following optimization program:

$$\Pi^M := \sup_{(T,\tau,d)\in\mathcal{P}} \mathbb{E}\left[\sum_{i=1}^{N}\int_0^\tau e^{-rt}\pi^i(X^i_{T^i(t)})dT^i(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)}, d_\tau\right)\right], \tag{Obj}$$

subject to the dynamic participation constraints: for all i and all possible histories h_t with $t \le \tau$,

$$\mathbb{E}\left[e^{-r(\tau-t)}g^i d^i_\tau - \int_0^\tau e^{-r(s-t)}c^i\left(X^i_{T^i(s)}\right)dT^i(s) \mid h_t\right] \ge 0.$$

As a first step in the proof, I consider the relaxed problem in which the principal can randomize over possible stopping point. To introduce it formally, I need to define a number of new objects. For a filtration $\mathcal{H} = \{\mathcal{H}_t\}_{t\ge 0}$, define the set of **$\mathcal{H}$-randomized** stopping times as

$$\mathcal{S}(\mathcal{H}) := \left\{S \in \mathcal{N}_0^\infty(\mathcal{H}) \,:\, dS \in \mathcal{M}_+^\infty(\mathcal{H}),\, S_{0^-} = 0,\, S_\infty \le 1\right\}.$$

$\mathcal{N}_0^\infty(\mathcal{H})$ is the set of $\mathcal{H}$-adapted process with values in $[0,\infty)$ such that $n \in \mathcal{N}_0^\infty(\mathcal{H})$ if n has nondecreasing paths $\mathbb{P}$-a.s.. $\mathcal{M}_+^\infty(\mathcal{H})$ is the set of $\mathcal{H}$-optional random measure. Observe that any randomized stopping time is equivalent to a $\mathcal{F}_t \otimes \mathcal{B}([0,1])$-stopping time defined on the enlarged filtered probability space $(\Omega \times [0,1], \mathcal{H} \times \mathcal{B}([0,1]), \{\mathcal{H}_t \times \mathcal{B}([0,1])\}_{t\geq 0}, \mathbb{P} \otimes \lambda)$, where λ is the Lebesgue measure on $[0,1]$.[25] Finally, let $\mathcal{C}$ be the set of $\bar{\mathcal{F}}$-measurable promotion rule:

$$\mathcal{C} := \left\{ d \ : \ \text{for all } t \geq 0,\ d_t \text{ is } \bar{\mathcal{F}}\text{-measurable and } \sum_{i=0}^{N} d_t^i = 1 \ \mathbb{P}\text{-a.s.} \right\},$$

and $\mathcal{C}^*$ be the set of nondecreasing promotion rule:

$$\mathcal{C}^* := \left\{ d \in \mathcal{C} \ : \ d^i\text{'s paths are càdlàg and nondecreasing } \mathbb{P}\text{-a.s. for } i = 1, \ldots, N \right\}.$$

The set of **randomized promotion contest** consists of all the promotion contests such that the promotion time τ is a randomized stopping time: $\tau \in \mathcal{S}(\mathcal{G}^T)$, and the decision rule d belongs to $\mathcal{C}^*$. It is denoted by $\mathcal{P}^r$.

Consider then the relaxed program:

$$\Pi := \sup_{(T,\tau,d)\in\mathcal{P}^r} \mathbb{E}\left[\sum_{i=1}^{N} \int_0^\tau e^{-rt} \pi^i \left(X^i_{T^i(t)} \right) dT^i(t) + e^{-r\tau} \bar{\pi} \left(X_{T(\tau)}, d \right) \right] \tag{RP}$$

subject to, for all $i \in \{1, \ldots, N\}$, for all $t \geq 0$, $\mathbb{P}$-a.s.,

$$\mathbb{E}\left[e^{-r(\tau - \tau\wedge t)} g^i d^i_\tau - \int_{\tau\wedge t}^{\tau} e^{-r(s-\tau\wedge t)} c^i \left(X^i_{T^i(s)} \right) dT^i(s) \mid \mathcal{F}^i_{T^i(t)} \right] \geq 0. \tag{DPC}$$

Proposition 9 *The value of* (Obj) *is weakly lower than the value of* (RP)*:* $\Pi^M \leq \Pi$.

Proposition 9 shows that the value of program (RP) is an upper bound on the principal's payoff for any implementable promotion contest. If an implementable promotion contest

[25] See, for example, Camboni and Durandard (2022).

achieves this upper bound, this is the principal's preferred one. It relaxes (Obj) in three ways. First, it replaces the feasibility set $\mathcal{P}^I$ by the set of all randomized promotion contests. This will allow to prove compactness. Secondly, it only requires that the workers have nonnegative continuation payoffs at all times $\mathbb{P}$-a.s. (and not necessarily after all possible histories). Thirdly, it pulls together all the $\mathcal{G}_t^T$ information sets that are not $\mathcal{F}^i_{T^i(t)}$ measurable, hence relaxing the constraints the principal faces. Its proof is in Appendix A.1.5.

The remaining of Section 1.5 is dedicated to the proof that the index contest achieves the optimum in (RP).

1.5.2 The $1\frac{1}{2}$-arm case

As in the classic bandit framework, the solution builds on the one arm problem. When there is only one worker, say worker i, the relaxed problem (RP) introduced above becomes

$$\Pi^i := \sup_{(\tau,d^i)\in\mathcal{P}^{r,i}} \mathbb{E}\left[\int_0^\tau e^{-rt}\pi^i\left(X_t^i\right)dt + e^{-r\tau}\left(d_\tau^i\int_\tau^\infty e^{-r(t-\tau)}\pi^i\left(X_t^i\right)dt + (1-d_\tau^i)W\right)\right] \tag{RP^i}$$

subject to, for all $t\geq 0$, $\mathbb{P}$-a.s.,

$$\mathbb{E}\left[e^{-r(\tau-\tau\wedge t)}g^i d_\tau^i - \int_{\tau\wedge t}^\tau e^{-r(s-\tau\wedge t)}c^i\left(X_s^i\right)ds \mid \mathcal{F}_t^i\right]\geq 0. \tag{DPC^i}$$

$\mathcal{P}^{r,i}$ is the set of all pairs (τ, d^i) such that τ is a (randomized) $\mathcal{F}^i$-stopping time and d^i is a $\mathcal{F}^i$-optional decision rule in $\mathcal{C}^*$. Define also $\mathcal{P}^{I,r,i}$: the set of all pairs $(\tau,d^i)\in\mathcal{P}^{r,i}$ that satisfy the constraints (DPC^i).

Recall that

$$U^i(x,\underline{x},\bar{x}) := \mathbb{E}\left[e^{-r\tau}g^i d_\tau^i - \int_0^\tau e^{-rt}c^i\left(X_t^i\right)dt \mid x\right]$$

is the continuation value of the worker with $X_0 = x$, $\tau = \inf\left\{t \geq 0 \,:\, X^i_t \notin (\underline{x}, \bar{x})\right\}$ and $d^i_\tau = \mathbb{1}_{\{X^i_\tau \geq \bar{x}\}}$. Define then

$$p^i(P) \coloneqq \inf\left\{x \in \mathcal{X}^i \,:\, \sup_{p \in \mathcal{X}^i} U^i(P, p, x) > 0\right\}.$$

$p^i(P)$ is the smallest value of $x \in \mathcal{X}^i$ at which the worker is willing to keep working if he is promoted only when his type exceed P. Finally also

Recall also that worker i's promotion threshold is given by the (nondecreasing) function $\bar{P}^i$ by

$$\bar{P}^i(\underline{x}) = \sup\left\{\bar{x} \geq \underline{x} \,:\, \lim_{x \to \underline{x}} U^i\left(x, \underline{x}, \bar{x}\right) \geq 0\right\}.$$

Finally define $\underline{p}^i(W)$ as the unique solution of

$$\Gamma^{s,i}\left(\underline{p}^i, \underline{p}^i\right) = W.$$

Theorem 2 characterizes the solution of the single worker promotion contest: (RP^i).

Theorem 2 *The promotion contest*

$$\tau \coloneqq \inf\left\{t \geq 0 \,:\, X^i_t \notin \left[\underline{p}^i(W), \bar{P}^i\left(\underline{X}^i_t\right)\right)\right\} \text{ and } d^i_\tau \coloneqq \mathbb{1}_{\left\{X^i_\tau \geq \bar{P}^i\left(\underline{X}^i_\tau\right)\right\}}$$

is optimal in the single worker problem (RP^i).

Theorem 2 states that it is optimal to delegate to worker 1 until his type either (i) reaches the promotion threshold $\bar{P}^i(\underline{X}^i_t)$, or (ii the principal becomes too pessimistic about him. To understand why that is, recall that the flow reward (conditional on worker 1's type) the principal obtains when worker 1 operates the project is the same before and after promotion. So the principal always wants to postpone her decision, as she gets more information about the worker at no cost if she waits. Since the worker's type is strongly Markovian, a likely candidate for the promotion time is the first hitting time of a threshold as high as possible. In particular, if the cost of effort is zero, the principal promotes the worker when his type reaches the upper boundary of $\mathcal{X}^i$. However, when effort is costly, this threshold is too high. So, the principal chooses the highest threshold for which the worker is willing to exert effort instead. If the agent's type increases, the promotion threshold stays constant: the principal needs to keep her promises. On the other hand, when the worker's type decreases, the worker becomes more pessimistic about his promotion chances. The principal then has to lower the promotion threshold to motivate the worker. The logic is the same as in McClellan (2017): the promotion threshold becomes laxer when the participation constraint binds.[26] Because of the monotonicity of the problem, this constraint binds precisely when the worker's type decreases.

Formally, the proof of Theorem 2 is based on the idea of the proof of Theorem 1 in McClellan (2017). It follows from the five steps below:

- First consider a relaxation of problem (RP^i) for which the constraint (DPC^i) only needs to hold for on a finite set of (stopping) times.
- Lemma 8 derives the Lagrangian associated with the relaxed problem as an application of Theorem 1 in Balzer and Janßen (2002).
- In the third step, useful properties of the solution of the relaxed problem introduced in step 1 are established.
- The fourth step identifies a promotion contest that guarantees the principal a payoff of at least the value of the relaxed problem introduced in the first step. It is enough to focus on promotion contests that promote worker i after good performances (as X^i

[26]See also Harris and Holmstrom (1982), Thomas and Worrall (1988), or Grochulski and Zhang (2011).

crosses an upper threshold from bellow) and take the outside option after bad outcomes (when X^i crosses a lower threshold from above).

- Putting everything together and letting the set of times at which (DPC^i) holds grow dense yields Theorem 2.

Steps 1, 2, and 5 are essentially the same as in the proof of Theorem 1 in McClellan (2017). Steps 3 and 4 are new and specific to our setting. The details are in Appendix A.1.6. Supporting Lemmas are in Appendix A.1.6.

Corollary 1 *Let* (τ, d^i) *be feasible in the single worker problem* (RP^i) *Then, for all* $\bar{W} \geq W$*,*

$$\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau} e^{-rt}\pi^i\left(X_t^i\right)dt + e^{-r\tau}\left(d_\tau^i \bar{\pi}^i\left(X_\tau^i\right) + (1-d_\tau^i)\bar{W}\right)\right] \\
&\quad \leq \mathbb{E}\Bigg[\int_0^{\tau^{s,i}\wedge\tau^i(\underline{p}^i(\bar{W}))} e^{-rt}\pi^i\left(X^i\right)dt \\
&\qquad + e^{-r\tau^{s,i}\wedge\tau^i(\underline{p}^i(\bar{W})}\left(\bar{\pi}^i\left(X_{\tau^{s,i}}^i\right)\mathbb{1}_{\{\tau^{s,i}<\tau^i(\underline{p}^i(\bar{W})\}} + \bar{W}\mathbb{1}_{\{\tau^{s,i}\geq\tau^i(\underline{p}^i(\bar{W})\}}\right)\Bigg].
\end{aligned}$$

Proof. Observe that the set $\mathcal{P}^{I,r,i}$ is independent of $\bar{W}$ and that Assumption 7 is satisfied for any $\bar{W} \geq W$. The result follows from Theorem 2. ■

1.5.3 Measurable stopping

The main result of this section shows that it is enough to focus on a subset of the implementable promotion contests such that the decision to promote worker i does not depend on the type of the other workers.

Proposition 10 *The supremum in* (RP) *is achieved by a (randomized) promotion contest* (T, τ, d)*. Moreover,* $\tau = \left(\bigwedge_{i=1}^N \tau^i\right) \wedge \tau^0$*, where* τ^i *is a* $\mathcal{F}^i$*-stopping time,* τ^0 *is a* $\mathcal{G}^T$*- randomized stopping time, and* $d_\tau^i = 1$ *only if* $\tau^i \leq \tau = \left(\bigwedge_{i=1}^N \tau^i\right) \wedge \tau^0$*.*

Proposition 10 has two parts. The first part states that the supremum in (RP) is achieved by a a promotion contest. It follows from Theorem 10 in Appendix A.1.7.

The second part characterizes the promotion time τ. It is the minimum of N $\mathcal{F}^i$-stopping times, τ^i's, and one $\mathcal{G}^T$-randomized stopping time τ^0. It follows from Corollary 9 in Appendix A.1.7.

1.5.4 An upper bound on the value of (RP)

Proposition 11 derives an upper bound on the principal's payoff in any implementable promotion contest.

Proposition 11 *The value of* (RP) *is bounded above by*

$$\mathbb{E}\left[\int_0^\infty re^{-rt}\bigvee_{i=1}^N \underline{\Gamma}^{s,i}_{T^{s,i}(t)}dt\right]. \tag{Π^M}$$

To build some intuition, it is useful to go back to the proof of indexability for superprocesses.[27] Start with N independent payoff processes $\tilde{\pi}^i_t$, one for each superprocess. To each of these payoff processes, associate the index process $\tilde{\Gamma}^i_t$ defined as the "equitable surrender value", i.e. the smallest W such that

$$W = \tilde{V}^i_t(W) := \sup_{\tau\geq t}\mathbb{E}\left[\int_t^\tau e^{-r(s-t)}\tilde{\pi}^i_s ds + e^{-r(\tau-t)}W \mid \mathcal{F}^i_t\right].$$

This index process has the desirable property that, for all $\tilde{W}$,[28]

$$\mathbb{E}\left[\int_0^\infty e^{-rt}r\underline{\tilde{\Gamma}}^i_t \vee W dt\right] = \sup_\tau \mathbb{E}\left[\int_0^\tau e^{-rt}\tilde{\pi}^i_t dt + e^{-r\tau}W\right],$$

[27]See Chapter 4 in Gittins et al. (2011) or Durandard (2022a), for example.

[28]See Proposition 3.2 in El Karoui and Karatzas (1994).

where $\underline{\tilde{\Gamma}}_t^i$ is the lower envelope of $\tilde{\Gamma}_t^i$. Whittle's condition guarantees that one of the payoff processes within each superprocess is such that its associated index process $\tilde{\Gamma}^{i,*}$ dominates the associated index process of all other possible index processes associated with this superprocess, i.e., for all W,

$$\mathbb{E}\left[\int_0^\infty e^{-rt} r \underline{\tilde{\Gamma}}_t^{i,*} \vee W dt\right] \geq \mathbb{E}\left[\int_0^\infty e^{-rt} r \underline{\tilde{\Gamma}}_t^{i} \vee W dt\right].$$

It is then possible to show that the optimal policy picks the dominating process for each superprocess and pulls the arm whose index is the highest at each instant t.

Here, start with an implementable promotion contest $(T, \tau, d) \in \mathcal{P}^I$ and find N single-arm implementable promotion contests $(\tau^i, d^i) \in \mathcal{P}^{I,i}$. This N single-arm implementable promotion contests generates N payoff processes for the principal:

$$h_t^i := \pi^i(X_t^i)\mathbb{1}_{\{t<\tau^i\}} + r\bar{\pi}^i\left(X_{\tau^i}^i\right)\mathbb{1}_{\{t\geq\tau^i\}}.$$

By the results of Section 1.5.3, each h^i can be chosen to be $\mathcal{F}^i$-adapted. As in the proof of indexability for superprocesses, one would like to associate to each of these payoff processes an *index process* Γ_t^i. However, the index process cannot be the "equitable surrender value" in the retirement problem:

$$\bar{V}_t^i(W) := \sup_{\tau\geq t}\mathbb{E}\left[\int_t^\tau e^{-r(s-t)}h_s^i ds + e^{-r(\tau-t)}W \mid \mathcal{F}_t^i\right].$$

Intuitively, this would allow the principal to break her promises and take the outside option while the worker's promised continuation utility is *strictly* positive. Hence, the retirement problem above does not take into account the worker's participation constraint. To overcome this issue, consider instead the optimal retirement problem in which the principal can take the outside option only on a set of decision times at which the continuation value of the worker is zero:

$$\tilde{V}^i\left(t, W; \tau^i, d^i\right) := \sup_{\rho \in \mathcal{T}^s(t;h^i)} \mathbb{E}\left[\int_t^\rho e^{-r(s-t)} h_s^i ds + e^{-r\rho} W \mid \mathcal{F}_t^i\right];$$

where

$$\mathcal{T}^s(t; \tau^i, d^i) = \left\{s \geq t \,:\, U_s^i(\tau^i, d^i) = 0\right\},$$

and $U_s^i(\tau^i, d^i)$ is worker i's continuation value at time s for the single-arm promotion contest $(\tau^i, d^i) \in \mathcal{P}^{I,i}$. The index process is then the "equitable surrender value" in this alternative retirement problem. One can therefore think of the problem as a multi-armed bandit problem in which the completion time of each task is the random duration between two times such that the worker's continuation is zero.

Finally, Corollary 1 guarantees that each index process is dominated (in the sense of Whittle) by the strategic index process. The conclusion then follows from the same arguments as in the nonstrategic case.

The proof of Proposition 11 is in Appendix A.1.8. The derivation of the index processes associated with our alternative retirement problem is in Appendix A.1.8.

1.5.5 Proof of Theorem 1

By Proposition 11, any implementable promotion contest gives a payoff weakly smaller than

$$\mathbb{E}\left[\int_0^\infty r e^{-rt} \bigvee_{i=1}^N \underline{\Gamma}^{s,i}_{T^{s,i}(t)} dt\right].$$

By Proposition 2, the principal obtains an expected payoff of

$$\mathbb{E}\left[\int_0^\infty re^{-rt}\bigvee_{i=1}^{N}\underline{\Gamma}^{s,i}_{T^{s,i}(t)}dt\right]$$

in the index contest. Thus the index contest is optimal.

1.6 Extensions

In this section, I discuss multiple extensions.

1.6.1 Relaxing Assumptions 5 and 7

Assumptions 5 and 7 simplify the analysis but rule out potentially interesting settings. In particular, Assumption 5 excludes Poisson learning with good news, a case that has received a lot of attention in the economic literature, while Assumption 7 excludes problems in which the principal has no outside option, i.e., in which the position has to fill internally.

However, both can be relaxed, as the Corollaries below establishes. Interestingly, both Corollaries rely on the continuity of the principal's value. Corollary 2 uses that the value is continuous in the payoff from the outside option, W, while Corollary 3 uses that the value is continuous in the process X^i (in the appropriate topology). Their proofs are in supplemental Appendix A.3.1.

Corollary 2 *The index contest is still optimal when Assumption 7 does not hold. The principal never takes the outside option.*

Corollary replaces Assumption 5 with the following assumption:

Assumption 8 *For all* $i \in \{1, \ldots, N\}$, *there exists a sequence* $(X^{i,n})_{n\in\mathbb{N}}$ *such that (i)* $X^{i,n}$ *satisfies Assumption 5, (ii)* $X^{i,n} - X^i$ *is* $\mathcal{F}^i$*-adapted, and (iii)* $X^i = \lim_{n\to\infty} X^{i,n}$ *uniformly on* *compact sets $\mathbb{P}$-a.s..*

Hence, any process satisfying Assumptions 2, 3, and 4, but not 5 can be approximated by a sequence of processes $X^{i,n}$ that satisfy 5. Assumption 8 simply guarantees that this sequence is $\mathcal{F}^i$-adapted. In particular, if, for all i, the probability space $\left(\Omega, \bar{\mathcal{F}}, \mathbb{P}\right)$ contains a $\mathcal{F}^i$-Brownian motion, Assumption 8 is satisfied. Define

$$\tau^0 := \inf\left\{t \geq 0 \,:\, \Gamma^{s,i}_{T^i(t)} \leq W \text{ for all } i\right\},$$

and

$$\tau^i := \inf\left\{t \geq 0 \,:\, T^i(t) > \bar{P}^i\left(\underline{X}^i_{T^i(t)}\right)\right\} \wedge \tau^{p,i},$$

where $\tau^{p,i}$ is the first tick of a Poisson clock that runs only on $\{X^i_t = \bar{P}^i(\underline{X}^i_t)\}$ which intensity is chosen to leave i indifferent between exerting effort or not when promoted at time τ^i.

Corollary 3 *Suppose that Assumption 5 is replaced with Assumption 8 and that the π^i's are continuous. Then the index contest associated with the strategic indices $\Gamma^{s,i}$ and the promotion time $\tau^* = \tau^0 \wedge \bigwedge_{i=1}^N \tau^i$ is optimal.*

Interestingly, Corollary 3 shows that when $\bar{P}^i(\underline{X}^i_t) = \underline{X}^i_t = X^i_t$ the strategic index associated to worker i is equal to the expected value of promoting i immediately: information has no value. This is the case, for example, if worker can be good or bad, the principal learns about worker i through the Poisson arrival of good news, the probability that worker i is good at time t (hence his type $X^i_{T^i(t)} := \mathbb{P}\left(\{i \text{ is good }\} \mid \mathcal{F}^i_{T^i(t)}\right))$ is too low.

1.6.2 Prize design

The main of this section establishes that when the principal can design the prize, the index contest is still optimal, i.e., the principal prefers to allocate the entire prize to one worker only. Moreover, there is no value in giving multiple "smaller" promotion to a worker.

The model is identical to the one presented in Section 1.2, except for the following two differences: (i) the prize is divisible, and (ii) the principal chooses (potentially) multiple times at which to promote workers. Formally, at time $t = 0$, the principal commits to a history-dependent promotion contest comprising of (i) a set of promotion time $\{\tau_k\}_{k=1}^{K}$ (with $K \in \mathbb{N} \cup \infty$) specifying when a fraction of the prize is allocated; (ii) a promotion decision d specifying which of the workers is promoted; and (iii) a delegation rule α that assigns at every instant the non-routine task to some worker. The promotion times, τ_k's, are $\mathcal{G}^T$-stopping time such that $\tau_0 = 0$ and $\tau_k < \tau_{k+1}$ $\mathbb{P}$-a.s.. The promotion decision is a $\mathcal{G}^T$-adapted (stochastic) process $d = \left(d^0 = \{d_t^0\}_{t\geq 0}, \ldots, d^N = \left\{d_t^N\right\}_{t\geq 0}\right) \in \mathcal{C}^*$. Again d^0 stands for the principal's decision to take her outside option. Finally, the delegation rule $T = \left(T^1 = \{T^1(t)\}_{t\geq 0}, \ldots, T^N = \left\{T^N(t)\right\}_{t\geq 0}\right) \in \mathcal{D}$ is a delegation process. The workers only decides to exert effort a_t^i in $\{0, 1\}$ when they are delegated the non-routine task.

Finally, the following additional assumption is maintained in this section.

Assumption 9 *(i) For all $i \in \{1, \ldots, N\}$, the process $\{\pi^i\left(X_s^i\right)\}_{s\geq 0}$ is a submartingale.*

(ii) For all $i \in \{1, \ldots, N\}$, the cost of effort is constant: $c^i(\cdot) := c^i$.

Assumption 9 (i) guarantees that upon promotion, the principal always wants there is no penalty from delegating the full project to the promoted worker. Assumption 9 (ii) simplifies the argument.

So, given a promotion contest $\left(T, \{\tau_k\}_{k=1}^{K}, d\right)$, the principal's expected payoff is

$$\Pi^M\left(T, \{\tau_k\}_{k=1}^{K}, d; W\right) := \mathbb{E}\left[\sum_{k=1}^{K}\left(\sum_{i=1}^{N}\int_{\tau_{k-1}}^{\tau_k} e^{-rt}\pi^i(X_{T^i(t)}^i)dT^i(t) + e^{-r\tau_k}\bar{\pi}\left(X_{T(\tau)}, d_{\tau_k}\right)\right)\right],$$

where

$$\bar{\pi}(x,d) := d^0 W + \sum_{i=1}^{N} \mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i\left(X^i_{d^i t}\right) d\left(d^i t\right) \mid X^i_0 = x^i\right].$$

The workers' expected payoffs are

$$U^i\left(T, \{\tau_k\}_{k=1}^K, d\right) := \mathbb{E}\left[\sum_{k=1}^{K} e^{-r\tau_k} g d^i_{\tau_k} - \int_0^\infty e^{-rt}(1 - \sum_{k=1}^{K} d^i_{\tau_k}\mathbb{1}_{\{t\geq\tau_k\}})c^i dT^i(t)\right].$$

The principal's objective is to design the promotion contest that maximizes her payoff among all implementable promotion contest. As above, this is equivalent to the maximization program:

$$\Pi^M := \sup_{(T,\{\tau_k\}_{k=1}^K,d)\in\mathcal{P}} \mathbb{E}\left[\sum_{k=1}^{K}\left(\sum_{i=1}^{N}\int_{\tau_{k-1}}^{\tau_k} e^{-rt}\pi^i(X^i_{T^i(t)})dT^i(t) + e^{-r\tau_k}\bar{\pi}\left(X_{T(\tau)}, d_{\tau_k}\right)\right)\right], \tag{Prize design}$$

subject to the dynamic participation constraints: for all i and all possible histories h_t,

$$\mathbb{E}\left[\sum_{k=1}^{K} e^{-r(\tau_k - t)} g d^i_{\tau_k}\mathbb{1}_{\{t\leq\tau_k\}} - \int_t^\infty e^{-rt}(1 - \sum_{k=1}^{K} d^i_{\tau_k}\mathbb{1}_{\{t\geq\tau_k\}})c^i dT^i(t) \mid h_t\right] \geq 0.$$

Theorem 3 *Suppose that Assumption 9 holds. Then the* ***index contest*** *solves* (Prize design).

Theorem 3 shows that optimal promotion contest grants the *entire prize to one worker at most*. The optimal contest is a winner-take-all. This is reminiscent of the classic result in Moldovanu and Sela (2001) of the optimality of a single prize. In our dynamic setting, fully allocating the prize to only one worker is also optimal. The index contest is meritocratic: the worker who performs the best (upon getting the opportunity) is promoted. This contrasts from recent results in dynamic contest theory in which the optimal contest was shown to be more egalitarian (see Halac et al. (2017) and Ely et al. (2021), for example).

In supplemental Appendix A.3.2, I indicate how to modify the proof of Theorem 1 to obtain Theorem 3. In particular, it follows the same steps. The only difference is in showing that one can focus on promotion contests in which the promotion times are measurable. Proposition 33 in supplemnetal Appendix A.3.2 replaces Proposition 10. The rest of the proof is identical.

1.6.3 Transfers

I ruled out transfers for three reasons in the main model. The first and most fundamental one was that I wanted to focus on the trade-off between the two classical promotion roles, i.e., incentives provision and sorting. The second reason for this restriction is empirical. In most organizations, compensation is promotion based.[29] This is the case in public administrations, where the salary grid is fixed, for example. Finally, the analysis developed in this article becomes intractable for general wages (although the main trade-off seems to be preserved when workers are protected by limited liability). So a complete analysis of transfers is well beyond the scope of my paper. Nevertheless, in this section, I point out how my model can accommodate restricted forms of transfers.

Suppose that the principal can only choose transfers that depend on the worker's current type and his effort decision (i.e., pay a flow wage $w^i_t = w^i(a^i_t, X^i_t)$ to worker i at every instant $t \geq 0$) and that the workers are protected by limited liability (i.e. $w^i_t \geq 0$). Then the index context is still optimal, under Assumption 4.(ii) (when the workers' types can only jump down), as long as $\pi^i(\cdot) - w^i(1, \cdot)$ is nondecreasing. This can be seen from the proof of Theorem 1 directly.

[29] Baker et al. (1988) find that "[m]ost of the average increases in an employee's compensation can be traced to promotions and not to continued service in a particular position.". See also Gibbs (1995) and Bernhardt (1995). It is also consistent with the observed separation of roles: Compensation and benefits managers within the human resource department have authority over the compensation structure, while the assignment of responsibilities and tasks are made within each department by managers that can closely monitor and supervise their team.

For example, if the wage paid to each of the workers is a constant fraction $\beta^i \in [0,1]$ of the flow payoff the principal obtains (i.e., $w_t^i = \beta^i \pi^i(X^i_{T^i(t)})dT^i(t))$, then the index contest is optimal. The strategic index are computed for the payoff process $(1-\beta^i)\pi^i(X^i(t))$, effort costs $c(X_t^i) - \beta^i \pi^i(X^i(t))$, and value of promotion $\tilde{g}^i\left(X_t^i\right) := g^i + \beta^i \bar{\pi}^i(X_t^i)$. One can then imagine that the principal engages in Nash bargaining with the workers (with threat points equal to their outside option) before the game starts to determine the β^i's.

1.6.4 Different information structures

Finally, the workers' types are assumed to be observable by all the players: by both the other workers and the principal. Interestingly, the index promotion contest remains optimal if each worker only observes his type and the principal does not observe the evolution of the types, but the workers can reveal their current type to the principal credibly. Hence it is easily seen that it is weakly dominant for the workers to reveal their type to the principal when $X_t^i = \underline{X}_t^i$ or $X_t^i = \bar{P}^i\left(\underline{X}_t^i\right)$, which is the only information the principal need to implement the index contest. Verifiability is important here: the same result cannot be obtained with cheap talk communication only.

1.7 Conclusion

I study the design of centralized dynamic contests in a general environment. Workers are heterogeneous and strategic. They have to be incentivized to exert effort, and their types evolve (stochastically) when they work. I showed that despite the richness of the model, the solution is simple and takes the form of an *index contest*.

My analysis is limited to the specific extension of the multi-armed bandit model I consider, and I do not suggest that my findings would hold in different environments. Some of the assumptions, such as the independence of the type processes, appear crucial and very hard to relax in a significant manner (although one could consider a particular form of conditional independence for multi-parameter processes, known as condition F4, see El Karoui and Karatzas (1997) or Walsh (1981) for example). However, the intuition behind the result is valid in other environments. For example, when the information about the project's success is private and cannot be credibly communicated but the uncertainty is small, results from the multi-armed bandit literature suggest that *index contest* would still perform well. This can be seen directly by inspecting the principal's payoff in the index contest (Π^M). When the uncertainty is small, the lower envelopes of the index processes associated with the case in which the principal observes the workers' types directly or observes a signal are close. Still, characterizing the specific form of the optimal mechanism when the workers have private information about the outcome of the delegation process would be interesting.

More generally, the idea that the endogenous allocation of opportunities or the endogenous acquisition of information affects the final decision when allocating an asset or promoting a worker is very natural and deserves more attention in future research.

Chapter 2

Under Pressure: Comparative Statics for Optimal Stopping Problems in Non-stationary Environments

2.1 Introduction

A large body of literature has employed optimal stopping problems to model many different decision processes, such as optimal pricing and information acquisition problems where a decision-maker must determine the optimal time to take a particular action. These problems are often nonstationary, as the decision environment can change suddenly or gradually over time. For example, in the optimal pricing setting, a firm's cash flow variance may suddenly rise in periods of crisis. Decision-makers may also receive information at varying speeds, with some news arriving gradually and others suddenly revealed. Additionally, decision-makers often operate under the pressure of a (possibly stochastic) deadline.

However, following the seminal contribution by Wald (1947) and Shiryaev (1967), most of the literature has focused on stationary environments where the decision-maker faces the same problem and has the same tools at every point in time. The reason for such restriction is mainly technical. Abandoning stationary environments poses new challenges to standard solution techniques and typically results in the lack of close-form solutions (see, for example, Fudenberg et al. (2018)). This paper aims to go beyond these technical limitations and show that interesting comparative statics can still be obtained in a large class of nonstationary optimal stopping problems, even without relying on close-form solutions.

Our paper has three main contributions. First, we formulate a general optimal stopping problem that can represent a wide variety of non-stationary environments (including those mentioned above) and find conditions under which this problem has a well-defined solution. Second, we show that in *monotone environments*, i.e., when the problem's time dependence is "monotone", we can still derive clean comparative statics results despite the lack of closed-form solutions. For example, we show that when the optimal stopping problem is monotone, the optimal continuation region enlarges or shrinks over time depending on whether the problem is *monotone increasing or decreasing.* Third, we specialize our model and use our comparative static results to analyze the tradeoff between timing and quality of the decisions in information acquisition problems where the environment is non-stationary, e.g., when the decision maker faces a (potentially stochastic) deadline, receives or processes information at variable speed (possibly even in chunks), or learns about a relevant parameter that is not binary (as in Fudenberg et al. (2018)).

We study a general optimal stopping problem where a decision maker chooses among different possible alternatives whose values evolve according to a general diffusion process. Formally our decision-maker solves

$$V(t,x) = \sup_{\tau} \mathbb{E}_{(t,x)} \left[\int_t^{\tau} e^{-\int_t^s r(u,X_u)du} f(s,X_s)ds + e^{-\int_t^{\tau} r(s,X_s)ds} g\left(X_{\tau}\right) \right], \tag{V}$$

subject to

$$X_{t+s} = X_t + \int_t^{t+s} \mu(u, X_u)du + \int_t^{t+s} \sigma(u, X_u)dB_u;$$

where the discount rate, as well as the drift μ and the variance σ of the diffusion process depend both on time t and on the state X_t.

In this general setting, standard methods to solve optimal stopping problems may fail, as we do not know a priori whether the Hamilton-Jacobi-Bellman equation associated with the problem has a smooth solution. Our first main result shows that, under some mild regularity conditions, the value function is smooth and is indeed the unique L^p-solution of the HJB equation. This technical result is crucial: it ensures we can derive properties and comparative statics of the value function and the stopping regions even without reaching a closed-form solution.

Our second contribution is to show that the continuation region of our optimal stopping problem (strictly) shrinks over time intervals where the value function (strictly) decreases with time and, on the contrary, enlarges over time intervals where the value function (strictly) increases with time.In other words, the set of states x at which the decision maker chooses to stop the drift-diffusion process (strictly) decreases or increases in the set inclusion order over a time interval when the value function decreases or increases in that time interval. Proving the strict part of this result is particularly challenging as, in principle, it requires knowledge of the cross derivative of the value function. To overcome this challenge, we develop a new argument based on partial differential equation results and the Hopf boundary lemma.

To illustrate the power of this comparative static, we focus on *(strictly) monotone (increasing or decreasing) optimal stopping problems*, where the value function $V(x, t)$ is globally (strictly) monotone and the continuation region is connected at each instant. In this case, our result implies that when the value function is (strictly) decreasing in time, an upper boundary that (strictly) decreases in time and a lower boundary that (strictly) increases in time delimit the continuation region. The upper and lower boundaries move in the opposite direction when the value function is (strictly) increasing in time. As we show, the boundaries' dynamic has clear empirical predictions, for example, regarding the speed and quality of decisions.

Next, we specialize our model to capture a classical information acquisition problem where a decision maker chooses between two possible alternatives while learning about some relevant parameter. In this setting, the link between the speed and accuracy of decisions has been at the core of recent developments in economics, psychology, and neuroscience. Accuracy here is the probability of making the correct decision given the true value of the parameter. Wald's classical sequential sampling model assumes that the relevant parameter is binary and the learning process is time-stationary. As a result, the belief process is time-stationary, and the stopping boundaries are constant (in the belief space), which implies that the time of a decision is uncorrelated with accuracy. However, multiple binary choice experiments in neuroscience documented interesting correlation patterns between the time and accuracy of decisions (see, for example, the survey by Ratcliff et al. (2016)), questioning the problem's time-stationarity.

In this setting, where shirking boundaries implies decreasing accuracy, and enlarging boundaries implies increasing accuracy, our results allow us to rationalize such speed-accuracy tradeoff and relate it to the problem's time dependence. If the problem is monotone increasing, then accuracy increases in the stopping time; if the problem is monotone decreasing, then accuracy decreases in the stopping time.[1] We then derive natural conditions on the primitives under which the optimal stopping problem is monotone.[2]

Accuracy is increasing (decreasing) over time when (i) learning speed increases (decreases)

[1]Decision accuracy always increases in the decision time when the decision time is exogenous. However, if the decision-maker optimally chooses when to stop, accuracy may be lower when she chooses to stop later.

[2]It is easy to check whether any given optimal stopping problem is monotone, using the same approach.

in time or (ii) the discount rate decreases (increases) over time (i.e., the decision maker values the future more over time). For example, our model predicts that slower investment choices are indicative of better decisions in financial markets that are getting more opaque (e.g., due to financial innovations or crises) and of worse decisions in financial markets that are getting more transparent (e.g. due to the regulatory effort). Also, if one cares progressively more about the future when they get older (as suggested, for example, by Trostel and Taylor (2001) or Kureishi et al. (2021)), then the decision quality increases over time.

Importantly, since our main result holds locally, we can also capture curvilinear relations between time and accuracy that consistently arise in both perceptual and cognitive testing (see, for example, Ratcliff et al. (2016) and Chen et al. (2018)). In our setting, accuracy can increase and then decrease over time due, for example, to non-monotone changes in learning speed, which can initially increase due to experience gains but eventually decreases as the decision maker becomes increasingly tired.

So far, we have only addressed gradual changes in the optimal stopping problem. However, considering abrupt changes is at least as crucial for many decisions where time pressure is of the essence. A decision maker can miss an opportunity if they do not seize it in time. For example, a competitor may preempt the decision-maker by hiring the worker or buying the asset the decision-maker was considering. Sometimes even when the possibility and the value of taking an action are constant over time, the learning process might change abruptly. For example, while during the quarter investors may gather information about the company performance and outlook gradually, it is typically in the earning release report and in the subsequent press conference that most information is disclosed to the market creating huge fluctuation in the market price.

Our third main contribution is to show that our environment accommodates also for all these abrupt changes in the optimal stopping problem both in terms of the information and the actions available to the decision maker. Thus our results can be extended to derive comparative statics even in settings where the decision maker operates under the pressure of a (potentially stochastic) deadline and in settings where information arrives both gradually and abruptly (according to a jump diffusion process).

The main challenge to overcome to prove the result is that the HJB equation in these settings is an integro-differential equation, which in turn would render any local argument not directly applicable. However, our key contribution is to show that we can use a time change argument to trace back these abrupt changes to our optimal stopping problem of a general diffusion process (without jumps). This dramatically simplifies the problem as it implies that we can find the solution by using the local HJB equation of our auxiliary stopping problem.

As a result, we show that the accuracy of slower decisions will be lower (higher) when the time pressure from a stochastic deadline increases (decreases) over time or when there is a fixed deadline.[3] From an applied perspective, this means, for example, that over time we should expect worse decisions in markets becoming increasingly competitive and better decisions when competition is fading away. Similarly, we show that slower decisions are more accurate in settings where the arrival rate of abrupt information increases (e.g., due to a rise in media coverage) and less accurate when this arrival rate decreases over time.

The last contribution of our paper is to show that our environment also accommodates sequential sampling problems whose relevant parameter is not binary. As the seminal paper by Fudenberg et al. (2018) highlighted, these settings are time nonstationary as information about the relevant parameter is contained not only in the value of the signal process but also in the time that has passed without the DM taking a decision. Fudenberg et al. (2018) focus on the speed-accuracy tradeoff when the relevant parameter is normally distributed.

[3] Here, we assume that the decision-maker must take action at the deadline. However, our results also apply to the case where the decision maker gets a fixed payoff at some (possibly stochastic) deadline. We show that slower decisions are less (more) accurate when the time pressure from a stochastic deadline increases (decreases) over time or when there is a fixed deadline if this deadline payoff is sufficiently low (high).

They show that, as time goes by, if no stopping boundary is hit, the decision makers become more convinced that there is not much difference in taking one action rather than the other, and the accuracy of the decision decreases. In section xxx, we show that we can embed their model into ours and obtain their result using our comparative statics. Also, we use our machinery to extend their result about decreasing accuracy also to settings where the expected value of taking the right decision remain constant, i.e., settings where the decision-makers do not become more pessimistic about the relevance of making the right decisions. We also highlight conditions under which accuracy increases in the stopping time.

Other relevant literature. Beyond Fudenberg et al. (2018), our paper is related to the vast literature on dynamic information. The seminal works by Wald (1947), Arrow et al. (1949), and Shiryaev (1967), analyze the optimal stopping problem of a decision maker that needs to choose when to stop learning and which decision to take in a time stationary environment, where the actions available and the learning process are constant over time. More recently, Moscarini and Smith (2001), Che and Mierendorff (2019), and Zhong (2022) endogenized the learning process by allowing the agent to select signal precision, direction of learning, and which type of signal to use respectively. Even these papers, however, focus on time-stationary problems where the actions and the learning processes available (with their associated costs) do not depend on time.[4]

Relatively few papers abandoned the tractability of stationary environments. Leaving the binary setting, Fudenberg et al. (2018) and Tajima et al. (2016) focus on problems where the decision maker gradually learns a relevant parameter that is normally distributed. In such a setting, the action of continuing learning loses value over time at any given state: more information has been gathered, and less uncertainty is left. As a result, these environments are non-stationary and display shrinking boundaries. Within the neuroscience literature,

[4]In these models, the decision maker optimally adapt their learning process to her belief. However, for any given belief, time has no impact.

Rapoport and Burkheimer (1971) studied, in discrete time, a sequential sampling problem with a deterministic deadline, and Frazier and Yu (2007) extended the analysis to consider a stochastic deadline whose arrival rate increases over time. To the best of our knowledge, our paper is the first to address, in full generality, optimal stopping problems in time-nonstationary environments, providing local (and global) comparative static results that apply to many different settings: e.g., with increasing or decreasing time pressure, discount rates, and speed of learning (including both gradual and abrupt arrival of information).[5]

The core of our applied results relates to the tradeoff between speed and accuracy, a tradeoff also demonstrated by several recent studies in psychology and neuroscience (Bolsinova et al. (2017); Molenaar et al. (2018); Chen et al. (2018); Goldhammer et al. (2014, 2015)). These studies document how, even after controlling for person and task, the relation between accuracy and response time can be negative, positive, and even curvilinear, with accuracy first increasing and then decreasing over time (or vice-versa) depending on the experimental setting. Most of this literature, however, took a positive stand, abstracting away from the question of whether such a pattern can arise from optimal decision-making.[6] Our model takes a normative stand, highlighting how all the dependencies mentioned above can be the result of optimal responses of a decision-maker to different information acquisition problems. In so doing, our model can shed light on the determinants of these patterns.

[5] Quah and Strulovici (2013) provide comparative statics about how the stopping time changes when the decision maker become more or less patient.

[6] For example, some papers simply assume that the decision-maker takes one choice or the other whenever the sampling process hits one of two constant boundaries. They focus on estimating the parameters (e.g., initial belief, possibly variable speed of learning, reaction time, etc.) that better fit this model without asking whether using two constant boundaries is optimal. Few exceptions in this literature take optimality into consideration, e.g., Drugowitsch et al. (2012), but they typically rely on numerical rather than analytical solutions.

2.2 Setting

We study an optimal stopping problem in continuous time, where a risk neutral decision maker sequentially chooses whether or not to stop a nonstationary diffusion process $X = \{X_t\}_{t\geq 0}$ on the open (possibly unbounded) interval $\mathcal{X} := [\underline{x}, \bar{x}] \subseteq \mathbb{R}$. To this end, let $\left(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, \mathbb{P}\right)$ be a filtered probability space whose filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions.[7] Denote by $\mathcal{Y}_T = [0, T) \times \mathcal{X}$, with $T \in (0, +\infty]$, the product set of times and states, by $\mathcal{T}$ the set of $\mathcal{F}$-stopping times taking values in $[0, T]$, and by $\mathcal{T}(t)$, $t \in [0, T)$, the set of $\mathcal{F}$-stopping time taking values in $[t, T]$.[8] If the decision maker chooses to stop the diffusion process at time t, she can take one of two alternative actions (a and b), delivering expected payoffs $g^a(X_t)$ and $g^b(X_t)$ respectively. For simplicity, we assume that $g^i : \bar{\mathcal{X}} \to \mathbb{R}$ belongs to $\mathcal{C}^{2,\alpha}(\bar{\mathcal{X}})$ and that $g^a - g^b$ has at most one zero at $x^c \in \mathcal{X}$. On the other hand, if the decision maker does not stop, she obtains a flow payoff of $f(t, X_t)$, with $f : [0, T) \times \mathcal{X} \to \mathbb{R}$ twice continuously differentiable, and the diffusion process evolves according to

$$X_t = X_0 + \int_0^t \mu(s, X_s) ds + \int_0^t \sigma(s, X_s) dB_s, \quad \mathbb{P}\text{-a.s.}, \tag{2.1}$$

where B is the standard one-dimensional Brownian motion. Both the drift $\mu : [0, T) \times \mathcal{X}$ and the volatility $\sigma : [0, T) \times \mathcal{X} \to \mathbb{R}_{++}$ are allowed to vary with time s and the current state of the process X_s. For simplicity, we assume that μ and σ are twice continuously differentiable, and that the endpoints $\underline{x}$ and $\bar{x}$ are (possibly unattainable) absorbing states.

[7] See for example Protter (2005).

[8] A $\mathcal{F}$-stopping time is a map $\tau : \Omega \to \mathbb{R}_+ \cup \{\infty\}$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$ and $\mathcal{T}(t) := \{\tau \text{ stopping time} : \tau \geq t \ \mathbb{P}\text{-a.s.}\}$.

Formally, at every time t and state $X_t = x$, the decision maker solves

$$V(t,x) = \sup_{\tau} \mathbb{E}_{(t,x)} \left[\int_t^{\tau} e^{-\int_t^s r(u,X_u)du} f(s,X_s)ds + e^{-\int_t^{\tau} r(s,X_s)ds} g\left(X_{\tau}\right) \right], \tag{V}$$

subject to

$$X_{t+s} = x + \int_t^{t+s} \mu(k,X_k)dk + \int_t^{t+s} \sigma(k,X_k)dB_k;$$

where $\mathbb{E}_{(t,x)}$ is the expectation operator associated with the process X starting at (t,x). $V(t,x)$ is the *value function* associated to our problem. Payoff are discounted at the (possibly stochastic) rate $r(t,x)$, where $r : [0,T) \times \mathcal{X} \to \mathbb{R}_+$ is twice continuously differentiable. We denote by $r := \inf_{(t,x)\in\mathcal{Y}_T} r(t,x)$ and, for simplicity, we assume that $\mathcal{X}$ is bounded if $r = 0$.

We assume that $f : (0,T] \times \mathcal{X} \to \mathbb{R}$ and $g : \mathcal{X} \to \mathbb{R}$ are such that

$$\mathbb{E}_{(t,x)} \left[\sup_{t\leq s\leq T} \int_t^{s} e^{-\int_t^s r(u,X_u)du} f(s,X_s)ds + e^{-\int_t^{s} r(s,X_s)ds} g\left(X_{t}\right) \right] < \infty$$

Remark 1 *A standard assumption in the study of control and stopping problems is uniform ellipticity: there exists $\lambda > 0$ such that, for all (t,x), $\sigma(t,x) > \lambda$. It is usually needed to guarantee that the Hamilton-Jacobi-Bellman equation has a solution. Here, we only assume that strict ellipticity holds, i.e., $\sigma(t,x) > 0$. It is enough for our purpose and we do not want to rule out the interesting case of Brownian learning or the possibility that learning slows down to zero as times goes to ∞ for example.*

Finally, we assume that at an endpoint $x^e \in \{\underline{x}, \bar{x}\}$, $f(t,x^e) - rg(x^e) \leq 0$ for all $t \geq 0$. This guarantees that it is optimal to stop immediately when the diffusion process exits $\mathcal{X}$.

As a first step we note that, the stochastic differential equation (2.1) has a unique strong solution and that the process X is Markovian (see Chapter 21 in Kallenberg (2006)). Therefore, following the classical theory in Markovian stopping problems we can focus on the stopping region:

$$\mathcal{S} := \{(t,x) \in \mathcal{Y}_T \,:\, V(t,x) = g(x)\}\,, \tag{2.2}$$

and its complement, the continuation region

$$\mathcal{C} := \{(t,x) \in \mathcal{Y}_T \,:\, V(t,x) > g(x)\} = \mathcal{S}^c. \tag{2.3}$$

By standard result in the theory of optimal stopping (e.g. Corollary 2.9 in Peskir and Shiryaev (2006)), the smallest optimal stopping time for (V) is

$$\tau_{\mathcal{S}} := \inf\left\{t' \geq 0 \,:\, \left(t+t', X_{t'}^{(t,x)}\right) \in \mathcal{S}\right\}. \tag{2.4}$$

Finally we define several functional spaces, which are used in the analysis. Let $W^{1,2,p}(\mathcal{Y}_T)$, $1 \leq p \leq \infty$ denote the space of functions that are (i) twice differentiable almost everywhere in space, (ii) once differentiable almost everywhere in time, and (iii) all of whose *weak* derivatives are in $L^p(\mathcal{Y}_T)$. It is often referred to as a Sobolev space. Also let $W^{1,2,p}_{loc}(\mathcal{Y}_T)$ denote the space of all functions f such that the restriction of f on any compact subset $\mathcal{Y}$ of $\mathcal{Y}_T$ is in the Sobolev space $W^{1,2,p}(\mathcal{Y})$. $\mathcal{C}^0(\mathcal{Y}_T)$ is the space of continuous functions on the domain $\mathcal{Y}_T$. For $k, k' \in \mathbb{N}$, $\alpha \in (0,1]$, $\mathcal{C}^{k,k',\alpha}$ is the space of functions that are k times continuously differentiable with respect to the time variable t with $\frac{\alpha}{2}$-Hölder continuous derivatives and k' times continuously differentiable with respect to the space variable x, with α-Hölder continuous derivatives. Formal definitions are given in Appendix B.1.

2.3 Analysis and results

Our main contribution is to identify a class of environments we call *monotone environments*, in which the continuation region's boundaries are strictly monotone. First, we introduce the tools we need for our analysis in Section 2.3.1. Then, we define *Monotone environments* and prove our main result in Section 2.3.2.

In this section, we maintain the following assumption.

Assumption 10 *The value function* (V) *is continuous on* $\bar{[}0, T) \times \mathcal{X}$.

Assumption 10 is usually easy to check in applications and holds under standard assumptions on the primitives. For example, Lemma 3 in Section 2.4 provides conditions under which V is continuous in our applications. The techniques used in the proof easily extends to different cases.[9]

2.3.1 Optimal stopping problems: toolbox

We show that the value function is the unique L^p-solution of the Hamilton-Jacobi-Bellman equation. This is a crucial to derive comparative static results on the shape of the continuation region without solving the problem explicitly. Consider

$$\begin{cases} \max\left\{g(x) - v(t,x), \left(\partial_t + \mathcal{L}^{(t,x)} - r(t,x)\right) v(t,x) + f(t,x)\right\} = 0 \text{ in } \mathcal{Y}_T, \\ v(t,x) = g(x) \text{ on } \partial\mathcal{Y}_T. \end{cases} \tag{HJB}$$

[9] Theorem 1 in Durandard and Strulovici (2022) provides alternative conditions that guarantee continuity of V in a wide variety of problems.

$\mathcal{L}$ is the infinitesimal generator associated with X. It is defined as follows: for all $(t,x) \in \mathcal{Y}_T$, for all $\phi(t,x) \in W^{1,2,p}(\mathcal{Y}_T)$,

$$\mathcal{L}^{(t,x)}\phi(t,x) = \frac{1}{2}\sigma^2(t,x)\phi_{xx}(t,x) + \mu(t,x)\phi_x(t,x).$$

So,

$$\left(\partial_t + \mathcal{L}^{(t,x)} - r(t,x)\right) v(t,x) = v_t(t,x) + \frac{1}{2}\sigma^2(t,x)v_{xx}(t,x) + \mu(t,x)v_x(t,x) - v(t,x).$$

Definition 9 *A function $v \in W^{1,2,p}_{loc}(\mathcal{Y}_T) \cap \mathcal{C}^0(\bar{[}0,T) \times \mathcal{X})$ is an L^p-solution of the Hamilton-Jacobi-Bellman equation* (HJB) *if (i) $u = b$ on $\partial[0,T) \times \mathcal{X}$, and (ii)*

$$\max\left\{g(x) - v(t,x), \left(\partial_t + \mathcal{L}^{(t,x)} - r(t,x)\right) v(t,x) + f(t,x)\right\} \geq 0$$

a.e. in $\mathcal{Y}_T$, where $v_t(t,x), v_x(t,x)$ and $v_{xx}(t,x)$ are the weak derivatives of u.[10]

Theorem 4 *Suppose the value function is continuous and grows at most linearly (Assumption 10). Then the value function* (V) *is the unique L^p-solution of the Hamilton-Jacobi-Bellman equation:*

$$\begin{cases} \max\left\{g(x) - v(t,x), \left(\partial_t + \mathcal{L}^{(t,x)} - r(t,x)\right) v(t,x) + f(t,x)\right\} = 0 \text{ in } \mathcal{Y}_T, \\ v(t,x) = g(x) \text{ on } \partial\mathcal{Y}_T. \end{cases} \tag{HJB}$$

Moreover, V is continuously differentiable and twice continuously differentiable with respect to x with α-Hölder continuous derivatives in the continuation region: $V \in \mathcal{C}^{1,2,\alpha}(\mathcal{C})$.

The proof of Theorem 4 relies on Theorem 1 in Durandard and Strulovici (2022).[11] Hence,

[10]The requirement that the inequality is satisfied almost everywhere is meaningful, because if $u \in W^{1,2,p}(\mathcal{Y}_T)$, then its weak derivatives up to the second order are uniquely well defined almost everywhere.

[11]As noted in Durandard and Strulovici (2022), the results in this paper applies for stochastic discount

we can apply this result locally: for any point (t,x) with $x \neq x^c$, Theorem 1 in Durandard and Strulovici (2022) then guarantees that the value function is the unique L^p solution of

$$\begin{cases} \max\left\{g(x) - v(t,x), \left(\partial_t + \mathcal{L}^{(t,x)} - r(t,x)\right)v(t,x) + f(t,x)\right\} = 0 \text{ in } C_\delta(t,x), \\ v(t,x) = V(t,x) \text{ on } \partial C_\delta(t,x), \end{cases}$$

where $C_\delta(t,x)$ is a cylindrical neighborhood of (t,x). If g is smooth, this conclude the proof.

However, when g is not differentiable at x^c, the above argument fails as the Assumptions of Theorem 1 in Durandard and Strulovici (2022) are not satisfied. We overcome this difficulty by proving that for all $t \in [0,T)$, there exists a small neighborhood of $(t,x^c) \in \mathcal{Y}_T$, $C_\delta(t,x^c)$, such that it is never optimal to stop in $C_\delta(t,x^c)$ (Proposition 12 below). The result then follows from Theorem 1 in Durandard and Strulovici (2022) applied to value function in this neighborhood.

Finally, the last statement follows from standard (interior) a priori estimates for the solutions of parabolic equations.

The following proposition 12 used in the second step is of independent interest too.

Proposition 12 *Suppose that ${g^1}'(x) \neq {g^2}'(x^c)$. Then $V(t,x^c) > g(x^c)$ for all $t \in [0,T)$, i.e., the DM never stops when $X_t = x^c$.*

Hence, the idea that the decision maker will never stop at a convex kink, as waiting a infinitesimal amount of time guarantee him a strictly higher payoff, has often been used in the study of stopping problem (see, for example, Dixit et al. (1994), Décamps et al. (2006), or Dixit (2013)).

In particular, it underlies the intuition behind smooth pasting and the smoothing effect of the Brownian motion, which goes as follows: Suppose that the value function exhibits an

factor when r is regular.

x-convex kink. Consider the following problem: at this point (t, x^c), the decision maker can choose to wait for a small interval of time dt or stop and obtain $V(t, x^c)$. If she waits, the decision-maker can observe the evolution of X and stop on either side of x^c, at which points, she resumes the optimal paths of play, hence obtaining $V(t + dt, X_{dt})$. The average payoff she obtains is greater than the payoff from investing at the indifference point itself since the value function V has a convex kink at x^c. This remains true even though this average payoff must be discounted and the decision-maker potentially has to pay a flow cost for waiting. The reason is the following one. By standard properties of non-degenerate diffusions, the expected gain from delaying the decision is proportional to $|g_x(x^{c+}) - g_x(x^{c-})| \sqrt{dt}$, while the cost due to discounting is of order dt. When dt is small the former effect dominates. As a result, the decision-maker is better off waiting: a "contradiction". So the value function cannot exhibit x-convex kinks and the stopping region $\mathcal{S}$ does not contain (t, x^c).

Proposition 12 formalizes this intuition. Its proof is technical and relies on a local time argument. It guarantees that when $g \notin C^2(\mathcal{X})$, its kink (which is necessary at x^c) is in the continuation region for all $t \geq 0$. This allows us to prove our first main result: Theorem 4.

2.3.2 Main result: monotone environments

Having introduced the main tool for our analysis: the Hamilton-Jacobi-Bellman equation, we are now ready to state and prove our main result. First we define locally monotone environments. They are characterized by a set of conditions, under which Theorem 5 and 6 shows that the boundaries of the continuation regions are locally monotone.

Definition 10 *The optimal stopping problem* (V) *is locally **monotone** on the time interval* $(\underline{t}, \bar{t})$ *if the value function* $V(t, x)$ *is monotone in time on* $(\underline{t}, \bar{t})$*; i.e., for all* $x \in \mathcal{X}$*,* $t \to V(t, x)$ *is monotone on* $(\underline{t}, \bar{t})$*.*

*It is locally **monotone increasing** on* $(\underline{t}, \bar{t})$ *if* $t \to V(t, x)$ *is nondecreasing on* $(\underline{t}, \bar{t})$ *for* *all* $x \in \mathcal{X}$*, and it is locally* ***monotone decreasing*** *on* $(\underline{t}, \bar{t})$ *if* $t \to V(t,x)$ *is nonincreasing on* $(\underline{t}, \bar{t})$ *for all* $x \in \mathcal{X}$*.*

We say that the optimal stopping problem (V) *is* ***monotone*** *if it is globally monotone (i.e.,* $(\underline{t}, \bar{t}) = (0, T)$*).*

Finally, we define strictly monotone environments.

Definition 11 *The optimal stopping problem* V *is locally* ***strictly monotone*** *on the time interval* $(\underline{t}, \bar{t})$ *if*

(i) It is locally monotone on the time interval $(\underline{t}, \bar{t})$*;*

(ii) The value function V *is* ***strictly monotone in time in the continuation region*** *on the time interval* $(\underline{t}, \bar{t})$*; i.e., for all* $(t,x) \in \mathcal{C} \cap (\underline{t}, \bar{t}) \times \mathcal{X}$, $V_t(t,x) < 0$ *or, for all* $(t,x) \in \mathcal{C} \cap (\underline{t}, \bar{t}) \times \mathcal{X}$, $V_t(t,x) > 0$*;*

(iii) $\sigma_t(t,x)V_{xx}(t,x) \leq 0$, $\mu_t(t,x)V_x(t,x) \leq 0$, $r_t(t,x)V(t,x) \geq 0$, *and* $f_t(t,x) \leq 0$ *for all* $(t,x) \in \mathcal{C} \cap (\underline{t}, \bar{t}) \times \mathcal{X}$ *if* $V_t(t,x) \leq 0$*; and* $\sigma_t(t,x)V_{xx}(t,x) \geq 0$, $\mu_t(t,x)V_x(t,x) \geq 0$, $r_t(t,x)V(t,x) \leq 0$, *and* $f_t(t,x) \geq 0$ *for all* $(t,x) \in \mathcal{C} \cap (\underline{t}, \bar{t}) \times \mathcal{X}$*, if* $V_t(t,x) \geq 0$.

If the value function is strictly decreasing in time in the continuation region, the stopping problem is ***strictly monotone decreasing****. If the value function is strictly increasing in time in the continuation region, the stopping problem is* ***strictly monotone increasing****.*[12]

We say that the optimal stopping problem (V) *is* ***strictly monotone*** *if it is globally strictly monotone (i.e.,* $(\underline{t}, \bar{t}) = (0, T)$*).*

Our first comparative static results shows that the continuation region is monotone (in the set inclusion order) in *monotone environments*. Theorem 5 shows that the continuation

[12]The derivatives in the statements (ii) and (iii) are standard derivatives. They are well-defined in the continuation region as a consequence of Theorem 4.

region is weakly increasing. It follows immediately from the local monotonicity of the value function.

Theorem 5 *Suppose the value function is continuous and grows at most linearly (Assumption 10). Then*

1. *If the optimal stopping problem is locally* ***monotone decreasing*** *on $(\underline{t}, \bar{t})$, the optimal continuation region $\mathcal{C}$ is nonincreasing (in the set inclusion order) on $(\underline{t}, \bar{t})$.*

2. *If the optimal stopping problem is locally* ***monotone increasing*** *on $(\underline{t}, \bar{t})$, the optimal continuation region $\mathcal{C}$ is nondecreasing (in the set inclusion order) on $(\underline{t}, \bar{t})$.*

Proof of Theorem 5. We only prove the Theorem in the locally monotone decreasing case, as the proof of the second case is similar. Consider the optimal stopping problem (V) and suppose that it is locally monotone decreasing on $(\underline{t}, \bar{t})$. Then the function

$$t \to V(t,x) - g(x)$$

is nonincreasing on $(\underline{t}, \bar{t})$. Therefore, for all $t \in (\underline{t}, \bar{t})$, $(t,x) \in \mathcal{C} \Rightarrow [\underline{t}, t] \times \{x\} \subseteq \mathcal{C}$. ■

Our second main Theorem 6 states that the continuation region is strictly monotone in the sense of Definition 12 below.

Definition 12 *[Strict set order] The continuation region is strictly increasing over time if, for $\bar{t} > \underline{t}$, the t-sections of the continuation region, $\mathcal{C}_t := \{(t,x) \in \mathcal{C} \,:\, x \in \mathcal{X}\}$, are such that $\mathcal{C}_{\underline{t}} \subset \mathcal{C}_{\bar{t}}$ and $\partial \mathcal{C}_{\underline{t}} \subset \mathcal{C}^{\circ}_{\bar{t}}$.*

The continuation region is strictly decreasing over time if, for $\bar{t} > \underline{t}$, the t-section of the continuation region are such that $\mathcal{C}_{\bar{t}} \subset \mathcal{C}_{\underline{t}}$ and $\partial \mathcal{C}_{\bar{t}} \subset \mathcal{C}^{\circ}_{\underline{t}}$.

Remark 2 *If the continuation region is strictly monotone in the sense of definition 12, its boundary can be written as the union of strictly monotone functions.*

Theorem 6 *Suppose the value function is continuous and grows at most linearly (Assumption 10).*

1. *If the optimal stopping problem is locally* ***strictly monotone decreasing*** *on* $(\underline{t}, \bar{t})$*, the optimal continuation region* $\mathcal{C}$ *is strictly decreasing.*

2. *If the optimal stopping problem is locally* ***strictly monotone increasing*** *on* $(\underline{t}, \bar{t})$*, the optimal continuation region* $\mathcal{C}$ *is strictly increasing.*

The proof of Theorem 6 is more challenging than that of Theorem 5, as it requires the knowledge of the cross derivative of the value function at the boundary. To overcome this issue, we develop a new argument based on partial differential equation results and the Hopf boundary lemma. **Proof of Theorem 6.** We prove the theorem in the case that the optimal stopping problem is locally strictly monotone decreasing on $(t_1, \bar{t}_2)$. The case in which the optimal stopping problem is locally strictly monotone increasing is similar. By Theorem 5, the stopping region is locally decreasing. So, there remains to show that the boundaries are strictly monotone when they are in $\mathcal{X}$.

The proof is by contradiction. Suppose that there exists $\underline{t}, \bar{t} \in [0, T)$ and $B \in \mathcal{X}$ such that $(\underline{t}, B)$ and $(\bar{t}, B) \in \partial\mathcal{C}$ and such that there exists $x < B$ with $(\bar{t}, x) \in \mathcal{C}$. Note that we can always find $\bar{t}$ such that this last condition holds. For the rest of the proof, we assume that for all $(t, x) \in \mathcal{C} \cap \underline{t}, \bar{t}) \times \mathcal{X}$, $x \leq B$, i.e. we are looking at an "upper boundary" of the continuation region. The proof when for all $(t, x) \in \mathcal{C} \cap \underline{t}, \bar{t}) \times \mathcal{X}$, $x \geq B$ is identical.

Let $\epsilon > 0$ and consider the rectangular domain $\mathcal{Y} := [\underline{t}, \bar{t}) \times (B - \epsilon, B)$ with parabolic boundary $\partial\mathcal{Y} := ([\underline{t}, \bar{t}] \times (\{B - \epsilon\} \cup \{B\})) \cup (\{T\} \times (B - \epsilon, B))$. We can choose ϵ such that the rectangular domain is contained in the continuation region. By Theorem 4, we know that the value function is the unique L^p-solution of the boundary value problem

$$\begin{cases} v_t(t,x) + \frac{\sigma(t,x)^2}{2} v_{xx} + \mu(t,x) v_x(t,x) - r(t,x) v(t,x) + f(t,x) = 0 \text{ if } (t,x) \in \mathcal{Y}, \\ v(t,x) = V(t,x) \text{ if } (t,x) \in \partial\mathcal{Y}. \end{cases}$$

By Corollary 3 in Durandard and Strulovici (2022), $V_t \in \mathcal{C}^0\left(\bar{\mathcal{Y}}\right)$. Then, Theorem 3.5.10 in Friedman (2008) guarantees that $V_t(t,x) \in \mathcal{C}^{1,2,\alpha}\left([\underline{t}, \bar{t}) \times (B - 2\epsilon, B)\right)$ and solves

$$\begin{cases} v_{tt}(t,x) + \frac{\sigma(t,x)^2}{2} v_{txx} + \mu(t,x) v_{tx}(t,x) - r(t,x) v_t(t,x) \\ \qquad = -\sigma(t,x)\sigma_t(t,x) v_{xx}(t,x) - \mu_t(t,x) v_x(t,x) + r_t(t,x) v(t,x) - f_t(t,x) \text{ if } (t,x) \in \mathcal{Y}, \\ v_t(t,x) = V_t(t,x) \text{ if } (t,x) \in \partial\mathcal{Y}. \end{cases}$$

Therefore, on $\mathcal{Y}$,

$$\begin{aligned} v_{tt}(t,x) + \frac{\sigma(t,x)^2}{2} v_{txx} + \mu(t,x) v_{tx}(t,x) - r(t,x) v_t(t,x) & \\ = -\sigma(t,x)\sigma_t(t,x) v_{xx}(t,x) - \mu_t(t,x) v_x(t,x) + r_t(t,x) v(t,x) - f_t(t,x) & \\ \geq 0, & \end{aligned}$$

where the inequality follows from condition (iii) in Definition 11. Then, by Hopf's boundary Lemma (Lemma 1.23 in Wang (2021)), $v_{tx}(t,B) > 0$ for all $t \in (\underline{t}, \bar{t})$ such that $v_{tx}(t,B)$ exists. But, for all $t \in (\underline{t}, \bar{t})$, $V(t,B) = g(x) \Rightarrow v_t(t,B) = V_t(t,B) = \frac{\partial}{\partial t} g(x) = 0$ and therefore $v_{tx}(t,B) = 0$. So if there exists $(t,B) \in (\underline{t}, \bar{t}) \times \{B\}$ such that $v_{tx}(t,B)$ is well defined, we have a contradiction. This follows from Theorem 2.1 in Wang (1992), since $V \in \mathcal{C}^{1,2,\alpha}\left([\underline{t}, \bar{t}) \times (B - 2\epsilon, B)\right)$ by Theorem 4.

Thus, the upper boundary of the continuation cannot be flat. This concludes the proof.

■

Remark 3 *From the proof of Theorem 6, one can see that condition (iii) in Definition 11 can be slightly relaxed. It is enough that, at all boundary points, there exists a neighborhood such that, in its intersection with the continuation region, we have*

$$\sigma_t(t,x)v_{xx}(t,x) + \mu_t(t,x)v_x(t,x) - r_t(t,x)v(t,x) + f_t(t,x) \leq 0,$$

when the stopping problem is locally monotone decreasing, and

$$\sigma_t(t,x)v_{xx}(t,x) + \mu_t(t,x)v_x(t,x) - r_t(t,x)v(t,x) + f_t(t,x) \geq 0.$$

when the stopping problem is locally monotone increasing. Condition (iii) in Definition 11 is sufficient for this.

Theorems 5 and 6 relates local conditions of the environment to local characteristics of the continuation and stopping regions. Therefore, they are only useful if one can check whether an environment satisfies the prerequisite conditions or if monotone problems are prevalent. This turns out to be the case. For example, finite time optimal stopping problem with time-stationary primitives are easily shown to be monotone. In Section 2.4, we also show that many information acquisition problems are monotone, and the techniques we develop can be used to check whether any specific stopping problem is monotone.

Finally, Theorems 5 and 6 gives conditions under which the dynamics of the boundaries of the continuation region are characterized locally. However, they only provide limited global information on the continuation region. For example, it may be connected or not.

To obtain a global characterization in monotone environment, we strengthen our assumptions. Assume that

Assumption 11 (SC) *For all* $t \in [0,T)$*, the mapping* $x \to f(t,x) + \left(\mathcal{L}^{(t,x)} - r(t,x)\right) g(x)$

is nondecreasing on $(\underline{x}, x^c]$ *and nonincreasing on* $[x^c, \bar{x})$.

Assumption 11 says that the value of waiting one extra instant is decreasing the further away we go from the indifference point x^c. To see this, formally, observe that, when g is smooth, by Itô's formula, for all $(t,x) \in \mathcal{Y}_T$,

$$V(t,x) - g(x) = \sup_\tau \mathbb{E}_{(t,x)} \left[\int_0^\tau e^{-\int_0^s r(u,X_u)du} \left(f(s,X_s) + \left(\mathcal{L}^{(s,X_s)} - r(t,x) \right) g(X_s) \right) ds \right].$$

It implies that t-section of the continuation region are convex.

Proposition 13 *Suppose that the single crossing Assumptions 10 and 11 hold. Then there exist functions* $\underline{b}, \bar{b} : \mathbb{R}_+ \to \mathcal{X}$ *such that*

$$\mathcal{S} = \left\{ (t,x) \in \mathbb{R}_+ \times \mathcal{X} \,:\, x \notin \left(\underline{b}(t), \bar{b}(t) \right) \right\}.$$

When g is smooth and $x \to f(t,x) + \left(\mathcal{L}^{(t,x)} - r(t,x) \right) g$ is monotone, the result is standard. See, e.g., Villeneuve (2007). Unfortunately the argument is not easily adaptable to our case. We provide a new proof based on the HJB equation in Appendix B.4.

As a corollary of Theorems 5, 6, and Proposition 13.

Corollary 4 *Suppose that Assumptions 10 and 11 hold, and that the problem is globally monotone.*

1. *If the optimal stopping problem is* ***monotone decreasing****, there exists a càglàd nondecreasing function* $\underline{b} : \mathbb{R}_+ \to \mathcal{X} \cup \{-\infty\}$ *and a càdlàg nonincreasing function* $\bar{b} ::$ $\mathbb{R}_+ \to \mathcal{X} \cup \{+\infty\}$ *with* $\underline{b}(T) \le x^c \le \bar{b}(T)$ *such that*

$$\mathcal{S} := \left\{ (t,x) \in \mathbb{R}_+ \times \mathcal{X} \,:\, x \notin \left(\underline{b}(t), \bar{b}(t) \right) \right\}.$$

When the optimal stopping problem is ***strictly monotone decreasing****, if* $\bar{b}(\underline{t}) < \bar{x}$*, then* $t \to \bar{b}(t)$ *is* ***strictly decreasing*** *on* $[\underline{t}, T)$*; and if* $\underline{b}(\underline{t}) > \underline{x}$*, then* $t \to \underline{b}(t)$ *is* ***strictly increasing*** *on* $[\underline{t}, T)$*.*

2. *If the optimal stopping problem is* ***monotone increasing****, there exists a càglàd nonincreasing function* $\underline{b} : \mathbb{R}_+ \to \mathcal{X}$ *and a càglàd nondecreasing function* $\bar{b} : \mathbb{R}_+ \to \mathcal{X}$ *with* $\underline{b}(0) < x^c < \bar{b}(0)$ *such that*

$$\mathcal{S} := \left\{(t,x) \in \mathbb{R}_+ \times \mathcal{X} \,:\, x \notin \left(\underline{b}(t), \bar{b}(t)\right)\right\}.$$

When the optimal stopping problem is ***strictly monotone increasing****, if* $\bar{b}(\bar{t}) < \bar{x}$*, then* $t \to \bar{b}(t)$ *is* ***strictly increasing*** *on* $[0, \bar{t})$*; and if* $\underline{b}(\bar{t}) > \underline{x}$*, then* $t \to \underline{b}(t)$ *is* ***strictly decreasing*** *on* $[0, \bar{t})$*.*

Proof of Corollary 4. The existence of the boundaries $\underline{b}$ and $\bar{b}$ follows from Proposition 13, and the bounds $\underline{b}(T) \le x^c \le \bar{b}(T)$ from Proposition 12. The monotonicity follows from Theorem 5 and 6. Finally since V is continuous, $\mathcal{S}$ is closed, and, hence, $\underline{b}$ is upper semincontinuous and $\bar{b}$ is lower semicontinuous. Together with their monotonicity properties, this guarantees that they are càdlàg or càglàd. ■

Finally, in monotone decreasing stopping problems, one can also show that the stopping boundaries are continuous under a small strengthening of assumption 11(SC).

Assumption 12 (SSC) *For all* $t \in [0,T)$*, the mapping* $x \to f(t,x) + \left(\mathcal{L}^{(t,x)} - r\right) g(x)$ *is nondecreasing on* $(\underline{x}, x^c]$ *and there exists a unique* $x^-(t) \in [\underline{x}, x^c]$ *such that, for all* $x \in (\underline{x}, x^c]$*,* $x < x^-(t) \Rightarrow f(t,x) + \left(\mathcal{L}^{(t,x)} - r(t,x)\right) g(x) < 0$ *and* $x > x^-(t) \Rightarrow f(t,x) + \left(\mathcal{L}^{(t,x)} - r\right) g(x) > 0$*.*

Similarly, for all $t \in [0,T)$*, the mapping* $x \to f(t,x) + \left(\mathcal{L}^{(t,x)} - r(t,x)\right) g(x)$ *nonincreasing on* $[x^c, \bar{x})$ *and there exists a unique* $x^+(t) \in [x^c, \bar{x}]$ *such that, for all* $x \in [x^c, \bar{x})$*,* $x > x^+(t) \Rightarrow$

$f(t,x)+\left(\mathcal{L}^{(t,x)}-r(t,x)\right)g(x)<0$ *and* $x>x^{+}(t)\Rightarrow f(t,x)+\left(\mathcal{L}^{(t,x)}-r(t,x)\right)g(x)<0.$

Assumption 12 yields useful properties of the stopping and continuation regions.

Lemma 2 *Suppose that Assumptions 10 and 12(SSC) hold, then*

$$\left\{(t,x)\in\mathcal{Y}_T\,:\,x\in\left(x^{-}(t),x^{+}(t)\right)\right\}\subseteq\mathcal{C}.$$

The proof of Lemma 2 is in Appendix B.4. So, if for all $t\in[0,T)$, the mapping $x\rightarrow f(t,x)+\left(\mathcal{L}^{(t,x)}-r(t,x)\right)g(x)$ satisfies strong single crossing on $(\underline{x},x^{c}]$ and on $[x^{c},\bar{x})$, we have

$$\mathcal{S}\subset\left\{(t,x)\in\mathcal{Y}_T\,:\,f(t,x)+\left(\mathcal{L}^{(t,x)}-r(t,x)\right)g(x)<0\right\}. \tag{2.5}$$

The above property of the stopping region has a surprising consequence for the regularity of the free boundary in monotone decreasing environments. It implies that the free boundary is continuous. This is established in Proposition 14.

Proposition 14 *Suppose that the optimal stopping problem is monotone decreasing and that Assumptions 10 and 12 hold. Then the stopping boundaries* $t\rightarrow\underline{b}(t)$ *and* $t\rightarrow\bar{b}(t)$ *are continuous on* $[\underline{t},T)$ *and on* $[\bar{t},T)$*, respectively; with* $\underline{t}=\inf\left\{t\geq 0\,:\,\underline{b}(t)>-\infty\right\}$ *and* $\bar{t}=\inf\left\{t\geq 0\,:\,\underline{b}(t)>-\infty\right\}$*.*

The proof of Proposition 14 is technical and can be found in Appendix B.4.

Remark 4 *The argument above works in monotone decreasing environment. When the environment is monotone increasing, it does not. That's because when the process* X *starts from a point* $(t_0+\epsilon,x)$ *with* $\epsilon>0$*, then, as time passes, it will move away from the boundary. On the contrary, in the proof above, we relied on the fact that* X *moves towards the boundary.*

Intuitively, the diffusion does not "see" the discontinuities of the boundaries in the increasing environment.

Remark 5 *Propositions 13 and 14 and Corollary 4 are proved for a time independent stopping reward g. They remain valid when g is non-stationary under the following additional assumptions: (i) The function $g : \bar{[}0,T) \times \mathcal{X} \to \mathbb{R}$ grows at most linearly: there exists C_g such that $|g(t,x)| < C_g(1+t+|x|)$. (iii) g can be represented as the maximum of two smooth functions: $g = g^1 \vee g^2$, with*

- *$g^i : \bar{[}0,T) \times \mathcal{X} \to \mathbb{R}$, $i = 1, 2$, belongs to $\mathcal{C}^{2,\alpha}(\bar{[}0,T) \times \mathcal{X})$.*
- *For all $t \in [0,T)$, $x \to g^1(t,x) - g^2(t,x)$ is strictly monotone and crosses zero at $x^c(t) \in \mathcal{X}$.*

Secondly, in Assumptions 11 and 12, we need $x \to f(x,t) + \left(\partial_t + \mathcal{L}^{(t,x)} - r\right)$ is nondecreasing on $(\underline{x}, x^c(t)]$ and nonincreasing on on $[x^c(t), \bar{x})$.

Finally, in the definition of a locally strictly monotone environment, we need that g^1 is supermodular in (t,x) and g^2 is submodular in (t,x) if the stopping problem is strictly decreasing, and that g^1 is submodular in (t,x) and g^2 is supermodular in (t,x) if the environment is strictly increasing.

When these conditions hold, our proofs extends without modification.

2.4 Applications

To illustrate the power of our main result, we use it to derive new and interesting predictions in non-stationary information acquisition problems with binary decision.

In these settings, there is an unknown payoff relevant state μ and two possible actions. The decision maker can learn about μ. She decides when to stop acquiring information and what action to take upon stopping. The payoff of the decision maker depends on the state of the world, her action, and how much information she acquired.

Sections 2.4.1 develop useful tools need to map information acquisition problems to our general setting. In Section 2.4.2, we introduce the problem of a decision maker who learns about a binary state and tries to match it with her action, similar to the classic Wald's problem. However, we assume that the information acquisition technology changes over time. When learning's speed increases, accuracy increases, and when learning's speed decreases, accuracy decreases. In Section 2.4.3, we show that when the relevant state is not binary and the information technology is stationary, the decision maker's decision becomes less accurate as time passes. A particular instance of this problem is studied in the seminal paper by Fudenberg et al. (2018). Finally in Section 2.4.4, we consider the effect abrupt changes at stochastic time have on the problem. For example, we look at the possibility that the decision maker faces a stochastic deadline at which she either have to take action immediately or the state is perfectly revealed.

2.4.1 Convex Stopping Problems

In stopping problems in which the decision maker makes a final decision upon stopping, the value function is typically convex (see Proposition 15). As a result, it becomes easy to check whether the environment is locally strictly monotone and thus whether Theorem 6 applies.

First we provide a formal definition for a *convex environment.*

Definition 13 *An environment is said to be convex if*

1. *The function* $g : \bar{\mathcal{X}} \to \mathbb{R}$, $x \to g(x)$ *is convex.*

2. *For all* $t \in [0, T)$, *the function* $x \to f(t, x)$ *is convex.*

3. *The discount factor r is constant: for all $(t,x) \in \mathcal{Y}_T$, $r(t,x) = r$.*

4. *If $r = 0$, then assume that (i) the value function* (V) *grows at most linearly and (ii) there exists $\epsilon > 0$ and $\bar{t} \in [0,\infty)$ such that $f(t,x) < -\epsilon$ for all $t \geq \bar{t}$ and $x \in \mathcal{X}$.*[13]

Then we prove that convex environments entail convex value functions.

Proposition 15 *In a convex environment the value function* (V) *is convex in x for all $t \in [0,T)$.*

The proof of Proposition 15 follows from a discrete time approximation argument and a useful result on the propagation of convexity due to Bergman et al. (1996). It is in Appendix B.5.1.

Moreover, we can show the following.

Lemma 3 *In a convex environment, if $f : \mathcal{Y}_T \to \mathbb{R}$ and $g : \bar{[}0,T) \times \mathcal{X} \to \mathbb{R}$ are Lipschitz continuous the value function* (V) *is continuous.*

Lemma 3, together with condition 4 of Definition 13, if $r = 0$, or Lemma 2 in Durandard and Strulovici (2022), if $r > 0$, guarantees that the value function is continuous and grows at most linearly in *convex stopping problems.* Its proof is in Appendix B.5.1.

As a corollary, we obtain conditions under which a convex optimal stopping problem is strictly monotone.

Corollary 5 *A convex stopping problem is (globally)* ***strictly monotone*** *if*

(i) It is monotone;

(ii) The value function V is ***strictly monotone in time in the continuation region****; i.e., for all $(t,x) \in \mathcal{C}$, $V_t(t,x) < 0$ or, for all $(t,x) \in \mathcal{C}$, $V_t(t,x) > 0$;*

(iii) $\sigma_t(t,x) \leq 0$, $\mu_t(t,x)V_x(t,x) \leq 0$, and $f_t(t,x) \leq 0$ for all $(t,x) \in \mathcal{C}$ if $V_t(t,x) \leq 0$; and $\sigma_t(t,x) \geq 0$, $\mu_t(t,x)V_x(t,x) \geq 0$, and $f_t(t,x) \geq 0$ for all $(t,x) \in \mathcal{C}$, if $V_t(t,x) \geq 0$.

[13] If $T < \infty$, condition (ii) holds vacuously.

2.4.2 Non-stationary learning about a binary state

One of the classical questions in Sequential Analysis concerns the testing of two simple hypotheses about the sign of the drift of an arithmetic Brownian motion. For a textbook treatment, see Peskir and Shiryaev (2006), Chapter VI. Formally, suppose that a decision maker observes a process Y_t

$$Y_t = \mu t + \sigma B_t,$$

whose drift μ is unknown and B is a standard Brownian motion. Based on observations of the process Y, one wants to test sequentially the hypotheses $H_0 : \mu < 0$ and $H_1 : \mu \geq 0$. In the Bayesian formulation of this sequential testing problem, the drift μ is a binary random variable taking values in $\{-1, 1\}$. X_0 is the decision maker's prior belief that $\mu = 1$. Moreover, μ and B are independent. The accuracy and urgency of a decision is governed by a gain function together with a constant cost $c > 0$ of observation per unit of time and a discount rate $r \geq 0$. The problem then consists of finding a decision time τ and rule $d \in \{-1, 1\}$ to maximize the expected total gain:

$$\mathbb{E}\left[e^{-r\tau}\left(a\mathbb{1}_{\{d=1,\mu=1\}} + b\mathbb{1}_{\{d=-1,\mu=-1\}}\right) - \int_0^\tau e^{-rt} c dt\right],$$

where a and b are nonnegative. Since Shiryaev (1967), it has been well-known that the sequential testing problem admits an equivalent formulation as the following stationary optimal stopping problem in which the state X_t is conditional probability of the drift taking value 1 at time t:[14]

$$\sup_\tau \mathbb{E}\left[X_\tau \vee (1 - X_\tau) - c\tau\right]$$

[14] There is a one-to-one correspondence between the observation and belief process.

where

$$X_{t+s} = X_t + \int_t^s 2\frac{\mu}{\sigma} X_u(1 - X_u) dB_u.$$

The solution to this problem consists of two constant boundaries in the belief space. When the posterior probability that $\mu = 1$ exceeds an upper threshold, the decision maker takes action $d = 1$, and when the posterior probability that $\mu = 1$ falls below a lower threshold, she takes action $d = 0$. A complete characterization of the solution can be found in Peskir and Shiryaev (2006), Chapter VI. As upper and lower threshold on the posteriors are constant over time, the accuracy of both decisions is also constant in time. So, in the classical model, the timing of decisions is uncorrelated with accuracy.

In this section, we extend the above problem to settings in which the information acquisition technology (i.e. the observation process) is not stationary, to capture observed relations between speed and accuracy, and relate them to primitives of the model.[15] By standard arguments, the problem is equivalent to a an optimal stopping problem, where the underlying process is the conditional probability that the drift is 1: $X_t := \mathbb{P}(\mu = 1 \mid \mathcal{F}_t)$. In particular, the state variable X is a $[0, 1]$-valued martingale. However the diffusion governing the evolution of X needs not be stationary anymore:

$$X_{t+s} = X_t + \int_t^s \sigma(u, X_u) dB_u,$$

where $\sigma(u, X_u)$ measures the speed of learning. Implicit in the above formulation is that information acquisition is gradual. Moreover, we will also assume that the decision maker can never learn the state perfectly and that the information acquisition technology is such that

[15] nnnnnLiterature

Assumption 13 *Suppose that (i)* $\sigma \in \mathcal{C}^{2,\alpha}([0,\infty) \times \mathcal{X})$, *(ii)* $\bar{\sigma}(x) := \sup_{t \in [0,T)} \sigma(t,x)$ *is Lipschitz, that (iii)* $\inf\left\{t \geq 0 \,:\, \bar{X}_t \in \{\underline{x}, \bar{x}\}\right\} = \infty$ $\mathbb{P}$*-a.s. where* $\bar{X}$ *is the unique strong solution of*

$$\bar{X}_t = x + \int_0^t \bar{\sigma}(X_s) dB_s.,$$

and that (iv), for all $\epsilon > 0$,

$$\lim_{x \to \underline{x}} \mathbb{P}_x\left(\bar{\tau}_{(x^c)} > M\right) \geq 1 - \epsilon \quad \text{and} \quad \lim_{x \to \bar{x}} \mathbb{P}_x\left(\bar{\tau}_{(x^c)} > M\right) \geq 1 - \epsilon,$$

where $\bar{\tau}_{(x^c)} = \inf\left\{t \geq 0 \,:\, \bar{X}_t = x^c\right\}$.

Assumption 13 guarantees that the continuation region in the sampling problem is bounded. The decision maker's problem is then

$$V^s(t,x) = \sup_{\tau} \mathbb{E}_{(t,x)}\left[e^{-r(\tau - t)} a X_\tau \vee b(1 - X_\tau) - \int_t^\tau e^{-r(s-t)} c ds\right], \tag{V^s}$$

subject to

$$X_{t+s} = X_t + \int_t^{t+s} \sigma(u, X_u) dB_u.$$

So the above problem can be seen as a particular instance of the general problem in Section 2.3 where the following holds:

Assumption 14 *(i)* $T = \infty$ *and* $\bar{x}$ *and* $\underline{x}$ *are inaccessible. (ii)* g^1 *and* g^2 *are affine and nonnegative. (iii) For all* $(t,x) \in \mathcal{Y}_t$, $f(t,x) \leq -c$ *for some* $c > 0$. *(iv) For all* $(t,x) \in \mathcal{Y}_T$, $\mu(t,x) = 0$.

In particular, assumption 12 (SSC) is satisfied. Moreover, we have

Proposition 16 *If $t \to \sigma(t,x)$ is strictly decreasing, then the optimal stopping problem (V^s) is **strictly monotone decreasing**.*

*If $t \to \sigma(t,x)$ is strictly increasing, then the optimal stopping problem (V^s) is **strictly monotone increasing**.*

Intuitively, when $\sigma(t,x)$ decreases over time, the speed at which the decision maker obtains information decreases over time. So starting from a later time, she is in a worse position: she needs to acquire information for a longer period of time to get the same amount of information. The proof of Proposition 16 is in Appendix B.5.2.

As a corollary of the above Proposition 16, Corollary 4, and Proposition 14, we have

Proposition 17 *Under assumption 13, the optimal stopping region $\mathcal{S}$ in (V^s) is given by*

$$\mathcal{S} \coloneqq \left\{(t,x) \in \mathbb{R}_+ \times \mathcal{X} \,:\, x \notin \left(\underline{b}(t), \bar{b}(t)\right)\right\},$$

where

- *$t \to \bar{b}(t)$ is **strictly decreasing** and continuous and $t \to \underline{b}(t)$ is **strictly increasing** and continuous on $[0,\infty)$ if $t \to \sigma(t,x)$ is strictly decreasing; and*
- *$t \to \bar{b}(t)$ is **strictly increasing** and $t \to \underline{b}(t)$ is **strictly decreasing** on $[0,\infty)$ if $t \to \sigma(t,x)$ is strictly increasing.*

The proof of Proposition 16 is in Appendix B.5.2. The above proposition summarizes how the decision boundaries, and therefore the posterior beliefs at the time of decision, change over time. As a result, Proposition 17 has direct implications about the evolution of accuracy over time. In particular, accuracy decreases (increases) over time when the upper boundary decreases (increases) or when the lower boundary increases (decreases) over time.

2.4.3 Learning about a non-binary state

Section 2.4.2 relates the speed-accuracy trade-off to the speed of learning. In this section, we consider an alternative explanation following the seminal work of Fudenberg et al. (2018). Hence, the fact that the belief process is stationary when the state is drawn from a binary distribution turns out to be special. For any other (non-degenerate) prior, the probability that the state μ is above or below zero for example is non-stationary (see Proposition 19 below). In this section, we therefore reconsider the sequential sampling problem when the state is drawn from a non-binary distribution.

We start with the "uncertain difference" Drift Diffusion Model of Fudenberg et al. (2018). An agent has to decide between two alternatives $\{l, r\}$. She is uncertain about the utilities $\theta = \left(\theta^l, \theta^r\right)$ associated with each choice and pays a constant running cost $c > 0$ to observe Brownian signals of the true utility. They show that, when the agent believe that θ is distributed according to a bivariate normal distribution (where each component's variance is $\sigma_0^2 > 0$), the stopping time is the first exit time from a shrinking continuation region. That the boundaries decreases implies that the accuracy of the decision is decreasing over time (Theorem 1 in Fudenberg et al. (2018)). Accuracy is defined as the probability of making the correct decision (i.e., choosing r when $\theta^r \geq \theta^l$ and conversely). Note that, this definition of accuracy does not take into account the magnitude of the difference. An error is defined similarly whether $\left|\theta^l - \theta^r\right|$ is large or not. This characteristic appears crucial in their setting with normal prior: agent's will make less accurate decisions in those problems for which $\left|\theta^l - \theta^r\right|$ is small, i.e. problems for which making the right decision is relative less important.

The purpose of this section is to highlight how we can view their setting as a special case of ours, and how, using our machinery, we can extend their result of accuracy being decreasing over time even to cases where the decision makers do not become pessimistic on the relevance of making the right decisions.

Formally, their problem can be seen to be equivalent to the following problem:

$$V(t,x) := \sup_{\tau} \mathbb{E}_{(t,x)} \left[X_\tau \vee 0 - c(\tau - t) \right],$$

subject to

$$X_s^{(t,x)} = x + \int_t^s \frac{\sqrt{2}\sigma_0^2 \alpha}{\alpha^2 + u\sigma_0^2} dB_u,$$

where B_k is a standard Brownian motion, $c > 0$ is a constant running cost, and $\alpha > 0$ is the noise of the Brownian signal in the original hypothesis testing problem. This problem can be interpreted as one in which the decision maker chooses between an alternative that delivers value 0 and one that delivers value $\theta^l - \theta^r$ unknown, where $E(\theta^l - \theta^r) = X$ and the speed of learning decreases over time.

Since the above problem satisfies Assumptions 13 and 14 (with $\bar{\sigma}(x) = \frac{\sqrt{2}\sigma_0^2\alpha}{\alpha^2}$), we can use Proposition 17 to characterize the shape of the continuation region.

Proposition 18 *The continuation region in the above problem is*

$$\mathcal{C} := \left\{ (t,x) \in [0,\infty) \times \mathbb{R} \, : \, \underline{b}(t) < x < \bar{b}(t) \right\},$$

where $t \to \bar{b}(t)$ and $t \to \underline{b}(t)$ are two continuous boundaries, with $\bar{b}(t) = -\underline{b}(t)$ for all $t \geq 0$. Moreover, $\bar{b}(\cdot)$ is strictly decreasing.

The proof is in Appendix B.5.3. As shown by Fudenberg et al. (2018), such shape of the boundaries corresponds to accuracy being decreasing in the stopping time (see Theorem 2 in Fudenberg et al. (2018)).

Note that in Fudenberg et al. (2018), two forces push toward the decrease in accuracy.

First, the speed of learning decreases, so there is less incentive for the DM to continue acquiring information as time passes by. Second, as time goes by and no decision is taken, the expected payoff difference between the two action fades away, so the benefit of taking the right decision also vanishes. In order to isolate the effect of the decrease in learning speed, we propose a *certain difference* drift diffusion model, where the payoff difference between the actions is constant and known. In this more conservative model, we show that the negative correlation between the time and accuracy of decisions still arises.

Our certain difference drift diffusion model is similar to the classical hypothesis testing problem. The value of choosing an alternative is either 0 or 1 depending on the sign of the true state μ. However, contrary to the classical Wald model, the unknown parameter μ is not binary. It is a real-valued random variable distributed according to $F \in \Delta(\mathbb{R})$. The decision maker observes

$$Y_t = \mu t + \sigma B_t,$$

and chooses a decision time τ and rule $d \in \{-1, 1\}$ to maximize the expected total gain:

$$\mathbb{E}\left[e^{-r\tau}\left(a\mathbb{1}_{\{d=1,\mu\geq 0\}} + b\mathbb{1}_{\{d=-1,\mu<0\}}\right) - \int_0^\tau e^{-rt} c dt\right],$$

where $a, b \geq 0$, $c > 0$, and $r \geq 0$. We assume that $0 < F(0) < 1$ to make the problem nontrivial.

From the filtering argument in Ekström and Vaicenavicius (2015), the problem admits the following equivalent optimal stopping formulation:

$$V^{nb}(t,x) = \sup_{\tau \geq t} \mathbb{E}_{(t,x)}\left[e^{-r(\tau-t)}\left(aX_\tau \vee b(1-X_\tau)\right) - \int_t^\tau e^{-r(s-t)} c ds\right], \qquad (V^{nb})$$

where

$$X_{t+s} = X_t + \int_t^s \sigma\left(u, X_u\right) dB_u$$

is the decision maker's belief that μ is nonnegative, and $\sigma(t,x) \in (0,1)$ is the volatility of the decision maker's beliefs. Moreover,

Proposition 19 *(i)* $\sigma \in \mathcal{C}^{2,\alpha}$. *(ii) For all* $x \in (0,1)$, $t \to \sigma(t,x)$ *is strictly decreasing if the support of* F *contains more than 2 elements.*

Proposition 19 follows immediately from the proof of Proposition 3.8 and Corollary 3.10 in Ekström and Vaicenavicius (2015).

Thus, as a consequence of Propositions 16 and 19 (which implies that σ is bounded above by a constant), we have

Corollary 6 *Suppose that* F *is not binary. The optimal stopping region* $\mathcal{S}$ *in* (V^s) *is given by*

$$\mathcal{S} := \left\{(t,x) \in \mathbb{R}_+ \times \mathcal{X} \,:\, x \notin \left(\underline{b}(t), \bar{b}(t)\right)\right\},$$

where $t \to \bar{b}(t)$ *is* ***strictly decreasing*** *and continuous and* $t \to \underline{b}(t)$ *is* ***strictly increasing*** *and continuous on* $[0,\infty)$.

Note that our result is not tied to one particular prior over μ (e.g., normal). Accuracy is decreasing in the decision time independently of the decision maker's prior F as long as it is not binary. In this sense, the predictions obtained in the classical Wald model (with binary prior) are not robust.

2.4.4 Learning with Stochastic Deadline

In this section, we consider information acquisition problems where the nature of the problem faced by the DM can change abruptly at a random deadline. For example, the DM may be forced to make a decision, may lose the opportunity to make the decision, or the true state may be revealed, when the deadline realizes. At any time before the deadline, the DM chooses whether to take action $d \in \{-1, 1\}$ or to acquire more information about an unknown parameter $\mu \in \{-1, 1\}$ that determines the action's payoff $\mathbb{1}_{\{\mu=d\}}$. If the decision maker chooses to acquire information (at a flow cost c), she observes the realization of a signal process and updates her belief. Formally, at time t, the DM's belief that $\mu = 1$ is

$$X_t = X_0 + \int_0^t \sigma(s, X_s) dB_s,$$

After the deadline realizes, learning stops.

We consider Markov deadlines that realize according to a Poisson time-dependent arrival rate $a(t, x)$ which can also depend on the DM's belief.[16] The DM's problem then consists of finding a decision time τ and a rule $d \in \{-1, 1\}$ to maximize the expected total gain given the presence of a stochastic deadline. The accuracy and urgency of a decision are determined by a gain function together with a stochastic deadline that arrives at a random time δ, and a payoff at the deadline $f(\delta, d, \mu)$. The expected total gain is

$$\mathbb{E}\left[\mathbb{1}_{\{\delta \geq \tau\}} \left(\mathbb{1}_{\{d=1,\mu=1\}} + \mathbb{1}_{\{d=-1,\mu=-1\}}\right) + \mathbb{1}_{\{\delta<\tau\}} f(\delta, d, \mu) - (\tau \wedge \delta)c\right].$$

As in the previous section, the optimal information acquisition problem admits the equiv-

[16] In the appendix B.5.4, we define the set of all Markov deadlines and show that it is well approximated by the class of Markov deadlines characterized by an arrival rate $a(t, x)$.

alent optimal stopping formulation:

$$V^\delta(t,x) \coloneqq \sup_{\tau\in\mathcal{T}} \mathbb{E}\left[X_\tau \vee (1-X_\tau)\chi_{\{\tau\leq\delta\}} + f(\delta,-1,X_\delta)\vee f(\delta,1,X_\delta)\chi_{\{\tau>\delta\}} - (\tau\wedge\delta)c\right], \quad (V^\delta)$$

subject to

$$X_t = X_0 + \int_0^t \sigma(s,X_s)dB_s,$$

where X_t is the decision maker belief that $\mu = 1$ at time t and, with a small abuse of notation $f(t,d,X) \coloneqq Xf(t,d,1) + (1-X)f(t,d,-1)$. We also assume that the volatility of the belief process is smooth (i.e., $\sigma \in \mathcal{C}^{2,\alpha}$).

This problem appears different from the stopping problems we considered above. However, in Appendix B.5.4, we show that we can reformulate it to fit our framework:

$$\begin{aligned} V^\delta(t,x) = \sup_{\tau\in\mathcal{T}(t)} \mathbb{E}_{(t,x)}\Bigg[& e^{-\int_0^\tau a(s,X_s)ds}\left(X_\tau \vee (1-X_\tau) - c\tau\right) \\ & + \int_0^\tau e^{-\int_0^t a(s,X_s)ds}\left(\bar{f}(t,X_t) - ct\right)a(t,X_t)dt\Bigg]. \end{aligned}$$

subject to

$$X_{t+s} = X_t + \int_t^{t+s} \sigma(u,X_u)dB_u.$$

Different f have different interpretations. First, we study the case where the true state is revealed at a stochastic deadline. Second, we consider the case where the DM is forced to make a decision at the stochastic deadline. Finally, we consider the case where the DM gets a zero payoff at the stochastic deadline. This last case is also equivalent to the problem in which the decision maker has to pay a constant flow cost c to acquire information and discount the future at rate $a(t, x)$.

Drift diffusion model with Eureka moment: Suppose that, at the deadline, the decision maker perfectly learns the true state μ. In this case, when the deadline realizes the DM stops immediately, makes the correct decision and obtains a payoff of 1. Then

Proposition 20 *Suppose that Assumption 13 holds. The optimal stopping region $\mathcal{S}$ of an optimal sampling problem is given by*

$$\mathcal{S} := \left\{(t, x) \in \mathbb{R}_+ \times \mathcal{X} \,:\, x \notin \left(\underline{b}(t), \bar{b}(t)\right)\right\},$$

where

- *$t \to \bar{b}(t)$ is **strictly decreasing** and continuous and $t \to \underline{b}(t)$ is **strictly increasing** and continuous on $[0, \infty)$ if $t \to a(t, x)$ is strictly decreasing and $t \to \sigma(t, x)$ is nonincreasing; and*
- *$t \to \bar{b}(t)$ is **strictly increasing** and $t \to \underline{b}(t)$ is **strictly decreasing** on $[0, \infty)$ if $t \to a(t, x)$ is strictly increasing and $t \to \sigma(t, x)$ is nondecreasing.*

The proof is in Appendix B.5.4.

Forced decision: Suppose that at the deadline, the decision maker has to make a decision, i.e., she cannot acquire additional information. This case corresponds to assuming $f(\delta, d, \mu) = \left(a\mathbb{1}_{\{d=1,\mu=1\}} + b\mathbb{1}_{\{d=-1,\mu=-1\}}\right)$. Thus, if the deadline realizes, the DM gets $X_\delta \vee (1 - X_\delta)$. In this case, we have

Proposition 21 *Suppose that Assumption 13 holds. The optimal stopping region $\mathcal{S}$ of an optimal sampling problem is given by*

$$\mathcal{S} := \left\{(t, x) \in \mathbb{R}_+ \times \mathcal{X} \,:\, x \notin \left(\underline{b}(t), \bar{b}(t)\right)\right\},$$

where

- $t \to \bar{b}(t)$ *is* ***strictly decreasing*** *and continuous and* $t \to \underline{b}(t)$ *is* ***strictly increasing*** *and continuous on* $[0, \infty)$ *if* $t \to a(t, x)$ *is strictly increasing and* $t \to \sigma(t, x)$ *is nonincreasing; and*

- $t \to \bar{b}(t)$ *is* ***strictly increasing*** *and* $t \to \underline{b}(t)$ *is* ***strictly decreasing*** *on* $[0, \infty)$ *if* $t \to a(t, x)$ *is strictly decreasing.*

The proof of Proposition 21 is the same as the proof of Proposition 20, hence omitted.

Time preferences and accuracy: Finally we study the case where the decision maker gets a zero at the deadline. This is equivalent to the problem in which the decision maker has to pay a constant flow cost c to acquire information and discount the future at rate $a(t, x)$, and therefore allows us to relate the decision's maker time preference to the accuracy of decisions.

Proposition 22 *Suppose that Assumption 13 holds. The optimal stopping region $\mathcal{S}$ of an optimal sampling problem is given by*

$$\mathcal{S} := \left\{(t, x) \in \mathbb{R}_+ \times \mathcal{X} \,:\, x \notin \left(\underline{b}(t), \bar{b}(t)\right)\right\},$$

where

- $t \to \bar{b}(t)$ *is* ***strictly decreasing*** *and continuous and* $t \to \underline{b}(t)$ *is* ***strictly increasing*** *and continuous on* $[0, \infty)$ *if* $t \to a(t, x)$ *is strictly increasing and* $t \to \sigma(t, x)$ *is nonincreasing; and*

- $t \to \bar{b}(t)$ *is* ***strictly increasing*** *and* $t \to \underline{b}(t)$ *is* ***strictly decreasing*** *on* $[0, \infty)$ *if* $t \to a(t, x)$ *is strictly decreasing and* $t \to \sigma(t, x)$ *is nondecreasing.*

Again, the proof of Proposition 22 is the same as the proof of Proposition 20, hence omitted.

Proposition 22 shows that the slope of the boundary is closely related to the slope of the discount rate. In particular, they coincides when the information acquisition technology is stationary. So the decrease of accuracy over time observed in many binary choice experiment in neuroscience is rationalized by an increasing discount rate. On the other hand, accuracy is increasing over time when the decision maker becomes more patient over time.

Chapter 3

A Best-Responses Approach to Robustness

3.1 Introduction

Real world mechanisms and contracts are often simple and hardly responsive to the details of the environment. However, most of the mechanism design literature identifies mechanisms that do not share these features: Optimal mechanisms usually respond dramatically to small changes, failing to rationalize standard organizational practices. Addressing this critique, the literature on robustness has successfully identified intuitive mechanisms that provide good performance guarantees and are largely independent of the environment. Yet, despite the simple structure of the robustly optimal mechanisms, the analysis largely lacks simplicity.

The difficulty often stems from the need to simultaneously determine mechanisms that achieve a good guarantee and the worst-case scenario. In this paper, we develop a new approach to identify classes of mechanisms that contain a robust optimum. Notably, our approach avoids the issues associated with explicitly solving for the worst-case scenario. While theoretically interesting, characterizing one specific robustly optimal mechanism does not need to be the ultimate goal. Instead, we are often interested in whether a robustly optimal mechanism possesses certain features. Is it linear, separable, or static? What uncertainty does it protect against? These are questions about *sets* of potentially-optimal mechanisms and *sets* of deviations by Nature. Such questions are the focus of our paper.

Our general framework considers a decision maker choosing an alternative from a set D, followed by an adversarial Nature choosing from a set N. The decision maker seeks to maximize her worst-case payoff $\pi(d, n)$. Our main results provide simple conditions that guarantee that an optimal worst-case mechanism belongs to some set $D^* \subseteq D$. We do so by considering a subset of Nature's responses $N^* \subseteq N$, such that D^* and N^* have a mutual-best-response property. Specifically, we require that when the decision maker chooses an element of D^*, Nature can minimize her payoff by choosing an element of N^*, i.e., Nature has a best-response in N^*. Conversely, the decision maker has a best-response in D^* when Nature plays an element of N^*. Notably, the conditions in Theorem 8 are tight, so this recipe can be applied to *any* robust decision problem.

The advantages of this approach are twofold. First, our proposed conditions are often easy to check and rely on familiar best-response reasoning or replication arguments. Second, identifying a class of mechanisms as containing an optimizer — rather than identifying the exactly optimal mechanism — is often the largest hurdle in the analysis. Once the analyst identifies a convenient D^* that contains a robustly optimal mechanism, it is often simple to identify an exact robustly optimal mechanism $d^* \in D^*$. Put differently, the tools we develop are specialized to their intended function: identifying particular *properties* of a robustly optimal mechanism. After presenting the main results, we show how to operationalize them across various examples and highlight instances where our results simplify the argument.

3.1.1 Related Literature

Our core methodological contribution lies within the fruitful literature on robust mechanism design, which identifies simple mechanisms providing payoff guarantees to the principal in unknown environments. This literature includes Frankel (2014), Garrett (2014), Carroll (2015), and Carroll and Meng (2016), as well as many more recent developments. Our starting point is different. Rather than identifying a particular optimal mechanism, we focus on sets of mechanisms. This approach presents one major advantage: When solving for a specific robustly optimal mechanism, one must find a strategy by Nature that constrains the principal to the lowest payoff guarantee. In other words, one must explicitly construct the saddle point of the game between Nature and the decision-maker. This is the approach followed in Carroll (2015), Carroll (2017), Libgober and Mu (2021), and Che and Zhong (2021), for example. By focusing on a class of mechanisms instead, we can often avoid solving for Nature's equilibrium strategy, simplifying the proofs.

Our approach is closer to the perspective put forward in Chassang (2013), Walton and Carroll (2022), or Deb and Roesler (2021), that provides conditions on the environment such that a robustly-optimal mechanism has desirable features. Contrary to these papers, which are still interested in directly characterizing the robustly optimal mechanism, we propose first determining the general properties the optimal mechanism should possess. As we demonstrate, identifying a class of good mechanisms instead of the optimal ones allows for bypassing the delicate constructive arguments these papers rely on by invoking different minimax theorems. Moreover, recent advances in developing such tools, including Brooks and Du (2021), further increase the scope of our methods.

Our approach also offers a critique of the robust design objective. Namely, optimal worst-case mechanisms are often determined by peculiar choices of Nature, which constrains the principal to some maximal payoff. Every mechanism that achieves this bound is worst-case optimal.

This begs two questions. First, by what criteria should we rank different robustly optimal mechanisms? Second — perhaps ironically — how sensitive are robustly optimal mechanisms to the model's description and the choices afforded to Nature? Some recent work has examined these questions. For example, see Dworczak and Pavan (2022) consider an information design problem in which multiple information structures achieve the payoff guarantee. An additional refinement that resembles admissibility selects the "best" among these worst-case optima.

Che and Zhong (2021) study a multi-dimensional screening similar setup to Deb and Roesler (2021). Rather than allowing Nature to choose any feasible information structure for the agent, they impose additional structure on the ambiguity set. As a result, they find that pure bundling need not be robustly optimal, and sub-bundling may emerge. Walton and Carroll (2022) proceed in a complementary direction. Rather than Nature choosing something that enters the agents' decisions (such as information or the available alternatives), Nature chooses among a set of possible functions that map a contract to the output distribution. They provide conditions directly on such functions that ensure that linear contracts are optimal without committing to any particular structure of the agent's underlying problem.

In addition to our methodological contribution, we use our methodology to analyze two new applications: robustness in a dynamic screening environment and robustness when contracting for search. In Section 3.4, we study the problem of a seller who is unsure about the evolution of a buyer's valuation but who knows the mean in each period. In this setting, we show that one robustly optimal dynamic mechanism consists of a sequence of static mechanisms. Each static mechanism coincides with the one identified by Carrasco et al. (2018). The argument is related to Baron and Besanko (1984): when the buyer's type evolves deterministically, an optimal dynamic mechanism is a memoryless sequence of optimal static mechanisms. However, if the agent accrues more information between interactions with the principal, static mechanisms need not be optimal. Our contribution is to show that when the principal faces ambiguity about the agent's type process, she protects herself by acting as if the evolution of the agent's type was deterministic, i.e., as if the information asymmetry between the seller and buyer were maximal.

The closest result is by Libgober and Mu (2021), but there are some differences in the setup. The primary difference is that their agent purchases the good at most once, whereas our agent interacts with the principal in every period. One could also view the multi-period interaction as similar to Carroll (2017), Deb and Roesler (2021), and Che and Zhong (2021) who consider a multi-dimensional type in a static setting. In their case and ours, the principal protects against the worst-case joint distribution of types. However, our setting introduces complexities in that the selling mechanism may be dynamic, the agent may be forward-looking, and both may face uncertainty about the future.

Finally, in Section 3.5, we show that debt contracts are robustly optimal when the agent can sequentially search à la Weitzman (1979) and the set of projects is unknown to the principal. Therefore, this application is related to a large literature that has tried to provide foundations for debt contracts. Most of these papers have focused on contracting frictions (see, e.g., Gale and Hellwig (1985), Povel and Raith (2004), Hébert (2018), and Chaigneau et al. (2022), and the references therein) or information acquisition (as, for example, Dang et al. (2013), Yang (2020), and Malenko and Tsoy (2020)). A notable exception is Antic (2014), which provides a robustness rationale for debt contracts when the agent's technology has the MLRP property but is unknown to the principal. In particular, his results build on and extend those of Innes (1990). Our approach suggests a different explanation: Debt contracts maximize potential experimentation and thus align the principal and agent's incentives when the agent has to search for the best option.

3.2 Main Result

Our framework is general and encompasses a large class of robust decision problems. A decision maker (she) has access to a set of alternatives D, with typical element denoted $d \in D$. She faces Knightian uncertainty about the possible actions available to an adversarial Nature. These actions belong to some set N. After the decision maker chooses $d \in D$, Nature chooses some $n \in N$. The resulting payoff of the decision maker is given by $\pi : D \times N \to \mathbb{R}$. The decision-maker wishes to maximize her worst-case payoff against Nature, so she seeks to solve

$$\sup_{d \in D} \inf_{n \in N} \pi(d, n) \tag{3.1}$$

At the outset, we place no assumptions on D, N, or π, so this framework captures various robust maximization problems. For instance, D could be a set of prices, wages, or mechanisms. N could denote the possible distributions of buyer values, dynamic information structures, or outside options. The payoff π could incorporate equilibrium selection, maximization by another agent, or regret relative to a full-information benchmark.

"Solving" such a problem involves identifying a set of alternatives $D^* \subseteq D$ within which an optimizer lives. That is, we wish to find D^* satisfying

$$\sup_{d \in D^*} \inf_{n \in N} \pi(d, n) = \sup_{d \in D} \inf_{n \in N} \pi(d, n).$$

Our main results provides conditions under which the above equality holds for a given D^*. We do so by considering a complementary set $N^* \subseteq N$ which include Nature's best-responses to elements of D^*.

Theorem 7 *Suppose there exist sets $D^* \subseteq D$ and $N^* \subseteq N$ such that*

(1) $\forall n^* \in N^*$, $\sup_{d \in D^*} \pi(d, n^*) = \sup_{d \in D} \pi(d, n^*)$

(2) $\inf_{n\in N^*}\sup_{d\in D^*}\ \pi(d,n) = \sup_{d\in D^*}\inf_{n\in N^*}\ \pi(d,n)$

(3) $\forall d^* \in D^*$, $\inf_{n\in N^*}\pi(d^*,n) = \inf_{n\in N}\pi(d^*,n)$

Then, $\sup_{d\in D^*}\ \inf_{n\in N}\pi(d,n) = \sup_{d\in D}\ \inf_{n\in N}\pi(d,n)$

Proof.

$$
\begin{aligned}
\sup_{d\in D}\inf_{n\in N}\ \pi(d,n) &\le \sup_{d\in D}\inf_{n\in N^*}\ \pi(d,n) && (N^*\subseteq N)\\
&\le \inf_{n\in N^*}\sup_{d\in D}\ \pi(d,n) && \text{(minimax inequality)}\\
&= \inf_{n\in N^*}\sup_{d\in D^*}\ \pi(d,n) && (1)\\
&= \sup_{d\in D^*}\inf_{n\in N^*}\ \pi(d,n) && (2)\\
&= \sup_{d\in D^*}\inf_{n\in N}\ \pi(d,n) && (3)\\
&\le \sup_{d\in D}\inf_{n\in N}\ \pi(d,n) && (D^*\subseteq D)
\end{aligned}
$$

■

Conditions (1) and (3) in Theorem 7 involve familiar "best response" logic on the part of the decision maker or Nature, taking the other as given. These conditions are often simple to check and follow by a replication argument. Condition (2) is a minimax equality that may not hold in all environments. However, a large class of problems involves an objective function π that is already bilinear. Alternatively, linearity may be inherited from the particular structure of D^* or N^*.

In cases where we do not expect the minimax equality (2) to hold, we can still proceed with a slightly weaker assumption, as described in the following result.

Theorem 8 *Fix $D^*\subseteq D$. There exists $N^*\subseteq N$ such that*

(1') $\sup_{d\in D}\inf_{n\in N^*}\pi(d,n) = \sup_{d\in D^*}\inf_{n\in N^*}\pi(d,n)$

(3) $\forall d^* \in D^*$, $\inf_{n\in N^*} \pi(d^*, n) = \inf_{n\in N} \pi(d^*, n)$

if and only if $\sup_{d\in D^*} \ \inf_{n\in N} \pi(d, n) = \sup_{d\in D} \ \inf_{n\in N} \pi(d, n)$

The proof parallels Theorem 7 and can be found in Appendix C.1.

Relative to Theorem 7, Theorem 8 replaces the joint requirements of (1) and (2) with a weaker requirement of (1'). (3) remains unchanged. In many cases, Nature's best response, when restricted to N^*, can be computed explicitly, simplifying the verification of (1').

Notably, the conditions (1') and (3) in Theorem 8 are tight. That is if we begin by assuming that for some $D^* \subseteq D$, $\sup_{d\in D^*} \ \inf_{n\in N} \pi(d, n) = \sup_{d\in D} \ \inf_{n\in N} \pi(d, n)$, then letting $N^* = N$ satisfies conditions (1') and (3). As a consequence, Theorem 8 pins down the essence of *every* result showing robust optimality: if robust optimality can be shown, Theorem 8 applies.

Finally, one caveat is that our results only imply ϵ-optimality of D^*. Because we did not impose any structure on the problem, a maximizer does not need to exist. However, in most practical applications, it is desirable to find an exactly-optimal mechanism and, hence, show that one exists. We conclude this section by closing the gap under some additional topological assumptions.

Corollary 7 *Suppose the conditions of Theorem 7 or 8 hold. Additionally suppose that D^* is compact and $\inf_{n\in N^*} \pi(\cdot, n)$ is upper semi-continuous. Then, there exists $d^* \in D^*$ such that*

$$\inf_{n\in N} \pi(d^*, n) = \sup_{d\in D} \inf_{n\in N} \pi(d, n)$$

In particular, if $\pi(\cdot, n)$ is upper semi-continuous for each $n \in N^*$, then $\inf_{n\in N^*} \pi(\cdot, n)$ is also upper semi-continuous and Corollary 7 applies. This assumption holds widely across applications. Finally, we provide an additional tool in Theorem 13 in Appendix C.7 to extend our results when D^* is not closed.

3.3 Applications

Having presented the main tools of our paper, we demonstrate how our approach applies to several well-known papers in the literature. We highlight instances where our focus on *sets* of mechanisms, in conjunction with Theorems 7 and 8, significantly simplifies the analysis.

3.3.1 Linear contracts are robustly optimal as in Carroll (2015)

In our first application, we show how to apply our results to obtain Carroll (2015)'s linearity result. Carroll's setup is as follows. A principal P contracts with an agent A, who takes a costly action $(c, F) \in \mathbb{R}_+ \times \Delta(\mathbb{R}_+)$ that produces some nonnegative stochastic output $y \in \mathbb{R}_+$. The action is not observable. Only the output is. Thus payment to the agent can depend only on the realization y and potential (cheap talk) messages exchanged between the agent and the principal before the agent takes action.

A technology $\mathcal{A}$ describes the agent's actions. A technology $\mathcal{A}$ is a compact subset of $\mathbb{R}_+ \times \Delta(\mathbb{R}_+)$. The agent knows $\mathcal{A}$, but the principal does not. Instead, the principal only knows a compact subset $\mathcal{A}_0$ of the available actions, and she believes that $\mathcal{A}$ can be any compact superset of $\mathcal{A}_0$. Moreover, Carroll imposes the following non-triviality assumption on $\mathcal{A}_0$: there exists $(c, F) \in \mathcal{A}_0$ such that $\mathbb{E}_F[y] - c > 0$. That is, the principal knows that she benefits from hiring the agent.

Next, we define the space of contracts the principal can offer. Carroll (2015) restricts attention to continuous deterministic payment functions $w : \mathbb{R}_+ \to \mathbb{R}_+$ (Theorem 1 in Carroll (2015)) and later shows that screening by offering menus of deterministic contracts cannot improve the principal's guarantee (Theorem 4 in Carroll (2015)). Our methodology allows us to consider a more general set of mechanisms: We allow the principal to offer any menu of randomized contracts. So, D is the set of all menus of lotteries over measurable functions $w : \mathbb{R}_+ \to \mathbb{R}_+$.

We can now summarize the timing of the game:

1. P offers a menus of randomized contracts.
2. A, knowing $\mathcal{A}$, decides which randomized contract G to accept.
3. A observes $g \sim G$ and chooses which action $(c, F) \in \mathcal{A}$ to take.
4. Output $y \sim F$ is realized.
5. Payoff's are received: $y - w_g(y)$ to P and $w_g(y) - c$ to A.

As in Carroll's, the agent's behavior is simple as he maximizes expected utility. Given technology $\mathcal{A}$, he picks the randomized contract G^* and action (c^*, F^*) that maximizes his payoff.[1]

From now on, our focus is on the principal's problem. P evaluates a menu by its worst-case expected payoff over technologies (and, when the agent is indifferent, equilibrium actions). So, P solves

$$\sup_{d \in D} \inf_{\mathcal{A} \supset \mathcal{A}_0} \pi\left(d, \mathcal{A}\right), \tag{3.2}$$

where

$$\pi(d, \mathcal{A}) := \inf_{(G_n^*, (c_n^*, F_n^*))_n} \lim_{n \to \infty} \mathbb{E}_{F_n^*}\left[y - \mathbb{E}_{G_n^*}\left[w_g(y)\right]\right].$$

subject to

$$\lim_{n \to \infty} \mathbb{E}_{F_n^*}\left[\mathbb{E}_{g \sim G_n^*}\left[w_g(y)\right] - c_n^*\right] = \sup_{G \in d,\, (c,F) \in \mathcal{A}} \mathbb{E}_F\left[\mathbb{E}_{g \sim G}\left[w_g(y)\right] - c\right].$$

[1] Because we did not assume that the offered menu was compact or that w is upper semi-continuous, there does not need to exist an agent optimal randomized contract and action. In that case, we let the agent chooses any maximizing sequence.

The principal's problem may appear daunting: the set of mechanisms is huge, Nature has many possible deviations, and even the payoff function π is complicated. Fortunately, Theorem 7 allows us to consider a simpler problem. Define

- $D^* := \{d \in D \,:\, d = \{w : \mathbb{R}_+ \to \mathbb{R},\, y \to \alpha y\},\, \alpha \geq 0\}$: the set of singleton menu that offers one linear contract, and
- $N^* := \left\{\mathcal{A} \,:\, \mathcal{A} = \mathcal{A}_0 \cup (0, \delta_{\{y\}}),\, y \in \left[0, \bigvee_{(c,F)\in\mathcal{A}_0} \left(\mathbb{E}_{\tilde{y}\sim F}\left[\tilde{y}\right] - c\right)\right]\right\}$: the set of technologies that includes the known sets $\mathcal{A}_0$ and a unique other options $(0, \delta_{\{y\}})$, where y is less than the maximum surplus of the known actions.

For $d \in D^*$ and $n \in N^*$, π rewrites

$$\pi\left(d, n\right) = (1-\alpha)y + i\left(\alpha y \geq \bigvee_{(c,F)\in\mathcal{A}_0} \left(\alpha\mathbb{E}_{\tilde{y}\sim F}\left[\tilde{y}\right] - c\right)\right).$$

Then, as a consequence of Theorem 7, we get

Proposition 23 *There exists a robustly optimal linear contract. Moreover, the principal does not benefit from offering menus over randomized contracts.*

The proof is in Appendix C.2.

3.3.2 Learning and robustness as in Libgober and Mu (2021)

For our second application, we provide a simple proof of the main result of Libgober and Mu (2021) using our techniques. As in the previous subsection, this again guarantees that the principal cannot benefit from using more complex mechanisms, thus extending the result in Libgober and Mu (2021) to the case where the principal can offer any dynamic mechanism.

We start by introducing their setup. They consider a standard durable good monopolist dynamic pricing problem and add the possibility of buyer learning. A seller sells a durable good at time $t = 1, 2, \ldots, T$, with $T \in \mathbb{N} \cup \{\infty\}$. A single buyer is present at time $t = 1$ and can choose whether and when to buy. The buyer and seller discount their payoff by a factor $\delta \in (0, 1)$. Finally, the product is costless for the seller to produce. The buyer has unit demand and obtains value $\delta^t v$ from purchasing the object at time t, where $v \in V \subset \mathbb{R}_+$ is drawn from a distribution $F \in \Delta(V)$ and fixed over time.

At time $t = 0$, the seller commits to a potentially random dynamic selling mechanism. A selling mechanism consists of a sequence of message spaces and associated mapping from messages to allocations that determines whether the buyer sells and the associated price: $(\mathcal{M}_t, \sigma : \mathcal{M}_t \to \{0, 1\} \times \mathbb{R}_+)_{t=1}^T$.[2] We denote by D the set of all (pure) dynamic selling mechanisms and by $\Delta(D)$ the set of all random dynamic selling mechanisms. Similarly, we denote by D^t the set of all (pure) dynamic selling mechanisms up to time t and by $\Delta(D^t)$ the set of all random dynamic selling mechanisms up to time t. Given a dynamic selling mechanism, the buyer then maximizes his expected payoff given the learning processes in the game generated by the mechanism.[3] In particular, they assume that the buyer does not directly know v, but instead learns through signals that arrive over time, via some information structure. Following them, we define a dynamic information structure as:

- A set of possible signals for every time $t \geq 1$, i.e., a sequence of sets $(S_t)_{t=1}^T$, and
- Probability distributions given by $I_t : V \times S^{t-1} \times D^t \to \Delta(S_t)$, for all t with $1 \leq t \leq T$.

To interpret the above distribution, note that the distribution over signals at time t could potentially depend on the buyer's valuation v, the history of signal realization up to t, and the realized mechanism up to time t. So, as in Libgober and Mu (2021), information flexibly depends on the realized mechanism. That is, we focus on the Stackelberg equilibrium

[2]Libgober and Mu (2021) restricts their analysis to dynamic random prices. Our approach allows for any dynamic selling mechanism. In particular, the principal can screen the private information the agent may have about the learning process.

[3]In particular, the buyer knows the information arrival process and is Bayesian about what information will be received in the future.

in the game played by the principal and Nature rather than the Nash equilibrium, which distinguishes our model from most papers in the robustness literature.

Together, the (random) dynamic selling mechanism $d \in \Delta(D)$ and the information process $\mathcal{I} = (I_t)_{t=1}^T$ determine a set of possible outcomes; i.e., times at which the buyer acquires the good with positive probabilities and associated prices. We are interested in the minimal seller payoff guarantee, i.e., the seller's payoff when evaluated as if the information process chosen by Nature and equilibrium selection were the worst possible, given the offered mechanism:

$$\sup_{d \in \Delta(D)} \inf_{\mathcal{I} \in N} \pi\left(d, I\right), \tag{3.3}$$

where

$$\pi\left(d, I\right) \coloneqq \inf_x \mathbb{E}\left[\sum_{t=1}^T x_t \delta^t p_t\right],$$

subject to the sequence x of payments maximizing the buyer's expected payoff in the game induced by d.

Finally, for simplicity, we make the following additional assumption to guarantee existence (which is absent from Libgober and Mu (2021)).

Assumption 15 *There exists $p > \inf\left(supp(F)\right)$ such that*

$$p\left(1 - \bar{G}(p)\right) \geq \inf\left(supp(F)\right),$$

where $\bar{G}$ is the "pressed" distribution associated to F, as defined in Libgober and Mu (2021), i.e., the distribution of $\mathbb{E}_F\left[v \mid v \leq r\right]$ when $r \sim F$.

Again, the problem faced by the principal appears complex. Fortunately, using the tools developed above, we can dramatically simplify it by considering pure "static" posted price mechanism. As a consequence of Theorem 7, we obtain:

Proposition 24 *There exists a robust optimal (random) dynamic selling mechanism that posts a constant price p^*. Moreover, the principal does not benefit from offering a more complex mechanism.*

The proof of Proposition 24 is in Appendix C.3.

3.3.3 Multidimensional screening as in Deb and Roesler (2021)

We highlight our approach in the multi-dimensional screening problem of Deb and Roesler (2021). We slightly simplify their setup to ease exposition.

A seller has K goods for sale, and the buyer's valuation is given by a type $\theta = (\theta_1, \ldots, \theta_K)$. We assume that the set of feasible types is $\Theta := [\theta_l, \theta_h]^K$, and θ is drawn from some exchangable distribution $F \in \Delta(\Theta)$. A type-θ buyer's valuation for a bundle $b \subseteq \{1, \ldots, K\}$ is assumed to be additive across items: $u(b, \theta) = \sum_{i \in b} \theta_i$.

The buyer does not directly observe her type. Rather, she observes a signal of her willingness-to-pay, which Deb and Roesler describe as "learning." Because the buyer's payoff is linear in θ, the only relevant statistic is the buyer's expected valuation for each component θ_i. Let N denote the set of distributions over posterior expected valuations that some signal structure induces. We will refer to the buyer's posterior expected valuation as her *interim* type.

The seller will design a sales mechanism to maximize his worst-case payoff against Nature's choice of N. A mechanism consists of a message space M, an allocation function q, and a transfer function t. The allocation describes a distribution over bundles that the buyer will receive, $q : M \to \Delta(2^K)$, and the transfer is the expected payment from the agent to the principal $t : M \to \mathbb{R}$. Finally, we impose an individual rationality constraint that, for all

θ, there exists some $m \in M$ that gives type θ a positive expected payoff. Let D denote the set of all such mechanisms.

Given a mechanism $d = (M, q, t)$, a reporting strategy is a function $\sigma : \Theta \to \Delta(M)$ that maps the buyer's interim type into chosen messages. Let $\Sigma(d)$ denote the set of best-response reporting strategies for the buyer. The seller's objective function is thus to solve

$$\max_{d \in D} \inf_{n \in N, \sigma \in \Sigma(d)} \mathbf{E}_{(n,\sigma)}[t(m)] \ , \tag{3.4}$$

where the expectation is taken over both Nature's draw of the interim type according to n and the buyer's reporting strategy σ. The goal is to find a class of mechanisms that maximizes this payoff.

With the setup concluded, we show how our framework easily surmounts this problem. While the setup does not immediately fit into our framework because of the additional constraint on the buyer's best-responses, it can be handled by appropriately defining the chosen mechanism.

We say that a mechanism is interim *direct* and *incentive compatible* if $M = \Theta$, and for all $\theta \in \Theta$,

$$\mathbf{E}_{b \sim q(\theta)} \sum_{i \in b} \theta_i - t(\theta) \geq \mathbf{E}_{b \sim q(\theta')} \sum_{i \in b} \theta_i - t(\theta') \tag{3.5}$$

The above is the standard notion of incentive compatibility applied to the buyer's expectation of her type after observing some signal. By the standard Revelation Principle argument, there is no loss in restricting attention to mechanisms of this form. The main advantage of looking at this class of mechanisms is that it directly incorporates the buyer's best-response into the seller's payoff. For a given interim type distribution $G \in \Delta(\Theta)$, the seller's expected payoff from a direct, incentive-compatible mechanism $d = (\Theta, q, t)$ is

$$\int_\Theta t(\theta) dG(\theta)$$

We will show the optimality of a restricted class of direct, incentive-compatible mechanisms. We say that a mechanism d is a *random pure bundling* mechanism if it is direct, incentive compatible, and additionally satisfies that t is continuous,

$$q(\theta)(b) > 0 \implies b = \{1, \ldots, K\} \quad \text{or} \quad b = \emptyset, \quad \text{and}$$
$$\sum_i \theta_i = \sum_i \theta_i' \implies q(\theta) = q(\theta')$$

A random pure bundling mechanism either sells the entire bundle or nothing to the buyer and is measurable with respect to the buyer's total valuation. Let D^* denote the set of random pure bundling mechanisms.

We now find the set of distributions by Nature, N^*, which rationalize D^* according to Theorem 7. We say that $G \in N$ is a *total value* signal if

$$\theta \in supp(G) \implies \theta_1 = \theta_2 = \ldots = \theta_K.$$

As the name suggests, a total-value signal can be generated by simply informing the buyer of her expected valuation for the grand bundle $\bar{\theta} = \sum \theta_i$. Due to the exchangeable prior assumption, after being informed of the total valuation, the buyer's posterior expected valuation for each good will be $\frac{\bar{\theta}}{K}$. Let N^* denote the set of total value signals.

Proposition 25 (Deb and Roesler (2021), Theorem 3) *Let D^* denote the set of random bundling mechanisms and N^* the set of total-value signals.*

1. *For any $G \in N^*$, $\sup_{d \in D, \sigma \in \Sigma(d)} \mathbf{E}_{(\sigma,G)}[t(m)] = \sup_{d \in D^*} \mathbf{E}_G[t(\theta)]$*

2. For any $d \in D^*$, $\inf_{G \in N} \mathbf{E}_G[t(\theta)] = \inf_{G \in N^*} \mathbf{E}_G[t(\theta)]$

3. $\sup_{d \in D^*} \inf_{G \in N^*} \mathbf{E}_G[t(\theta)] = \inf_{G \in N^*} \sup_{d \in D^*} \mathbf{E}_G[t(\theta)]$

As a result, there exists a worst-case optimal random pure bundling mechanism.

The complete proof is in Appendix C.4. A few technicalities aside, the argument is a straightforward application of Theorem 7. First, given any total-value signal of Nature, the seller's problem simplifies to a single-dimensional maximization for the grand bundle. In this case, we already know that a posted price for the grand bundle is optimal, which can be approximated by a continuous mechanism with arbitrary precision. Second, given any random pure-bundling mechanism, it is without loss to restrict attention to Nature informing the buyer of the valuation for the total good. Finally, the Sion minimax theorem conditions are satisfied due to the continuity of t.

3.4 Robust dynamic screening

In this section, we consider a new application and show how our results can be applied to simplify dynamic robust contracting problems. A principal can sell one unit of a good to an agent every period $1 \leq t \leq T$, with $T \in \mathbb{N} \cup \{\infty\}$. The principal, however, does not know the distribution of the agent's valuations or the process through which they evolve. Instead, she only knows the mean of the distribution at each time t. That is, we assume that the principal faces distributional uncertainty.

Formally, let $v_t \in V_t \subseteq [0, \bar{V}]$ with $\bar{V} \in \mathbb{R}_+$ denote the agent's type in period t. The seller is uncertain about the distribution $F \in \Delta\left(\prod_{t=1}^{T} V_t\right)$ of $(v_t)_{t\in\mathbb{N}}$, but knows the sequence of first moments: $(\bar{v}_t)_{t\in\mathbb{N}}$. So the set of distributions the seller entertains is

$$N := \left\{ F \in \Delta\left(\prod_{t=1}^{T} V_t\right) \; : \; \mathbb{E}_F\left[v_t\right] = \bar{v}_t, \;\; \text{for all } 1 \leq t \leq T \right\}.$$

We assume that the principal commits to a (potentially random) dynamic selling mechanism to maximize her worst-case payoff guarantee:

$$\sup_{d\in D}\inf_{n\in N}\pi\left(d,n\right),$$

where

$$\pi(d,n):=\inf_{(x_t)_{t=1}^T}\mathbb{E}_n\left[\sum_{t=1}^T\delta^t p_t x_t\right],$$

and $(x_t)_{t=1}^T$ is a sequence of buying probabilities that is optimal for the agent in the problem defined by mechanism d.

Proposition 26 *There exists a robust ϵ-optimal dynamic selling mechanism that charges the static robustly optimal (random) price each period.*

The proof of Proposition 26 is in Appendix C.5.

Corollary 8 *The sequence of random posted prices that induces the distribution P_t over prices in period t, $1\leq t\leq T$, with*

$$P_t(p):=\begin{cases}0 \;\; \textit{if } p\leq \underline{p}_t\\ \frac{\ln(p)-\ln\left(\underline{p}_t\right)}{\ln\left(\bar{V}\right)-\ln\left(\underline{p}_t\right)} \;\; \textit{if } \underline{p}_t\leq p\leq\bar{V},\end{cases}$$

and $\underline{p}_t$ is the solution to $\underline{p}\left(1+\ln\left(\bar{V}\right)-\ln\left(\underline{p}_t\right)\right)=\bar{v}_t$, is robustly optimal.

The proof of the above Corollary is in Appendix C.5.

3.5 Robust Contracting for Search

We conclude the present paper with a more extensive application to a novel robust contracting problem: how to incentivize an agent to search through alternatives. Robustness enters due to the principal's limited knowledge of the agent's available alternatives.

We show that debt contracts maximize a principal's worst-case payoff against all sets of alternatives that an agent may have access to. In particular, debt contracts neither distort the order in which an agent searches through alternatives nor provide an incentive to terminate the search early. Debt contracts are, therefore, a natural class of contracts when the moral hazard problem of the agent is not a one-time effort decision but a *dynamic* optimization problem that involves uncertainty. This provides a rationale for the use of debt contracts for financing in risky environments such as R&D and entrepreneurship.

3.5.1 Model

A principal ("the investor") contracts with an agent ("the entrepreneur"), who engages in costly sequential search to produce a profitable project. The agent's efforts and search are not observable, only the realized value of the project he presents to the principal.

A project is described by a pair $(c, F) \in \mathbb{R}^+ \times \Delta(Y)$, where $Y = [0, Y^{max}] \subseteq \mathbb{R}^+$. In the language of Weitzman (1979), this is a "Pandora's box." The interpretation is that the agent exerts a cost of c to learn the realized value of the project, which is a priori distributed according to F.

At the outset, the agent is aware of the set of feasible projects $\mathcal{A} = ((c_i, F_i))_{i=0}^n$, while the principal only knows one of one possible project $\mathcal{A}_0 := \{(c_0, F_0)\}$. In an extension, we describe this knowledge as stemming from the agent's strategic disclosure decision. For now, we leave this knowledge as exogenous. We assume $\mathcal{A}$ is finite, and $\mathcal{A}_0$ satisfies a non-triviality assumption that $\mathbf{E}_{F_0}[y] - c_0 > 0$.

The principal's only incentive tool is a wage contract $w : Y \to \mathbb{R}^+$. The contract describes the agent's monetary payment for a given project realization. When a project of value y is realized, the agent receives $w(y)$, and the principal receives $y - w(y)$. We assume that the contract w satisfies two-sided limited liability, so $w(y) \in [0, y]$. That is, the principal decides what share of the profits each player receives. Both players are risk neutral.

Given a wage contract w and a true set of available projects $\mathcal{A}$, the agent engages in Weitzman search (with recall). The agent's strategy will be described as a function of two state variables: the agent's best realization of y and the collection of unsampled projects $\bar{S}$. Thus, a strategy will be a function $\sigma : Y \times 2^{\mathcal{A}} \to 2^{\mathcal{A}} \cup \emptyset$. Where $\sigma(y, \bar{S}) = s$ means that the agent will sample project $s \in \mathcal{A} \setminus \bar{S}$ next, and $\sigma(y, \bar{S}) = \emptyset$ means the agent has chosen to stop searching and present y to the principal. Let $\Sigma(w, \mathcal{A})$ denote the set of optimal search strategies for the agent. For a given strategy σ, we write $\mathbf{E}_\sigma$ to denote the expectation with respect to the induced distribution over the agent's search. We abusively write $i \in \sigma$ to denote the event that project (c_i, F_i) is explored according to strategy σ.

We summarize the timing of the model as follows:

1. The principal observes $(c_0, F_0) \in \mathcal{A}$
2. The principal sets contract w
3. Knowing $\mathcal{A}$, the agent sequentially searches according to $\sigma \in \Sigma(w, \mathcal{A})$
4. Agent payoff of $\mathbf{E}_\sigma[w(y) - \sum_i c_i \mathbf{1}_{[i \in \sigma]}]$ and principal payoff of $\mathbf{E}_\sigma[y - w(y)]$

The principal's objective is to determine a wage contract w that maximizes the worst-case payoff against all supersets of the known project $\mathcal{A}_0$. Formally, given a realized set of projects $\mathcal{A}$ and wage contract w, the principal's payoff is

$$V_P(w \mid \mathcal{A}) = \sup_{\sigma \in \Sigma(w, \mathcal{A})} \mathbf{E}_\sigma[y - w(y)].$$

The principal evaluates wage contracts by their worst-case realization of $\mathcal{A}$, and therefore seeks to solve

$$V_P = \sup_w \inf_{\mathcal{A} \supseteq \mathcal{A}_0} V_P(w \mid \mathcal{A}).$$

Our focus is on identifying what wage contract solves the principal's problem. In the next section, we present the analysis and introduce the form of an optimal contract: the debt contract.

3.5.2 Analysis

We begin by recalling the solution to the sequential search problem in Weitzman (1979). For a given project (c_i, F_i) and wage contract w define the *reservation value* r_i^w (or *index*) as the unique solution to

$$c_i = \int [w(y) - r_i^w]^+ dF_i(y).$$

We write r_i with no subscript to denote the reservation value when the agent collects the entire profit $w(y) = y$. The agent's optimal strategy is to search projects in descending order of their reservation value. The agent concludes search whenever she has opened a prize y for which $w(y)$ is larger than the remaining reservation values. Notice that the agent assigns no value to realizations below the index r^w when she considers the order in which to search. This motivates the inspection of a special class of contracts.

A contract w is a $z-$*debt contract* if $w(y) = [y - z]^+$. In such a contract, the principal collects all returns up to the debt level z, after which the agent is the residual claimant and collects the rest. We are now ready to state the main result.

Theorem 9 *Let r_0 denote the index of (c_0, F_0). The r_0-debt contract maximizes V_p.*

The proof of Theorem 9 proceeds in two steps. Denote the r_0-debt contract by w_0. First, we show that for any $\mathcal{A} \supseteq \mathcal{A}_0$, $V_P(w_0 \mid \mathcal{A}) \geq V_P(w_0 \mid \mathcal{A}_0)$. Then, we show that when $\mathcal{A} = \mathcal{A}_0$, w_0 attains the maximal surplus. Together, we conclude that w_0 must be worst-case optimal.

We say that a contract w is *order-preserving* if for all pairs (c_1, F_1) and (c_2, F_2) and corresponding $r_1 \geq r_2$, if $r_2^w > 0$ then $r_1^w \geq r_2^w$. In words, if the agent preferred to search project 1 before 2 when she collected the entire profit, as long as project 2 is profitable under w, she still prefers to search 1 first.

Notice that a contract being order-preserving does not directly say anything about the values at which an agent will stop searching and is, therefore, not explicitly a statement about the principal's payoff. We consider this in a related definition. We say that a contract *w satisfies Independence of Irrelevant Alternatives (IIA)* if for all $\mathcal{A}_0 = (c_0, F_0)$ and all $\mathcal{A} \supseteq \mathcal{A}_0$, if $\mathbf{E}_{F_0}[w(y)] - c_0 \geq 0$, then $V_P(w \mid \mathcal{A}) \geq V_P(w \mid \mathcal{A}_0)$.

The following proposition uniquely ties these two properties to debt contracts.

Proposition 27 *The following are equivalent:*

1. *w is order-preserving*

2. *w satisfies IIA*

3. *w is a debt contract*

Proof.

We show that a debt contract must be order-preserving and satisfies IIA, and leave the converse to Appendix C.6. These properties boil down to a simple observation. Let w_z be a z-debt contract. Then, for any (c, F) with index r,

$$r^{w_z} = r - z. \tag{3.6}$$

Equation (3.6) follows immediately from the construction of the index and the definition of a debt contract. Because debt contracts have an additive effect on any project's index, they must be order-preserving.

We can also directly argue that any debt contract satisfies IIA. Consider any $z-$ debt contract w, and any $\mathcal{A}_0 = (c_0, F_0)$ such that $\mathbf{E}_{F_0}[w(y)] - c_0 > 0$. This implies $r_0^w > 0$, and the agent searches project (c_0, F_0) when restricted to $\mathcal{A}_0$. Now consider any superset $\mathcal{A} \supseteq \mathcal{A}_0$ and an optimal strategy $\sigma \in \Sigma(w, \mathcal{A})$ that presents the highest realization when stopping.

Fix any sequence of draws and optimal stopping decision of the agent. There are two cases. If project (c_0, F_0) was searched, then the distribution of the maximal draw must first order stochastically dominate the case in which the agent only has access to $\mathcal{A}_0$. Since $y - w(y)$ is weakly increasing, this results in a higher payoff for the principal.

Alternatively, if (c_0, F_0) was never searched, the agent opened another project (c_i, F_i) and realized some y_i such that $w(y_i) > r_0^w > 0$. This implies that the agent surpassed the debt level, as otherwise $w(y_i) = 0$. Therefore, $y_i - w(y_i) = z$. Since z is the principal's maximal payoff with a z-debt contract, this dominates the case when just $\mathcal{A}_0$ is available. ■

Proposition 27 clears the first hurdle for Theorem 9: when using the r_0-debt contract, the principal's worst-case is that $\mathcal{A} = \mathcal{A}_0$. Second, it is immediate to see that the r_0-debt contract w_0 leaves the agent with no expected surplus:

$$\mathbf{E}_{F_0}[w_0(y)] - c_0 = \mathbf{E}_{F_0}[(y - r_0)^+] - c_0 = 0. \tag{3.7}$$

Therefore, when restricted to $\mathcal{A}_0$, the principal attains the maximum value from the r_0-debt contract:

$$V_P(w_0 \mid \mathcal{A}_0) = \mathbf{E}_{F_0}[y] - c_0 = \max_w V_P(w \mid \mathcal{A}_0).$$

Together with Proposition 27, this proves Theorem 9 and identifies the principal's worst-case value is simply the total surplus of the known project:

$$V_P = \mathbf{E}_{F_0}[y] - c_0.$$

3.5.3 Suboptimality of Linear Contracts

To highlight the differences in our setup relative to Carroll (2015), it is useful to study how linear contracts perform in this dynamic environment. Again fix some $\mathcal{A}_0 = (c_0, F_0)$ and consider the wage contract $w^{lin}(y) = \alpha y$, which attempts to extract the whole surplus from the agent. This implies

$$\alpha = \frac{c_0}{\mathbf{E}_{F_0}[y]}.$$

When there are no other projects, performs as well as the r_0-debt contract by construction. However, this contract does not satisfy IIA and is susceptible to being "crowded out" by an alternative project which is better for the agent but worse for the principal. The index r^{lin} for the known project solves

$$c_0 = \int (w^{lin}(y) - r^{lin})^+ dF(y).$$

Plugging in the expression for α, we see that $r^{lin} = 0$. Now consider the alternative project $(0, \delta_x)$, where δ_x is a Dirac mass on some $x > 0$. This project has a strictly positive index *and* $\alpha x > 0$. Therefore, under w^{lin}, the agent will search $(0, \delta_x)$ and never proceed to (c_0, F_0). Since this is true regardless of x, we get:

$$\inf_{\mathcal{A} \supseteq \mathcal{A}_0} V_P(w^{lin} \mid \mathcal{A}) = 0.$$

In fact, we showed that there is *no* linear contract which achieves the principal's value V_P. Within the set of linear contracts, it is impossible to simultaneously extract the agent's full surplus when only $\mathcal{A}_0$ is realized and guarantee a positive payoff against any $\mathcal{A} \supseteq \mathcal{A}_0$.

This leads to a natural question: is the r_0-debt contract uniquely worst-case optimal? Unfortunately, not quite, as the following proposition shows. Recall that V_P is the principal's payoff guarantee under r_0.

Proposition 28 *A contract w satisfies $\inf_{\mathcal{A} \supseteq \mathcal{A}_0} V_P(w \mid \mathcal{A}) = V_P$ if and only if*

1. *$w(y) = 0$ for all $y \leq V_P$*

2. *$\mathbf{E}_{F_0}[w(y)] = c_0$*

Proposition 28 identifies the key properties for robust optimality, and the proof is identical to Theorem 9. In particular, w must impose a minimum debt level of V_P, but need not go as high as r_0. This guarantees that (c_0, F_0) cannot be crowded out by an alternative project at a loss to the principal. If the agent stops before searching (c_0, F_0), the principal must attain the maximal value of V_P.

This implies that the principal's worst case is that no other projects are available to the agent. Condition 2 says that, in this case, the principal takes the entire surplus.

We conjecture that the non-zero debt level generalizes readily to more intricate contracting environments. For example, the principal could know that $\mathcal{A}_0$ contains more than a single project or may be able to condition payments on multiple realizations that the agent provides. We find this to be an intriguing direction to pursue in future work.

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Appendix A

Appendix to Chapter One

A.1 Appendix A

A.1.1 Jump-Diffusion Processes

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, for all $i \in \{1, \dots, N\}$, let $B^i = \{B_t^i\}$ be Brownian motion adapted to $\mathcal{F}^i$ and $P^i = \{P_t^i\}$ be a homogeneous Poisson point process adapted to $\mathcal{F}^i$, such that P^i and B^i are independent, and the (B^i, P^i) and (B^j, P^j) are mutually independent for $i \neq j$. Let $\tilde{N}^i(ds, dz) = N^i(ds, dz) - m^i(dz)ds$ be the Poisson martingale measure generated by P^i, where $N^i(ds, dz)$ is the (homogeneous) Poisson counting measure generated by P^i and $m^i(dz)$ is the Lévy measure on $\mathbb{R} \setminus \{0\}$ generated by P^i.

The type of worker i, X^i, is a stochastic process with values in the open set $\mathcal{X}^i \subseteq \mathbb{R}$[1] and evolves according to the (potentially degenerate) jump-diffusion stochastic differential equation:

$$X_{t^i}^i = x^i + \int_0^{t^i} \mu^i(X_s^i)ds + \int_0^{t^i} \sigma^i(X_s^i)dB_s^i + \int_0^{t^i} \int_{\mathbb{R}_+} k^i(X_{s-}^i, z)\tilde{N}^i(ds, dz), \tag{A.1}$$

[1]For simplicity, assume that X^i either does not reach the boundary of the set or that they are absorbing.

where $x^i \in \mathcal{X}^i$, and t^i is the cumulative effort worker i has put into the project up to time t: $t^i = T^i(t) = \int_0^t \alpha_s^i a_s^i dt$. If $k^i = 0$, (A.1) is a continuous stochastic differential equation and the type of worker i is a diffusion process. The σ-field $\mathcal{F}_{t^i}^i$ contains all the information accumulated on worker i when he has put total effort t^i into the project.

I will make the following assumptions:

Assumption 16 *For all $i \in \{1, \ldots, N\}$, $\mu^i : \mathcal{X}^i \to \mathbb{R}$, $\sigma^i : \mathcal{X}^i \to \mathbb{R}$ is locally Lipschitz continuous and grows at most linearly.*

Assumption 17 *For all $i \in \{1, \ldots, N\}$,*

- $\int_{\mathbb{R}\setminus\{0\}} \frac{|z|^2}{1+|z|^2} dm^i(z) < \infty$;
- *There exists $\rho : \mathbb{R} \setminus \{0\} \to \mathbb{R}_+$ such that $\int_{\mathbb{R}\setminus\{0\}} \rho(z)^2 d\pi(z) < \infty$ and such that $|k^i(x,z) - k^i(y,z)| \leq \rho(z) |x - y|$; and*
- *For all $x, y \in Q$ and $z \in \mathbb{R} \setminus \{0\}$, $x + k^i(x,z) \geq y + k^i(y,z)$.*

The first condition in Assumption 17 is a restriction on the set of possible jump processes. It is standard in the theory of jump-diffusion and is satisfied by all stable processes. It is satisfied if m is a finite measure which admits a second-order moment, for example.

Assumptions 16 and 17 are sufficient for the existence of a strong solution of (A.1) and the validity of a comparison theorem.

Lemma 4 *Under Assumptions 16 and 17, the stochastic differential equation* (A.1) *has a unique strong solution.*

Proof of Lemma 4. This follows from Theorem 310 in Rong (2006). ■

An immediate consequence of Lemma 4 and Proposition 2.1 in Wang (2010) is that the process X^i is Feller.

Next, I show that it also satisfy the other Assumptions I made in Section 1.2.2, under some additional conditions. Assumption 3 holds by Theorem 295 in Rong (2006). Assumption 4 is satisfied if, for all $i \in \{1, \dots, N\}$, for all $x \in \mathcal{X}^i$, (i) either $k^i(x, z) \geq 0$; or (ii) $k^i(x, z) \leq 0$. Finally, Assumption 5 holds if $\sigma^i(x) > 0$ for all $x \in \mathcal{X}^i$ or if $\mu^i(x) > 0$ for all x such that $\sigma^i(x) = 0$. This can be seen from an application of Girsanov's theorem, Bluementhal's 0-1 law, and the Dvoretzky, Erdos, and Kakutani theorem (Theorem 9.13 in Karatzas and Shreve (1998)).

A.1.2 Proof of Lemma 1

I first show that in any implementable promotion contest, each worker's continuation value after any history is nonnegative. I prove the contrapositive. Let (T, τ, d) be a promotion contest and suppose that $U_t^i(T, \tau, d) < 0$ for some $i \in \{1, \dots, N\}$ and $t \geq 0$ after some history. I claim that the promotion contest (T, τ, d) is not implementable, i.e., there is no (weak) Perfect Bayesian equilibrium that generates it. To see this, note that sequential rationality is violated for worker i: were he to stop exerting effort forever (which is an admissible strategy by condition (ii)), his continuation payoff would be nonnegative. So (T, τ, d) is not implementable.

Next I show that any promotion contest that gives a nonnegative continuation value to each worker after any history is implementable. Let (T, τ, d) be such a promotion contest. Consider the strategies $\{a_t^i = 1\}_{t \geq 0}$, for all $i \in \{1, \dots, N\}$ and the principal's choosing the contest defined by $\left(\left\{\alpha_t^i = \frac{dT^{s,i}(t)}{dt}\right\}_{t \geq 0}, \tau, d\right)$ as long as $a_t^i = 1$ for all workers and taking her outside option immediately otherwise. The above strategy profile is admissible. By condition (i), it is feasible. Moreover, it is immediate to see that no worker has a profitable deviation: if worker i deviates, he gets a continuation payoff of 0, while, if he does not, he gets a continuation payoff of $U_t^i \geq 0$. The result then follows.

A.1.3 Proof of Proposition 2

I first show that the index contest can be implemented in a (weak) Perfect Bayesian equilibrium. Consider the following strategies $\left\{a_t^i = \mathbb{1}_{\{t \leq \tau^0 \wedge \bigwedge_{i=1}^N \tau^{s,i}\}} + d^i_{\tau^0 \wedge \bigwedge_{i=1}^N \tau^{s,i}} \mathbb{1}_{\{t > \tau^0 \wedge \bigwedge_{i=1}^N \tau^{s,i}\}}\right\}_{t \geq 0}$, for all $i \in \{1, \ldots, N\}$, and $\left(\left\{\alpha_t^i = \frac{dT^{s,i}(t)}{dt}\right\}_{t \geq 0}, \tau^0 \wedge \bigwedge_{i=1}^N \tau^{s,i}, d^s\right)$.

By Theorem 7.1 in El Karoui and Karatzas (1997), the principal has no profitable deviation. There remains to show that no worker has a profitable deviation. This follows from the structure of the index contest.

The delegation rule T^s is an index delegation rule, and, hence, each $T^{s,i}$ is flat of the set

$$\left\{t \geq 0 \,:\, \underline{\Gamma}^{s,i}_{T^i(t)} = \bigvee_{j=1}^N \underline{\Gamma}^{s,j}_{T^j(t)}\right\} \; \mathbb{P}\text{-a.s..}$$

But, for all $i = 1, \ldots, N$, $\Gamma^i_{T^{s,i}(t)}$ decreases only on the set $\left\{X^i_{T^{s,i}(t)} = \underline{M}^i_{T^{s,i}(t)}\right\}$ by lemma 10, and, therefore, using Assumption 5, for almost every t, if worker i is delegated the project, then $\Gamma^i_{T^{s,i}(t)} > \underline{\Gamma}^i_{T^{s,i}(t)}$. In this case, only worker i is delegated the project (i.e., at most one worker exerts effort at almost every instant $t \geq 0$.).

To see this, note that, by Proposition 10 in Kaspi and Mandelbaum (1998), the sets $\mathcal{D}^i = \left\{t \geq 0 \,:\, \sigma^i(\Gamma^i_{t^-}) > t\right\}$ are $\mathcal{F}^i$-totally inaccessible. So two arms pulled simultaneously cannot start an excursion by an upward jump of at least one of their indices (necessarily from the value of their common minimum $\underline{\Gamma}$). This can be seen this by contradiction. Suppose not, i.e., two workers i and j are delegated simultaneously at time t. This implies that both $T^i(t)$ belongs to $\mathcal{D}^i$ and $T^j(t)$ belongs to $\mathcal{D}^j$, and both $\underline{\Gamma}^i_{T^{s,i}(t)}$ and $\underline{\Gamma}^j_{T^{s,j}(t)}$ are strictly decreasing at t. So let $\underline{\Gamma}$ be a point from which $\Gamma^i_{T^{s,i}(t)}$ starts an excursion from its minimum. Then, if $\Gamma^j_{T^{s,j}(t)}$ jumps upward from its minimum $\underline{\Gamma}$, $\left\{\underline{\Gamma}^j_{t^-} = \underline{\Gamma}, \, \underline{\Gamma}^j_{t-\epsilon} > \underline{\Gamma} \text{ for all } \epsilon > 0\right\} \cap \mathcal{D}^j \neq \emptyset$. But $\left\{\underline{\Gamma}^j_{t^-} = \underline{\Gamma}, \, \underline{\Gamma}^j_{t-\epsilon} > \underline{\Gamma} \text{ for all } \epsilon > 0\right\}$ is a predictable set, and therefore $\left\{\underline{\Gamma}^j_{t^-} = \underline{\Gamma}, \, \underline{\Gamma}^j_{t-\epsilon} > \underline{\Gamma} \text{ for all } \epsilon > 0\right\} \cap \mathcal{D}^j = \emptyset$: a contradiction.

Therefore if the indices of two arms start an excursion from a level set $\underline{\Gamma}$ at the same time, it must be from a point of continuity for both $\Gamma^i_{T^{s,i}(t)}$ and $\Gamma^j_{T^{s,j}(t)}$. But, then, the priority rule defined in (A.8) specifies which of the two arms is pulled first and until the end of its excursion.

It follows that the continuation value of worker i during an excursion when his type is X and his minimum is $\underline{X}$ coincides with the continuation value in the single agent problem when his type is X and his minimum is $\underline{X}$, which, by theorem 2, is nonnegative.

Finally, for all $j \neq i$, $\Gamma^j_{T^{s,j}(t)} = \underline{\Gamma}^j_{T^{s,j}(t)}$. But, on $\Gamma^j_{T^{s,j}(t)} = \underline{\Gamma}^j_{T^{s,j}(t)}$, $\left\{X^i_{T^{s,j}(t)} = \underline{X}^j_{T^{s,j}(t)}\right\}$. So the continuation value of worker j is zero by Theorem 2, and no unemployed worker wants to quit.

Thus no worker has a profitable deviation, as it is easily seen that the cost from delaying the moment they get the reward by shirking is always greater than the saved effort cost when the continuation value of a worker is nonnegative as, for all $s \geq 0$,

$$e^{-rs}U^i_t \leq U^i_t,$$

and the continuation value U^i_t only depends on the current state of the game which remains fixed when the worker shirks.

The second part of Proposition 2 follows from the Martingale argument in El Karoui and Karatzas (1994). It is reproduced in Lemma 28 in Appendix A.2.1 for completeness.

A.1.4 Proof of Proposition 1

Let $c^i(\cdot) := 0$ for all $i \in \{1, \ldots, N\}$. Then $\bar{P}^i(x) = \sup\{x \in \mathcal{X}^i\}$ for all $i \in \{1, \ldots, N\}$, and the result follows from Theorem 1.

A.1.5 Proof of Proposition 9

I show that any implementable promotion contest (T, τ, d) is feasible for the relaxed program. Let $(T, \tau, d) \in \mathcal{P}^i$ be an implementable promotion contest. Let $i \in \{1, \ldots, N\}$ and $t \geq 0$. By the law of iterated expectations, one has

$$\begin{aligned}
&\mathbb{E}\left[e^{-r(\tau-\tau\wedge t)} g \mathbb{1}_{\{d=i\}} - \int_{\tau\wedge t}^{\tau} e^{-r(s-\tau\wedge t)} c dT^i(s) \mid \mathcal{F}^i_{T^i(t)}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[e^{-r(\tau-\tau\wedge t)} g \mathbb{1}_{\{d=i\}} - \int_{\tau\wedge t}^{\tau} e^{-r(s-\tau\wedge t)} c dT^i(s) \mid \mathcal{G}^T_t\right] \mid \mathcal{F}^i_{T^i(t)}\right] \\
&= \mathbb{E}\left[U^i_t \mid \mathcal{F}^i_{T^i(t)}\right] \\
&\geq 0;
\end{aligned}$$

where the second equality is by definition and the inequality follows from lemma 1. Therefore any promotion contest $(T, \tau, d) \in \mathcal{P}^I$ is feasible in (RP).

By Lemma 5 below, for any promotion contest, we can find a payoff equivalent promotion contest with $d \in \mathcal{C}^*$. So for any d is feasible in the original problem, there is a d feasible in the new problem that gives the same payoff to the principal. Finally, by Lemma 6 below, the set of stopping times $\mathcal{T}$ is a subset of $\mathcal{S}\left(\mathcal{G}^T\right)$. Therefore the set of randomized promotion contest $\mathcal{P}^r$ is a superset of $\mathcal{P}$, and, hence, and the result follows: $\Pi \geq \Pi^M$.

Lemma 5 *For all promotion contest (T, τ, d), there exists an alternative promotion contest $(T, \tau, \tilde{d})$ with $\tilde{d}^i_t$ monotone $\mathbb{P}$-a.s. for all $i \in \{0, 1, \ldots, N\}$ such that (T, τ, d) and $(T, \tau, \tilde{d})$ give the same payoff to the principal and to all the workers.*

Proof of Lemma 5. Let (T, τ, d) be a promotion contest. Define $\tilde{d}$ as follows: for all $t \geq 0$,

$$\forall i \in \{1, \ldots, N\}, \tilde{d}^i_t = d^i_\tau \mathbb{1}_{\{t\geq\tau\}} \text{ and } \tilde{d}^0_t = d^0_\tau \mathbb{1}_{\{t\geq\tau\}} + \mathbb{1}_{\{t<\tau\}}.$$

Clearly, $\tilde{d}^i$ is $\mathbb{P}$-a.s. monotone. Furthermore, (T, τ, d) and $(T, \tau, \tilde{d})$ gives the same payoff to all the players since $d_\tau = \tilde{d}_\tau$. Finally $\tilde{d}$ is $\mathcal{G}^T$-adapted, and therefore $(T, \tau, \tilde{d})$ is a promotion contest. ■

Lemma 6 *The set of $\mathcal{G}$-stopping times $\mathcal{T}(\mathcal{G})$ can be identified with the set of extreme points of $\mathcal{S}(\mathcal{G})$.*

Proof of Lemma 6. Any $\mathcal{G}$-stopping time τ can be identified with the random stopping time S^τ defined as $S^\tau(t) := \mathbb{1}_{\{t \geq \tau\}}$, which is easily seen to be an extreme point of $\mathcal{S}(\mathcal{G})$. On the other hand, if $S(\cdot) \neq \mathbb{1}_{\{\cdot \geq \tau\}}$ for some $\mathcal{G}$-stopping time, there exists $\bar{s} \in (0, 1)$ such that the processes S^1 and S^2 defined by

$$S^1_t := \frac{1}{\bar{s}} S(t) \wedge \bar{s} \text{ and } S^2_t := \frac{1}{1 - \bar{s}} S(t) \vee (1 - \bar{s})$$

are different elements of $\mathcal{S}(\mathcal{G})$. But then, S is not an extreme point of $\mathcal{S}(\mathcal{G})$. ■

A.1.6 Omitted Proofs for Section 1.5.2

Proof of Theorem 2

Step 1: Define $\mathcal{Q}^i := \left\{x^i_q,\, 0 \leq q \leq Q + 1\right\} \subseteq \mathcal{X}^i$, a grid of points in the state space such that $x^i_0 := X^i_0$ and $x^i_{q+1} < x^i_q$ for all $i = 1, \ldots, N$ and $q = 0, \ldots Q$.

I solve the stopping problem with constraints:

$$\Pi^i_{\mathcal{Q}} := \sup_{(\tau, d^i)} \mathbb{E}\left[\int_0^\tau e^{-rt} \pi^i\left(X^i_t\right) dt + e^{-r\tau}\left(d^i_\tau \int_\tau^\infty e^{-r(t-\tau)} \pi^i\left(X^i_t\right) dt + (1 - d^i_\tau) W\right)\right] \quad (\mathrm{RRP}^i)$$

subject to, for all $q = 0, \ldots, Q$,

$$\mathbb{E}\left[e^{-r\tau} g^i d^i_\tau + \int_{\tau \wedge \tau^i(x^i_q)}^\tau e^{-rt} c^i\left(X^i_t\right) dt\right] \geq 0, \qquad (\mathrm{RDP}(x^i_q))$$

where $\tau^i(x_q^i) = \inf\left\{t \geq 0 \,:\, X_t^i \leq x_q^i\right\}$.

Lemma 7 *The value of relaxed problem* (RRP^i) *is weakly greater than the value of problem* (RP^i)*:* $\Pi_{\mathcal{Q}}^i \geq \Pi^i$.

Proof of lemma 7. Any feasible (τ, d) in (RP^i) satisfies all the (RDP(x_q^i)) constraints by Lemma 21 in Appendix A.1.7. So the choice set in (RRP^i) is weakly larger than the choice set in RP^i, and $\Pi_{\mathcal{Q}}^i \geq \Pi^i$. ∎

Step 2: To accommodate the constraints, set up the Lagrangian associated with (RRP^i).

Lemma 8 *There exists* $(\lambda_0, \ldots, \lambda_Q) \in \mathbb{R}_+^{Q+1}$ *with* ($\lambda_q > 0$ *if and only if* (RDP(x_q^i)) *is binding for* $\tau^i(x_q^i)$) *such that problem* (RRP^i) *is equivalent to the unconstrained pure stopping problem:*

$$\begin{aligned}\Pi_{\mathcal{Q}}^i = \sup_{(\tau, d^i)} \mathbb{E}\Bigg[\int_0^\tau e^{-rt}\left(\pi^i(X_t^i) - \sum_{q=0}^{Q} \lambda_q c^i\left(X_t^i\right) \mathbb{1}_{\{t \geq \tau(x_q^i)\}}\right) dt \\ + e^{-r\tau} d_\tau^i \left(\sum_{q=0}^{Q} \lambda_q g^i + \int_\tau^\infty e^{-r(t-\tau)} \pi^i(X_t^i) dt\right) + e^{-r\tau}(1 - d_\tau^i) W\Bigg]\end{aligned} \tag{A.2}$$

Lemma 8 follows from Theorem 1 in Balzer and Janßen (2002).

Proof of lemma 8. For $(\tau := \epsilon, d^i = 1)$, $\epsilon > 0$ small, all constraints are slack. So Theorem 1 in Balzer and Janßen (2002) applies: there exist Lagrange multipliers $(\lambda_q)_{q=0}^{Q} \in \mathbb{R}_+^{Q+1}$ such that the optimal promotion time and decision rule, (τ, d^i), solve

$$\begin{aligned}\Pi_{\mathcal{Q}}^i = \sup_{(\tau, d^i)} \mathbb{E}\left[\int_0^\tau e^{-rt}\pi^i(X_t^i)dt + e^{-r\tau}\left(d_\tau^i \int_\tau^\infty e^{-r(t-\tau)}\pi^i(X_t^i)dt + (1 - d_\tau^i)W\right)\right] \\ + \sum_{q=0}^{Q} \lambda_q \mathbb{E}\left[e^{-r\tau} g^i d_\tau^i - \int_{\tau \wedge \tau^i(x_q^i)}^{\tau} e^{-rt} c^i\left(X_t^i\right) dt\right]\end{aligned}$$

Rearranging yields (A.2). ∎

Step 3: Identify a promotion contest that gives the principal a payoff weakly higher than the value of (RRP^i).

Let $\tau_q^i := \inf\left\{t \geq 0 \,:\, X_t^i \notin \left(x_{q+1}^i, x_q^i\right)\right\}$. Define $\tilde{g}^i := g^i$ if X^i only jumps up and

$$\bar{g}^i := \inf\left\{\tilde{g} \geq 0 \,:\, \inf_{q\in\{0,\ldots,Q-1\}} \inf_{x\in\left(x_{q+1}^i, x_q^i\right)} \mathbb{E}\left[e^{-r\tau_q^i}\tilde{g} - \int_0^{\tau_q^i} e^{-rt}c^i\left(X_t^i\right)dt \mid x\right] \geq g^i\right\}. \quad \text{(A.3)}$$

if X^i only jumps down. Finally, let $\tilde{P}^i(\underline{x})$ be defined as $\bar{P}^i(\underline{x})$ but for g^i replaced by $\tilde{g}^i$. Then

Proposition 29 *Let*

$$\tilde{P}_{\mathcal{Q}}^i\left(\underline{x}\right) := \sum_{q=0}^{Q} \tilde{P}^i(x_q^i)\mathbb{1}_{\{\underline{x}\in\left(x_{q+1}^i, x_q^i\right]\}}.$$

The pair $\left(\bar{\tau}_{\mathcal{Q}}, d^{\mathcal{Q}}\right) \in \mathcal{P}^i$ *with*

$$\tau_{\mathcal{Q}} := \inf\left\{t \geq 0 \,:\, X_t^i \notin \left[\underline{p}_{\mathcal{Q}}^i, \tilde{P}_{\mathcal{Q}}^i\left(\underline{X}_t^i\right)\right)\right\} \text{ and } d_{\tau_{\mathcal{Q}}}^{i,\mathcal{Q}} := \mathbb{1}_{\left\{X_{\tau_{\mathcal{Q}}}^i \geq \tilde{P}_{\mathcal{Q}}^i\left(\underline{X}_{\tau_{\mathcal{Q}}}^i\right)\right\}}$$

gives a weakly greater payoff to the principal than any feasible promotion contest in (RRP^i).

The intuition for the above proposition is clear. The principal always wants to wait and obtains as much information as possible before making a final and irreversible decision. Her option value of waiting is always positive. On the other hand, her cost of waiting is null since, conditionally on choosing to promote the worker, her continuation value is a martingale. The promotion contest $\left(\tau_{\mathcal{Q}}, d^{\mathcal{Q}}\right)$ guarantees that the principal waits as long as possible before making a decision.

Proof of Proposition 29. Distinguish two cases corresponding to the two cases of Assumption 4.

- X^i **only jumps up.** Let (τ^*, d^*) be the optimal promotion contest in (RRP^i) given by Lemma 13 in Appendix A.1.6. The payoff the principal gets from $\left(\tau_{\mathcal{Q}}, d^{\mathcal{Q}}\right)$ is weakly greater than the value of (DRRP^i) since $\tau_{\mathcal{Q}} \geq \tau^*$ $\mathbb{P}$-a.s. and the principal takes her outside option when X^i falls below the same level $\underline{p}^i_{\mathcal{Q}}$ in both cases. This concludes the proof.

- X^i **only jumps down.** Let $\bar{\tau}_{\mathcal{Q}}$ be the optimal stopping time in (DRRP^i) given by Lemma 16 in Appendix A.1.6. By Lemma 15 in Appendix A.1.6, the value associated with $\bar{\tau}_{\mathcal{Q}}$ is weakly greater than the value of (RRP^i). But the payoff the principal gets from $\left(\tau_{\mathcal{Q}}, d^{\mathcal{Q}}\right)$ is weakly greater than the value of (DRRP^i) since $\tau_{\mathcal{Q}} \geq \bar{\tau}_{\mathcal{Q}}$ $\mathbb{P}$-a.s. and the principal takes her outside option when X^i falls below the same level $\underline{p}^i_{\mathcal{Q}}$ in both cases. This concludes the proof.

■

Step 4: Finally Theorem 2 is obtained by letting the grid $\mathcal{Q}$ become finer and finer.

Proof of Theorem 2. Let $(\mathcal{Q}^n)_{n\in\mathcal{N}} \subseteq 2^{\mathcal{X}^i}$ be a sequence of grids in $\mathcal{X}^i$ such that $\mathcal{Q}^n \subseteq \mathcal{Q}^{n+1}$ for all $n \in \mathbb{N}$ and such that $\lim\limits_{n\to\infty} \mathcal{X}^n$ is dense in $\mathcal{X}^i$. Let $(\tau^n, d^n)_{n\in\mathbb{N}}$ be the pair given by Proposition 29. Define

$$\tau^* := \inf\left\{t \geq 0 \,:\, X_t \notin \left[\underline{p}^*, \bar{P}^i(\underline{X}_{\tau^*})\right)\right\},$$

$$\text{and } d^*_{\tau^*} := \mathbb{1}_{\{X_{\tau^*} = P(\underline{X}_{\tau^*})\}};$$

where $\underline{p}^*$ is an accumulation point of $\left(\underline{p}^i_{\mathcal{Q}^n}\right)_{n\in\mathbb{N}}$. Along a subsequence, $(\tau^n, d^n)_{n\in\mathbb{N}}$ converges to (τ^*, d^*) $\mathbb{P}$-a.s. (as $\tilde{g}^i_{\mathcal{Q}^n} \to g^i$). Since, for all $n \in \mathbb{N}$, $\Pi^i_{\mathcal{Q}^n} \geq \Pi^i$, it follows that (τ^*, d^*) yields a value greater than Π^i to the principal.

But, (τ^*, d^*) is feasible in (RP^i); i.e., (τ^*, d^*) satisfies the dynamic participation constraint (DPC^i). This follows from Lemma 18. Therefore (τ^*, d^*) is optimal in (RP^i).

There remains to show that $\underline{p}^* := \underline{p}^i$, which follows from noting that otherwise $(\tilde{\tau}^*, \tilde{d}^*)$

with

$$\tilde{\tau}^* := \inf\left\{t \geq 0 \,:\, X_t \notin \left[\underline{p}^i, \bar{P}^i(\underline{X}_{\tau^*})\right)\right\},$$
$$\text{and } \tilde{d}^*_{\tau^*} := \mathbb{1}_{\{X_{\tau^*} = P(\underline{X}_{\tau^*})\}}$$

yields a greater payoff to the principal. ■

Supporting Lemmas for the proof of Theorem 2

Supporting Lemmas for Step 3: First characterize the solution $\left(\tau_{\mathcal{Q}}, d^{\mathcal{Q}}\right)$ of (RP^i). Let

$$x^i_W := \sup\left\{x \in \mathcal{X}^i \,:\, \mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i\left(X^i_t\right)dt \mid x\right] + \sum_{q=0}^{Q}\lambda_q g^i \leq W\right\}.$$

Lemma 9 *There exists* $\left(\tau_{\mathcal{Q}}, d^{\mathcal{Q}}\right)$ *that solves* (A.2) *with (i)* $\tau_{\mathcal{Q}} := \inf\{t \geq 0 \,:\, X^i_t \notin S(t)\}$, *where* $S(t)$ *is a correspondence constant on* $[\tau^i(x^i_q), \tau^i(x^i_{q+1}))$ *for all* $q \in \{0, \dots, Q\}$, *such that* $S(t) \cap \{x \in \mathcal{X}^i \,:\, x < x^i_W\} = \left(-\infty, \underline{p}^i_{\mathcal{Q}}\right)$ *for some threshold* $\underline{p}^i_{\mathcal{Q}} \in \mathcal{X}^i$ *and* $S(\tau^i(x^i_q)) = S(\tau^i(x^i_{q+1}))$ *if* (RDP(x^i_q)) *is not binding at* $\tau^i(x^i_{q+1})$, *and (ii)* $d^{i,\mathcal{Q}}_{\tau_{\mathcal{Q}}} = \mathbb{1}_{\{X^i_{\tau_{\mathcal{Q}}} \geq x^i_W\}}$.

Moreover $\mathbb{P}\left(d^i_\tau = 0\right) > 0$.

Proof of Lemma 9. Let $\bar{A} = (A_0, \dots, A_Q)$ with $A = \mathbb{1}_{\{\underline{X}^i_t \in \left(x^i_{q+1}, x^i_q\right]\}}$. The process $(X^i_t, \bar{A})$ on the extended state space $\left\{(x, t_0, \dots, t_Q) \in \mathcal{X}^i \times \{0,1\}^{Q+1} \,:\, x \geq \underline{x}\right\}$ inherits the Feller property from X^i under $\mathbb{P}$. The result then follows from Theorem 11 in Appendix A.2.2. In particular, for all $q \in \{0, \ldots, Q\}$, the value function with $A^q = 1$ is given by

$$\begin{aligned} V_q(x) \coloneqq \sup_{\tau, d^i} \mathbb{E}\Bigg[&\int_0^{\tau \wedge \tau^i(x^i_{q+1})} e^{-rt} \left(\pi^i(X^i_t) - \sum_{k=0}^{q} \lambda_k c^i \left(X^i_t\right) \right) dt \\ &+ e^{-r\tau \wedge \tau^i(x^i_{q+1})} \Bigg\{ V_{q\left(\underline{X}^i_{\tau^i\left(x^i_{q+1}\right)}\right)} \left(X^i_{\tau^i(x^i_{q+1})} \right) \mathbb{1}_{\{\tau \geq \tau^i(x^i_{q+1})\}} \\ &+ \mathbb{1}_{\{\tau < \tau^i(x^i_{q+1})\}} \left[d^i_\tau \left(\sum_{q=0}^{Q} \lambda_q g^i + \int_\tau^\infty e^{-r(t-\tau)} \pi^i(X^i_t) dt \right) + (1 - d^i_\tau) W \right] \Bigg\} \mid x \Bigg]. \end{aligned} \tag{A.4}$$

It is clear that $d^i_\tau = 1$ if and only if $\mathbb{E}\left[\sum_{q=0}^{Q} \lambda_q g^i + \int_\tau^\infty e^{-r(t-\tau)} \pi^i(X^i_t) dt \mid \mathcal{F}^i_\tau\right] \geq W$, and therefore that d^i only depends on X^i_τ, as X^i is Feller, and, hence, has the strong Markov property. So

$$\begin{aligned} V_q(x) = \sup_{\tau} \mathbb{E}\Bigg[&\int_0^{\tau \wedge \tau^i(x^i_{q+1})} e^{-rt} \left(\pi^i(X^i_t) - \sum_{k=0}^{q} \lambda_k c^i \left(X^i_t\right) \right) dt \\ &+ e^{-r\tau \wedge \tau^i(x^i_{q+1})} \Bigg\{ V_{q\left(\underline{X}^i_{\tau^i\left(x^i_{q+1}\right)}\right)} \left(X^i_{\tau^i(x^i_{q+1})} \right) \mathbb{1}_{\{\tau \geq \tau^i(x^i_{q+1})\}} \\ &+ \mathbb{1}_{\{\tau < \tau^i(x^i_{q+1})\}} \left(\sum_{q=0}^{Q} \lambda_q g^i + \int_\tau^\infty e^{-r(t-\tau)} \pi^i(X^i_t) dt \right) \vee W \Bigg\} \mid x \Bigg]. \end{aligned}$$

From Theorem 11 in Appendix A.2.2 again, the smallest optimal stopping time is given by

$$\tau^{\mathcal{Q}} \coloneqq \inf \left\{ (x, q) \,:\, V_q(x) = \sum_{q=0}^{Q} \lambda_q g^i + \mathbb{E}\left[\int_0^\infty e^{-rt} \pi^i \left(X^i_t\right) dt \mid x \right] \right\}.$$

There remains to show that the stopping region S_q on $\left[\tau^i(x^i_q), \tau^i(x^i_{q+1})\right)$ can be taken to be such that

$$
\begin{aligned}
S_q \cap \left\{x \in \mathcal{X}^i \,:\, x < x^i_W\right\} = \Bigg\{ & x \in \mathcal{X}^i \,:\, x \geq x^i_{q+1} \text{ and} \\
& \qquad V_q(x) = \sum_{q=0}^{Q} \lambda_q g^i + \mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i\left(X^i_t\right) dt \mid x\right]\Bigg\} \\
= & \left[\underline{p}^i_q, \bar{P}^i_q\right).
\end{aligned}
$$

By Lemma 10, $V_q(x)$ is nondecreasing in x. It follows that, if $x' \leq x^i_W$ is such that $x' \in S_q$, then

$$
\left\{x \in \mathcal{X}^i \,:\, x \leq x'\right\} \subseteq S_q
$$

To see this, note that by Lemma 10, for all $x'' \leq x'$, $W = V_q(x') \leq V_q(x'') \geq W$, and, hence, $V_q(x'') = W$. Therefore, it is optimal for the principal to stop at x'' and take her outside option, and $S_k \cap \{x \in \mathcal{X}^i \,:\, x \leq x^i_W\} = (-\infty, \underline{p}^i_k)$ or $(-\infty, \underline{p}^i_q]$ with $\underline{p}^i_q \coloneqq \sup\{x \in S_q \,:\, x \leq x^i_W\}$.

Furthermore, I claim that $\underline{p}^i_q \leq \underline{p}^i_{q+k}$ for all $k \in \{1, \ldots, Q-q\}$. This follows from the definition of V^q since $c^i \geq 0$ and $\lambda_j \geq 0$ for all $j \geq 0$. Letting $\underline{p} \coloneqq \inf\left\{\underline{p}^i_q \,:\, \underline{p}^i_q > -\infty\right\}$ if there exists $\underline{p}^i_q > -\infty$, one sees that the principal stops and takes her outside option (if she ever does) the first time X^i_t enters $\left(-\infty, \underline{p}^i_Q\right)$ or $\left(-\infty, \underline{p}^i_Q\right]$.

Note here that $V_q(x)$ is right-continuous and increasing, hence upper semicontinuous. Therefore, on $(-\infty, x^i_W)$, the stopping region is open, and hence $S_q \cap (-\infty, x^i_W) = \left(-\infty, \underline{p}^i_Q\right)$.

There remains to show that $\mathbb{P}\left(d^i_\tau = 0\right) > 0$. The proof is by contradiction. So suppose not. Then the value of the principal is given by $\mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i\left(X^i_t\right) dt\right]$, which contradicts Assumption 7. This concludes the proof. ■

Lemma 10 *The V_q's defined in* (A.4) *are nondecreasing in x.*

Proof of lemma 10. The proof is by induction. By definition $V_{Q+1}(x) \coloneqq W$ for all x and hence is nondecreasing.

Now let $0 \leq q \leq Q$ and assume that, for $k \geq q+1$, V_k is nondecreasing in x. I show that

$$\begin{aligned} V_q(x) = \underset{\tau \geq 0}{esssup}\, \mathbb{E}\Bigg[\int_0^{\tau \wedge \tau(x^i_{q+1})} & e^{-rt} \left(\pi^i(X^i_t) - c^i\left(X^i_t\right) \sum_{k=0}^{q} \lambda_k \right) dt \\ & + e^{-r\tau \wedge \tau^i(x^i_{q+1})} \Bigg\{ V_{q\left(\underline{X}^i_{\tau^i\left(x^i_{q+1}\right)}\right)} \left(X^i_{\tau^i(x^i_{q+1})} \right) \mathbb{1}_{\{\tau \geq \tau^i(x^i_{q+1})\}} \\ & + \mathbb{1}_{\{\tau < \tau^i(x^i_{q+1})\}} \left(\sum_{q=0}^{Q} \lambda_q g^i + \int_\tau^\infty e^{-r(t-\tau)} \pi^i(X^i_t) dt \right) \vee W \Bigg\} \mid x \Bigg] \end{aligned}$$

is weakly increasing in x.

Let $\bar{x} \geq \underline{x}$, and let $\tau^{\underline{x}}$ be an optimal stopping time associated with $V(\underline{x})$, which exists by Theorem 11. From the definition of V^q in (A.4),

$$\begin{aligned} V_q(\bar{x}) \geq \mathbb{E}\Bigg[\int_0^{\tau^{\underline{x}} \wedge \tau^i(x^i_{q+1})} & e^{-rt} \left(\pi^i(X^i_t) - c^i\left(X^i_t\right) \sum_{k=0}^{q} \lambda_k \right) dt \\ & + e^{-r\tau^{\underline{x}} \wedge \tau^{\underline{x}}(x^i_{q+1})} \Bigg\{ V_{q\left(\underline{X}^i_{\tau^{\underline{x}}\left(x^i_{q+1}\right)}\right)} \left(X^i_{\tau^{\underline{x}}(x^i_{q+1})} \right) \mathbb{1}_{\{\tau^{\underline{x}} \geq \tau^{\underline{x}}(x^i_{q+1})\}} \\ & + \mathbb{1}_{\{\tau^{\underline{x}} < \tau^{\underline{x}}(x^i_{q+1})\}} \left(\sum_{q=0}^{Q} \lambda_q g^i + \int_{\tau^{\underline{x}}}^\infty e^{-r(t-\tau^{\underline{x}})} \pi^i(X^i_t) dt \right) \vee W \Bigg\} \mid \bar{x} \Bigg], \end{aligned}$$

where $\tau^{\underline{x}}(x^i_{q+1}) \coloneqq \inf \left\{ t \geq 0 \,:\, X^{i,\underline{x}}_t \leq x^i_{q+1} \right\} \leq \tau(x^i_{q+1})$ $\mathbb{P}$-a.s. by Assumption 3.

But, from the definition of the V_k's, since $c^i\left(X^i_t\right) \geq 0$ and $\lambda_{q+k} \geq 0$, $k \in \{1, \ldots, Q-q\}$, note that $V_q(x) \geq V_{q+k}(x)$ for all $k \in \{1, \ldots, Q-q\}$ and all $x \geq x^i_{q+1}$. Using that $X^{i,\bar{x}}_{\tau^{\underline{x}}(x^i_{q+1})} \geq$

$X^{i,\underline{x}}_{\tau^{\underline{x}}(x^i_{q+1})}$,

$$\begin{aligned}V_q(\bar{x}) \geq \mathbb{E}\Big[& \int_0^{\tau^{\underline{x}}\wedge\tau^i(x^i_{q+1})} e^{-rs}\pi^i(X^i_s)ds - \sum_{k=0}^{q}\lambda_k \int_0^{\tau^{\underline{x}}\wedge\tau^i(x^i_{k+1})} e^{-rs}c^i\left(X^i_s\right)ds \\ & + e^{-r\tau^{\underline{x}}\wedge\tau(X^{p+1})}\left(V_{q\left(\underline{X}^{i,\underline{x}}_{\tau^{\underline{x}}\left(x^i_{q+1}\right)}\right)}(X^i_{\tau^{\underline{x}}(x^i_{q+1})})\mathbb{1}_{\{\tau^{\underline{x}}(x^i_{q+1})\leq\tau^{\underline{x}}\}}\right. \\ & \left. + \mathbb{1}_{\{\tau^{\underline{x}}<\tau^{\underline{x}}(x^i_{q+1})\}}\left(\sum_{q=0}^{Q}\lambda_q g^i + \int_{\tau^{\underline{x}}}^{\infty} e^{-r(t-\tau^{\underline{x}})}\pi^i(X^i_t)dt\right)\vee W\right) \mid \bar{x}\Big].\end{aligned}$$

Since the V_{q+k}'s are nondecreasing by the induction hypothesis, π^i is nondecreasing and $c^i\left(X^i_t\right)$ is nonincreasing by Assumption 3,

$$\begin{aligned}V_q(\bar{x}) \geq \mathbb{E}\Big[& \int_0^{\tau^{\underline{x}}\wedge\tau^i(x^i_{q+1})} e^{-rs}\pi^i(X^i_s)ds - \sum_{k=0}^{q}\lambda_k \int_0^{\tau^{\underline{x}}\wedge\tau^i(x^i_{q+1})} e^{-rs}c^i\left(X^i_s\right)ds \\ & + e^{-r\tau^{\underline{x}}\wedge\tau^i(x^i_{q+1})}\left(V_{q\left(\underline{X}^{i}_{\tau^i\left(x^i_{q+1}\right)}\right)}\left(X_{\tau^i(x^i_{q+1})}\right)\mathbb{1}_{\{\tau^i(x^i_{q+1})\leq\tau^{\underline{x}}\}}\right. \\ & \left. + \mathbb{1}_{\{\tau^{\underline{x}}<\tau^i(x^i_{q+1})\}}\left(\sum_{q=0}^{Q}\lambda_q g^i + \int_{\tau^{\underline{x}}}^{\infty} e^{-r(t-\tau^{\underline{x}})}\pi^i(X^i_t)dt\right)\vee W\right) \mid \underline{x}\Big] \\ = V_q(\underline{x}), &\end{aligned}$$

as the integrand is $\mathbb{P}$-a.s. smaller. Since $\bar{x} \geq \underline{x}$ are arbitrary, V_q is nondecreasing in x.

Thus, by induction, V_q is nondecreasing for all $0 \leq q \leq Q$, and the proof is complete. ■

Next distinguish two cases, corresponding to the two cases of Assumption 4.

X^i only jumps up:

Lemma 11 *Assume that X^i only jumps up. There exists $\left(\tau_{\mathcal{Q}}, d^{\mathcal{Q}}\right)$ that solves* (RRP^i) *with (i) $\tau_{\mathcal{Q}} := \inf\left\{t \geq 0 : X^i_t \notin \left[\underline{p}^i, \bar{P}^i_{\mathcal{Q}}(t)\right)\right\}$, for some threshold $\underline{p}^i \in \mathcal{X}^i$ and process $\bar{P}^i_{\mathcal{Q}}(\cdot)$ constant on $[\tau^i(x^i_q), \tau^i(x^i_{q+1}))$ for all $q \in \{0, \ldots, Q\}$ with $\bar{P}^i_{\mathcal{Q}}(\tau^i(x^i_q)) = \bar{P}^i_{\mathcal{Q}}(\tau^i(x^i_{q+1}))$ if* (DRDP(x^i_q)) *is not binding at x^i_{q+1}., and (ii) $d^{i,\mathcal{Q}}_{\tau_{\mathcal{Q}}} = \mathbb{1}_{\{X^i_{\tau_{\mathcal{Q}}}\geq\bar{P}^i_{\mathcal{Q}}(\tau_{\mathcal{Q}}))\}}$.*

Proof of Lemma 11. After applying Lemma 9, there remains to show that $S(t) \cap \{x \in \mathcal{X}^i : x \geq x_W^i\} = \left[\bar{P}_Q^i(\underline{x}), \infty\right)$ for some $\bar{P}_Q^i(\cdot)$ constant on $[\tau^i(x_q^i), \tau^i(x_{q+1}^i))$ for all $q \in \{0, \ldots, Q\}$. For all q, define

$$\bar{P}_q^i := \inf\left\{x \in S_q : x \geq x_W^i\right\},$$

But $x \geq \bar{P}_q^i$ implies that $x \in S_q$. To see this, note that, starting from $x \geq \bar{P}_q^i$, $d_\tau^i = 1$ $\mathbb{P}$-a.s.. So, at x, the principal's continuation value is

$$\begin{aligned} V_q(x) = \mathbb{E}\Bigg[&\int_0^{\tau_Q} e^{-rs}\pi\left(X_s^i\right) ds - \sum_{k=0}^{q} \lambda_k \int_0^{\tau_Q} e^{-rs} c^i\left(X_s^i\right) ds \\ &+ e^{-r\tau_Q}\left[\int_{\tau_Q}^{\infty} e^{-r(t-\tau_Q)}\pi^i\left(X_t^i\right) dt + \sum_{q=0}^{Q}\lambda_q g^i\right] \Bigg| \, x\Bigg] \\ \leq \mathbb{E}&\left[\int_0^\infty e^{-rt}\pi^i\left(X_t^i\right) dt \mid x\right] + \sum_{q=0}^{Q}\lambda_q g^i, \end{aligned}$$

Therefore it is optimal for the principal to stop at x and promote the worker.

Thus the principal stops and promotes the worker the first time X_t^i enters $\left(\bar{P}^i_{q\left(\underline{X}_t^i\right)}, \infty\right)$ or $\left[\bar{P}^i_{q\left(\underline{X}_t^i\right)}, \infty\right)$. Note here that $\mathbb{P}\left(\left\{\tau_{(\bar{P}_q^i,\infty)} = 0\right\} \mid X_t^i = \bar{P}_q^i\right) = 1$, the stopping times $\tau_{(\bar{P}_q^i,\infty)}$ and $\tau_{[\bar{P}_q^i,\infty)}$ are indistinguishable. Moreover, $\bar{\pi}^i$ is right-continuous by the Portmanteau theorem, as π^i is increasing upper-semicontinuous and X^i is Feller, and therefore the two stopping times give the same payoffs to the principal and to the worker. So, one can assume that $S_q^j \cap [x_W^i \infty) = [\bar{P}_q^i, \infty)$.

Letting $\bar{P}_Q^i\left(\underline{X}_t^i\right) := \bar{P}^i_{q\left(\underline{X}_t^i\right)}$ yields the desired result. ■

Lemma 12 *Every constraint* (RDP(x_q^i)) *is binding in the problem* (RRP^i).

Proof. Let (τ, d) be the solution of (RRP^i) given by Lemma 11.

Observe that at least one constraint is binding, for otherwise the solution would coincide with that of the unconstrained problem, i.e., the worker is never promoted, which violates all the constraints.

Next, I show that (RDP(x_q^i)) for $q = 0$ is binding. Let q^* be the first binding constraint. If $q^* = 0$, I am done. So suppose not. Then

$$\mathbb{E}\left[e^{-r\tau} g^i d_\tau^i - \int_0^\tau e^{-rt} c^i\left(X_t^i\right) dt\right] > 0.$$

By Lemma 11, on the random interval $[0, \tau^i(x_{q^*}^i))$, the solution consists in a stationary promotion threshold $\bar{P}_0$ and a stationary threshold $\underline{p}_Q^i$ such that the principal takes her outside option at $\underline{p}_Q^i$. However, since (RDP(x_q^i)) is not binding at x_0^i, there exists $P > \bar{P}_0$ such that

$$\mathbb{E}\left[e^{-r\tau^i_{[P,\infty)} \wedge \tau^i(x_1^i)} g^i \mathbb{1}_{\{\tau^i_{[P,\infty)} < \tau^i(x_1^i)\}} - \int_0^{\tau^i_{[P,\infty)} \wedge \tau^i(x_1^i)} e^{-rt} c^i\left(X_t^i\right) dt\right] = 0.$$

Let $\tau(P)$ be defined as $\tau(P) \coloneqq \inf\left\{t \geq 0 \,:\, X_t^i \notin \left[\underline{p}_Q^i, P\right)\right\}$ on $[0, \tau^i(x_1^i))$ and $\tau(P) = \tau$ on $[\tau^i(x_1^i), \infty)$. Note that $\tau(P)$ is feasible in the relaxed problem (RRP^i) and yields a higher payoff for the principal (strictly if $W > 0$); a contradiction. So (RDP(x_q^i)) for $q = 0$ is binding.

Similarly, one can show that (RDP(x_q^i)) for $q = 1, \ldots, Q$ are binding. To see this, suppose not, i.e., (RDP(x_q^i)) is not binding for some $q \in \{1, \ldots, Q\}$. Let $\underline{q} \geq 1$ be the smallest q such that (RDP(x_q^i)) is not binding. Let $\tilde{q} > \underline{q}$ be the next binding constraint, with $\tilde{q} = Q + 1$ if all constraints $q \geq \underline{q}$ are lax. Then on the random interval $\left[\tau^i(x_{\underline{q}-1}^i), \tau^i(x^{\tilde{q}})\right)$, by Proposition 11, the optimal stopping time is stationary and the worker is promoted if and only if his type exceeds $\bar{P}^i_{\mathcal{Q}}\left(x^i_{\underline{q}-1}\right)$. At $\tau^i(x^i_{\underline{q}-1})$, the continuation value of the worker is zero:

$$\mathbb{E}\left[e^{-r\tau\wedge\tau^i(x^i_{\tilde{q}})}g^i\mathbb{1}_{\{\tau<\tau^i(x^i_{\tilde{q}})\wedge\tau^i(\underline{p}^i_{\mathcal{Q}})\}}\int_0^{\tau\wedge\tau^i(x^i_{\tilde{q}})}e^{-rt}c^i\left(X^i_t\right)dt \mid X^i_{\tau^i(x^i_{\underline{q}-1})}\right]=0.$$

But, by Assumption 3,

$$\begin{aligned}&\mathbb{E}\left[e^{-r\tau\wedge\tau^i(x^i_{\tilde{q}})}g^i\mathbb{1}_{\{\tau<\tau^i(x^i_{\tilde{q}})\wedge\tau^i(\underline{p}^i_{\mathcal{Q}})\}}-\int_0^{\tau\wedge\tau^i(x^i_{\tilde{q}})}e^{-rt}c^i\left(X^i_t\right)dt \mid X^i_{\tau^i(x^i_{\underline{q}-1})}\right]\\ &\quad\geq\mathbb{E}\left[e^{-r\tau\wedge\tau^i(x^i_{\tilde{q}})}g^i\mathbb{1}_{\{\tau<\tau^i(x^i_{\tilde{q}})\wedge\tau^i(\underline{p}^i_{\mathcal{Q}})\}}-\int_0^{\tau\wedge\tau^i(x^i_{\tilde{q}})}e^{-rt}c^i\left(X^i_t\right)dt \mid X^i_{\tau^i(x^i_{\underline{q}})}\right]\\ &\quad>0,\end{aligned}$$

a contradiction.

So all (RDP(x^i_q)) constraints are binding. ■ The next lemma describes the solution to (RRP^i).

Lemma 13 *Assume that X^i only jumps up. Define the process*

$$\bar{P}^i_{\mathcal{Q}}(t):=\sum_q^Q\bar{P}^i\left(\underline{X}^i_{\tau^i(x^i_q)}\right)\mathbb{1}_{\{[\tau^i(x^i_q),\tau^i(x^i_{q+1})\}}.$$

The pair (τ^*,d^*) with

$$\begin{aligned}\tau^*&=\inf\left\{t\geq 0\;:\;X^i_t\notin\left(\underline{p}^i,\bar{P}^i_{\mathcal{Q}}(t)\right)\right\},\\ \textit{and }d^*_{\tau^*}&=\mathbb{1}_{\{X^i_{\tau^*}=\bar{P}^i_{\mathcal{Q}}(\tau^*)\}}\end{aligned}$$

solves (RRP^i).

Proof. From Lemma 16, there remains to show that $\bar{P}^i_q=\bar{P}^i\left(X^i_{\tau^i(x^i_q)}\right)$. This follows from Lemma 12: because the (RDP(x^i_q)) constraint is binding at every $q\in\mathcal{Q}$, the continuation value of worker is zero at $\tau^i(x_q^i)$, so the worker is indifferent between quitting and continuing at $\tau^i(x_q^i)$. Observing that $\bar{P}^i(\underline{x})$ is increasing in $\underline{x}$, the result follows. ■

X^i **only jumps down:** Introduce a new stopping problem with constraints. Here one can assume, without loss of generality, that, for all $q \in \{0, \ldots, Q\}$, $\tilde{P}^i(x_q^i) \in \mathcal{Q}$ or $\tilde{P}^i(x_q^i) \geq x_0^i$ (using Assumption 5), and that $x_Q^i < \underline{p}_{\mathcal{Q}}^i$. Otherwise, consider a sequence of grids $\mathcal{Q}^m$ with $x_{Q^n}^i \to \inf \mathcal{X}^i$. By the last statement of Lemma 9, eventually $x_{Q^n}^i < \underline{p}_{\mathcal{Q}^n}^i$.

Consider then

$$\begin{aligned}\bar{\Pi}_{\mathcal{Q}}^i \coloneqq \sup_{\tau} \mathbb{E}\Bigg[& \int_0^{\tau \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)} e^{-rt} \pi^i \left(X_t^i \right) dt \\ & + e^{-r\tau \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)} \left(\mathbb{1}_{\{\tau \leq \tau^i(\underline{p}_{\mathcal{Q}}^i)\}} \tilde{\bar{\pi}}^i \left(X_\tau^i \right) + \mathbb{1}_{\{\tau > \tau^i(\underline{p}_{\mathcal{Q}}^i)\}} W \right) \Bigg] \end{aligned} \tag{DRRP^i}$$

subject to, for all $q = 0, \ldots, Q$ such that $x_q^i \geq \underline{p}_{\mathcal{Q}}^i$,

$$\mathbb{E}\left[e^{-r\tau} g^i \mathbb{1}\{\tau \leq \tau^i(\underline{p}_{\mathcal{Q}}^i)\} + \int_{\tau \wedge \tau^i(x_q^i)}^{\tau} e^{-rt} c^i \left(X_t^i \right) dt \right] \geq 0; \tag{DRDP(x_q^i)}$$

where $\underline{p}_{\mathcal{Q}}^i$ is the threshold optimal threshold obtained in Lemma 9, and

$$\tilde{\bar{\pi}}^i(x) \coloneqq \begin{cases} \mathbb{E}\left[\int_\tau^\infty e^{-r(t-\tau)} \pi^i \left(X_t^i \right) dt \mid x \right] & \text{if } x \in \mathcal{Q}, \\ -\infty \text{ otherwise.} \end{cases}$$

Exactly as in step 2 and 3, by Theorem 1 in Balzer and Janßen (2002) and Theorem 11 in Appendix A.2.2,

Lemma 14 *There exists $\bar{\tau}_{\mathcal{Q}}$ that solves* (DRRP^i) *with $\bar{\tau}_{\mathcal{Q}} \coloneqq \inf\{t \geq 0 : X_t^i \notin S(t)\}$, where $S(t)$ is a correspondence constant on $[\tau^i(x_q^i), \tau^i(x_{q+1}^i))$ for all $q \in \{0, \ldots, Q\}$ with $S(\tau^i(x_q^i)) = S(\tau^i(x_{q+1}^i))$ if* (DRDP(x_q^i)) *is not binding at $\tau^i(x_{q+1}^i)$. Moreover, for all $t \geq 0$ and all $q \in \{0, \ldots, Q\}$, $S(t) \cap \left(x_{q+1}^i, x_q^i\right) = \emptyset$.*

The value of problem (DRRP^i) is weakly greater than the value of (RRP^i).

Lemma 15 *The value of* (DRRP^i) *is (weakly) greater than the value of* (RRP^i)*:* $\bar{\Pi}^i_{\mathcal{Q}} \geq \Pi^i_{\mathcal{Q}}$.

Proof of Lemma 15. Consider the optimal stopping time $\tau_{\mathcal{Q}}$ for problem (RRP^i) given by Lemma 9. I claim that there exists a feasible stopping time in (DRRP^i) that gives a payoff of $\Pi^i_{\mathcal{Q}}$ to the principal.

To see this, observe that if $S_q \cap (x^i_q, x^i_{q+1}) \cap \{x \in \mathcal{X}^i : x \geq x^i_W\} = \emptyset$ for all q, I am done. So suppose not, i.e., $S_q \cap (x^i_q, x^i_{q+1}) \cap \{x \in \mathcal{X}^i : x \geq x^i_W\} \neq \emptyset$ for some $q \in \{0, \ldots, Q\}$. So, when $\underline{X}^i_t \in (x^i_{q+1}, x^i_q]$, the principal finds it optimal to stop at $\tilde{x} \in S_q \cap (x^i_q, x^i_{q+1}) \cap \{x \in \mathcal{X}^i : x \geq x^i_W\}$. Her continuation value is

$$\mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i\left(X^i_t\right)dt \mid \tilde{x}\right],$$

which is equal to

$$\begin{aligned}\mathbb{E}\Bigg[&\int_0^{\tau^i_{\left(x^i_{q+1},x^i_q\right)}\wedge\tau^i(\underline{p}^i_{\mathcal{Q}})} e^{-rt}\pi^i\left(X^i_t\right)dt \\ &+ e^{-r\tau^i_{\left(x^i_{q+1},x^i_q\right)}\wedge\tau^i(\underline{p}^i_{\mathcal{Q}})}\int_{\tau^i_{\left(x^i_{q+1},x^i_q\right)}\wedge\tau^i(\underline{p}^i_{\mathcal{Q}})}^\infty e^{-r\left(t-\tau^i_{\left(x^i_{q+1},x^i_q\right)}\wedge\tau^i(\underline{p}^i_{\mathcal{Q}})\right)}\pi^i\left(X^i_t\right)dt \mid \tilde{x}\Bigg].\end{aligned}$$

Thus the the stopping time $\tilde{\tau}$ defined by

$$\tilde{\tau} := \tau_{\mathcal{Q}} + \inf\left\{t \geq 0 : X^i_{\tau+t} \notin \left(x^i_{q+1}, x^i_q\right)\right\}\mathbb{1}_{\left\{\underline{X}^i_\tau \in \left(x^i_{q+1},x^i_q\right)\right\}}.$$

gives the same payoff to the principal as $\tau_{\mathcal{Q}}$. Next, note that it is feasible. This follows from noting that, by definition of $\tilde{g}^i$,

$$\begin{aligned}
&\mathbb{E}\left[e^{-r\tilde{\tau}}\tilde{g}^i\mathbb{1}_{\{X^i_{\tau_\mathcal{Q}}\in S\left(\underline{X}^i_{\tau_\mathcal{Q}}\right)\}}-\int_{\tilde{\tau}\wedge\tau^i(x^i_q)}^{\tilde{\tau}}e^{-rt}c^i\left(X^i_t\right)dt\right]\\
&\geq\mathbb{E}\left[e^{-r\tau_\mathcal{Q}}g^i\mathbb{1}_{\{X^i_{\tau_\mathcal{Q}}\in S\left(\underline{X}^i_{\tau_\mathcal{Q}}\right)\}}-\int_{\tau_\mathcal{Q}\wedge\tau^i(x^i_q)}^{\tau_\mathcal{Q}}e^{-rt}c^i\left(X^i_t\right)dt\right]\\
&\geq 0;
\end{aligned}$$

where the second inequality follows from (RDP(x^i_q)).

Repeating the above construction for all q such that $S_q\cap(x^i_q,x^i_{q+1})\cap\{x\in\mathcal{X}^i\,:\,x\geq x^i_W\}\neq\emptyset$ proves the claim.

Therefore the value of (DRRP^i) is (weakly) greater than the value of (RRP^i): $\bar{\Pi}^i_\mathcal{Q}\geq\Pi^i_\mathcal{Q}$.

■

Next note that the optimal stopping region in (DRRP^i) can be taken to be $[\bar{P}^i_q,\infty)$ for some $\bar{P}^i_q\geq x^i_q$ on each random interval $[\tau^i(x^i_q),\tau^i(x^i_{q+1}))$.

Lemma 16 *There exists $\bar{\tau}_\mathcal{Q}$ that solves* (DRRP^i) *with $\tau_\mathcal{Q}:=\inf\left\{t\geq 0\,:\,X^i_t\notin\left[\underline{p}^i_\mathcal{Q},\bar{P}^i_\mathcal{Q}(t)\right)\right\}$, where $\bar{P}^i_\mathcal{Q}(\cdot)$ is a process constant on $[\tau^i(x^i_q),\tau^i(x^i_q))$ for all $q\in\{0,\ldots,Q\}$ with $\bar{P}^i_\mathcal{Q}(\tau^i(x^i_q))=\bar{P}^i_\mathcal{Q}(\tau^i(x^i_{q+1}))$ if* (DRDP(x^i_q)) *is not binding at $\tau^i(x^i_{q+1})$.*

Proof of Lemma 16. After applying Lemma 14, there remains to show that $S(\underline{t})\cap\{x\in\mathcal{X}^i\,:\,x\geq x^i_W\}=\left[\bar{P}^i_\mathcal{Q}(t),\infty\right)$ for some process $\bar{P}^i_\mathcal{Q}(\cdot)$ constant on $[\tau^i(x^i_q),\tau^i(x^i_q))$, for all $q\in\{0,\ldots,Q\}$.

For all q, define

$$\bar{P}^i_q:=\inf\left\{x\in S_q\,:\,x\geq x^i_{q+1}\right\},$$

By Lemma 14, the optimal stopping time $\bar{\tau}_\mathcal{Q}$ is such that, on $[\tau^i(x^i_q),\tau^i(x^i_{q+1}))$, $\bar{\tau}_\mathcal{Q}$ is the first entry time in S_q, with $\bar{P}^i_q:=\inf\left\{x\in S_q\,:\,x\geq x^i_{q+1}\right\}\geq x^i_q$. Therefore the first entry time into $S_q \cap \{x \in \mathcal{X}^i : x \geq x_W^i\}$ (if it happens) occurs when X^i crosses $\bar{P}_q^i$ from below (as X^i only jumps down). Since S_q is a subset of the finite grid $\mathcal{Q}$, the inf is attained and one can take $S_q \cap [x_W^i \infty) = [\bar{P}_q^i, \infty)$.

Letting $\bar{P}_{\mathcal{Q}}^i(t) := \bar{P}_{q(\underline{X}_t^i)}^i$ gives the desired result. ■

Lemma 17 *Every constraint* (DRDP(x_q^i)) *is binding in the problem* (DRRP^i).

Proof. Let $\bar{\tau}_{\mathcal{Q}}$ be the optimal stopping time for (DRRP^i) given by Lemma 16. Observe first that at least one constraint is binding, for otherwise the solution would coincide with that of the unconstrained problem, i.e., the worker is never promoted, which violates all the constraints.

Next, I show that (DRDP(x_q^i)) for $q = 0$ is binding. Let q^* be the first binding constraint. If $q^* = 0$, I am done. So suppose not. Then

$$\mathbb{E}\left[e^{-r\tau}\tilde{g}^i d_\tau^i - \int_0^\tau e^{-rt} c^i\left(X_t^i\right) dt\right] > 0.$$

By lemma 14, on the random interval $[0, \tau^i(x_{q^*}^i))$, the solution consists in a stationary promotion threshold $\bar{P}_0$. However, since (DRDP(x_q^i)) is not binding at x_0^i, there exists $P > \bar{P}_0$ such that

$$\mathbb{E}\left[e^{-r\tau_{[P,\infty)}^i \wedge \tau^i(x_1^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)} \tilde{g}^i \mathbb{1}_{\{\tau_{[P,\infty)}^i < \tau^i(x_1^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)\}} - \int_0^{\tau_{[P,\infty)}^i \wedge \tau^i(x_1^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)} e^{-rt} c^i\left(X_t^i\right) dt\right] = 0.$$

But then, choosing

$$\tilde{\tau}(\omega) = \begin{cases} \inf\left\{t \geq 0 \,:\, X_t^i \notin \left[\underline{p}_{\mathcal{Q}}^i, P\right)\right\} \text{ if } inf\left\{t \geq 0 \,:\, X_t^i \notin \left[\underline{p}_{\mathcal{Q}}^i, P\right)\right\} \leq \tau^i(x_1^i), \\ \bar{\tau}_{\mathcal{Q}} \text{ otherwise,} \end{cases}$$

instead of $\bar{\tau}_{\mathcal{Q}}$ is feasible in the relaxed problem (DRRP^i) and yields a higher payoff for the principal. So (DRDP(x_q^i)) for $q = 0$ is binding.

Similarly, I can show that (DRDP(x_q^i)) for $q = 1, \ldots, Q$ such that $x_q^i \geq \underline{p}_{\mathcal{Q}}^i$ are binding. To see this, suppose not, i.e., (DRDP(x_q^i)) is not binding for some $q \in \{1, \ldots, Q\}$. Let $\underline{q} \geq 1$ be the smallest q such that (DRDP(x_q^i)) is not binding. Let $\tilde{q} > \underline{q}$ be the next binding constraint, with $\tilde{q} = Q + 1$ if all constraints $q \geq \underline{q}$ are lax. Then on the random interval $\left[\tau^i(x_{\underline{q}-1}^i), \tau^i(x^{\tilde{q}})\right)$, by Proposition 11, the optimal stopping time is stationary and the worker is promoted if and only if his type exceeds $\bar{P}_{\mathcal{Q}}^i\left(\tau^i(x_{\underline{q}-1}^i)\right)$. At $\tau^i(x_{\underline{q}-1}^i)$, the continuation value of the worker is zero:

$$\mathbb{E}\Bigg[e^{-r\bar{\tau}_{\mathcal{Q}} \wedge \tau^i(x_{\tilde{q}}^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)} \tilde{g}^i \mathbb{1}_{\{\bar{\tau}_{\mathcal{Q}} < \tau^i(x_{\tilde{q}}^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)\}} - \int_0^{\bar{\tau}_{\mathcal{Q}} \wedge \tau^i(x_{\tilde{q}}^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)} e^{-rt} c^i\left(X_t^i\right) dt \mid X_{\tau^i(x_{\underline{q}-1}^i)}^i\Bigg] = 0.$$

But, by Assumption 3,

$$\begin{aligned}
&\mathbb{E}\Bigg[e^{-r\bar{\tau}_{\mathcal{Q}} \wedge \tau^i(x_{\tilde{q}}^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)} \tilde{g}^i \mathbb{1}_{\{\bar{\tau}_{\mathcal{Q}} < \tau^i(x_{\tilde{q}}^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)\}} - \int_0^{\bar{\tau}_{\mathcal{Q}} \wedge \tau^i(x_{\tilde{q}}^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)} e^{-rt} c^i\left(X_t^i\right) dt \mid X_{\tau^i(x_{\underline{q}-1}^i)}^i\Bigg] \\
&\geq \mathbb{E}\Bigg[e^{-r\bar{\tau}_{\mathcal{Q}} \wedge \tau^i(x_{\tilde{q}}^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)} \tilde{g}^i \mathbb{1}_{\{\bar{\tau}_{\mathcal{Q}} < \tau^i(x_{\tilde{q}}^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)\}} - \int_0^{\bar{\tau}_{\mathcal{Q}} \wedge \tau^i(x_{\tilde{q}}^i) \wedge \tau^i(\underline{p}_{\mathcal{Q}}^i)} e^{-rt} c^i\left(X_t^i\right) dt \mid X_{\tau^i(x_{\underline{q}}^i)}^i\Bigg] \\
&> 0,
\end{aligned}$$

a contradiction.

So all (DRDP(x_q^i)) constraints are binding. ■

Supporting Lemma for Step 4:

Lemma 18 *For $\underline{x} \in \left(x^i_{q+1}, x^i_q\right]$ and $x \geq x^i_q$, worker i's continuation value after any history before promotion with $\left(X^i_t, \underline{X}^i_t\right) = (x, \underline{x})$ is nonnegative.*

Proof of Lemma 18. Worker i's continuation value is

$$U^i_t := \mathbb{E}\left[e^{-r(\tau-t)} g^i d^i_\tau - \int_t^\tau e^{-r(s-t)} c^i \left(X^i_s\right) ds \mid \mathcal{F}^i_t\right].$$

Since $\tau^*_{\mathcal{Q}}$ and $d^*_{\mathcal{Q}}$ only depends on $\left(X^i, \underline{X}^i\right)$ and $\left(X^i, \underline{X}^i\right)$ has the strong Markov property (as X is a Feller process), the continuation value of worker i is a function of $\left(X^i_t, \underline{X}^i_t\right)$: $U^i_t :- U^i\left(X^i_t, \underline{X}^i_t\right)$.

Moreover, by construction, for all $\underline{x}$, $U^i\left(\underline{x}, \underline{x}\right) = 0$ and $x \to U^i(x, \underline{x})$ is nondecreasing on $\left[\underline{x}, \bar{P}^i\left(\underline{x}\right)\right)$. This follows from Assumption 3 and from $c^i\left(\cdot\right)$ being nonincreasing. Therefore, after any history before promotion with $\left(X^i_t, \underline{X}^i_t\right) = (x, \underline{x})$,

$$U^i_t = U^i\left(X^i_t, \underline{X}^i_t\right) = U^i\left(x, \underline{x}\right) \geq U^i\left(\underline{x}, \underline{x}\right) = 0.$$

This concludes the proof. ■

A.1.7 Omitted Proofs for Section 1.5.3

The proof of the first part of Proposition 10 is provided in Appendix A.1.7. Appendix A.1.7 presents supporting lemmas needed in the proof. The second part is proved in Appendix A.1.7.

First part of Proposition 10: Existence of an optimal promotion contest in (RP)

The goal of this section is to prove the first part of Proposition 10, i.e., that a (randomized) promotion contest that promotes solves (RP).

Theorem 10 *A solution to* (RP) *exists.*

The logic of the proof is standard. It relies on the following two properties: (i) the feasible set is compact and (ii) the objective is upper semi-continuous. However the proof is technical. The set of all randomized promotion contests is a complicated object. Showing that it is compact in a suitable topology is not immediate. In particular, because the information the principal has at time t, $\mathcal{G}_t^T$, is endogenous, we cannot prove existence directly from a weak* compactness argument as is done in Bismut (1979) or Pennanen and Perkkiö (2018) for stopping problems. We cannot guarantee that the *weak** limit of the maximizing sequence of stopping times and promotion decision is adapted to the "right" filtration. To overcome this issue, I will use the concept of weak convergence of filtration from the theory of "extended weak convergence" introduced now.[2]

Let $\mathcal{R}(\mathcal{H})$ the set of $\mathcal{H}$-**regular** processes. A process $A = \{A_t\}_{t\geq 0}$ is **regular** if it is of class (D) and its left-continuous version A_- and its predictable projection ${}^pA$ are indistinguishable.[3] The space of $\mathcal{H}$-regular processes is a Banach space, whose dual can be identified with the space of $\mathcal{H}$-optional random measure. A formal statement is found in Pennanen and Perkkiö (2018), Theorem 1.[4]

Definition 14 *A sequence of filtrations* $\left(\mathcal{F}^n = \{\mathcal{F}_t^n\}_{t\geq 0}\right)_{n\in\mathbb{N}}$ ***converges weakly*** *to a filtration* $\mathcal{F} = \{\mathcal{F}_t^n\}_{t\geq 0}$ *if, for every* $A \in \mathcal{F}_\infty$, $\mathbb{E}\left[\mathbb{1}_A \mid \mathcal{F}_\cdot^n\right] \to \mathbb{E}\left[\mathbb{1}_A \mid \mathcal{F}_\cdot\right]$ *in probability for the Skorokhod topology. We write* $\mathcal{F}^n \to^w \mathcal{F}$*.*

I then proceed in two steps:

- First we show that for all delegation rule T and randomized promotion decision d, there exists an optimal $\mathcal{G}^T$-(randomized) stopping time S^*. This part is standard and builds on the duality results derived in Bismut (1978),[5] and used in Bismut (1979) and

[2]See Coquet et al. (2001) for an introduction.
[3]For a more detailed presentation, see Dellacherie and Meyer (1982), remark 50 d), or Bismut (1978).
[4]See also the section on random measure of Dellacherie and Meyer (1982), Theorem 2 in Bismut (1978), or Proposition 1.3 in Bismut (1979).
[5]See also Dellacherie and Meyer (1982).

Pennanen and Perkkiö (2018) to obtain both the weak compactness of the feasible set and the (weak) continuity of the objective. This is done in Lemma 22.

- Next we construct a maximizing sequence of promotion contest (T, τ, d) that has a convergent subsequence. We show, in Lemma 23, that the limit of this subsequence is a solution of (RP) using results from the theory of "extended weak convergence" derived in Coquet et al. (2001) and Coquet and Toldo (2007). This is done in the proof of Theorem 10.

Proof of theorem 10. The value Π of problem (RP) is equal to

$$\Pi = \sup_{(T,d)\in\mathcal{D}\times\mathcal{C}^* \text{ such that } d \text{ is } \mathcal{G}^T\text{-optional}} \Pi\left(T, d\right),$$

where $\Pi(T, d)$ is defined by (RP(T,d)) in Lemma 22.

Consider a maximizing sequence $\left(T^n, d^n\right)_{n\in\mathbb{N}} \subseteq \mathcal{D}\times\mathcal{C}^*$ such that

$$\lim_{n\to\infty} \Pi\left(T^n, d^n\right) = \Pi.$$

By Lemmas 19 and 20, the set $\mathcal{D}\times\mathcal{C}^*$ is (sequentially) compact in the product topology. So there exists a subsequence $\left(T^{n_k}, d^{n_k}\right)_{k\in\mathbb{N}} \subseteq \left(T^n, d^n\right)_{n\in\mathbb{N}}$ that converges to some $(T^*, d^*) \in \mathcal{D}\times\mathcal{C}^*$. Furthermore, d^* is $\mathcal{G}^{T^*}$-optional.

To see this, observe that, for all K and all $\bar{k} \geq K$, $d^{n_{\bar{k}}}$ is $\left(\mathcal{G}_t^{T^*} \vee \bigvee_{k\geq K} G_t^{T^{n_k}}\right)$-adapted and, therefore, d^* is $\left(\mathcal{G}^{T^*} \vee \bigvee_{k\geq K} \mathcal{G}^{T^{n_k}}\right)$-adapted, for all K. By Proposition 1 in Coquet et al. (2001), $\mathcal{G}^{T^{n_k}} \to \mathcal{G}^{T^*}$ as $k\to\infty$. So $\left(\mathcal{G}^{T^*} \vee \bigvee_{k\geq K} \mathcal{G}^{T^{n_k}}\right) \to^w \mathcal{G}^{T^*}$ as $K\to\infty$. Then, for all t, $\mathbb{E}\left[d_t^* \mid \left(\mathcal{G}_t^{T^*} \vee \bigvee_{k\geq K} G_t^{T^{n_k}}\right)\right] \to \mathbb{E}\left[d_t^* \mid \mathcal{G}_t^{T^*}\right]$ in probability, and hence $\mathbb{P}$-a.s. along a subsequence, as $K\to\infty$. But, for all $t\geq 0$ and all K, $d_t^* = \mathbb{E}\left[d_t^* \mid \left(\mathcal{G}_t^{T^*} \vee \bigvee_{k\geq K} G_t^{T^{n_k}}\right)\right]$. So $d_t^* = \mathbb{E}\left[d_t^* \mid \mathcal{G}_t^{T^*}\right]$ for all $t\geq 0$. The result then follows from the optional projection theorem (Theorem 2.7 in Bain and Crisan (2008)) as d^* is càdlàg.

Therefore, by Lemma 23,

$$\lim_{n\to\infty} \Pi\left(T^{n_k}, d^{n_k}\right) = \Pi\left(T^*, d^*\right) = \Pi.$$

The conclusion then follows from Lemma 22. ■

Supporting Lemma for Theorem 10

Lemma 19 *The set of optional increasing paths, $\mathcal{D}$, is (sequentially) compact for the sequential convergence defined by: for all K compact subset of $\mathbb{R}_+$, all continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ and all $i \in \{1, \ldots, N\}$,*

$$T^n \to T \Leftrightarrow \int_K f(t) dT^{i^n}(t) \to \int_K f(t) dT^i(t) \ \mathbb{P}\text{-}a.s.,$$

uniformly in K.

Proof of Lemma 19. Recall that we can identify the set of feasible delegation rule with the set of $\mathcal{F}_s$-adapted multi-process $\alpha = \left(\alpha^1, \ldots, \alpha^N\right)$, where $\alpha_t = \left(\alpha_t^1, \ldots, \alpha_t^N\right) \in \Delta^N$ is the Radon-Nikodym derivative of T evaluated at t: $\frac{dT(t)}{dt}$.

By Theorem 2.2 and the first Corollary in Becker and Mandrekar (1969), the set of progressively measurable multi-parameter random measures taking values in Δ^N is sequentially compact under the sequential convergence defined by: for all K compact subset of $\mathbb{R}_+$, all continuous function $f : \mathbb{R}_+ \times \Delta^N \to \mathbb{R}$,

$$A^n \to A \text{ if and only if, } \forall i \in \{1, \ldots, N\}, \quad \int_K \int_0^1 f\left(t, \alpha_t\right) dA_t^{i^n}(\omega) dt \to \int_K f\left(t, \alpha^i\right) dA_t^i(\omega) dt \ \mathbb{P}\text{-a.s.},$$

uniformly in K.

In particular, this implies that the set of delegation rule is sequentially precompact under the sequential convergence defined by: for all K compact subset of $\mathbb{R}_+$, all continuous function f and all $i \in \{1, \ldots, N\}$,

$$T^n \to T \Leftrightarrow \int_K f(t) dT^{i^n}(t) \to \int_K f(t) dT^i(t) \text{ } \mathbb{P}\text{-a.s.};$$

uniformly in K, since any control $A \in \mathcal{D}$ generates a unique delegation rule T.

There remains to show that it is closed: So far, we have obtained the limit T in the sense of the above as a progressively measurable process. We still need to verify that the limit T is an increasing optional path, i.e., that it satisfies condition 1.-3. of definition 1. Conditions 2. and 3. are easily seen to hold. Condition 1. follows from the $\mathbb{P}$-almost sure convergence of $T^{i,n}(t) \to T^i(t)$ for all t, which is seen to hold by choosing the constant function $f(t) = 1$. ■

Lemma 20 *The set of nondecreasing randomized promotion decision $\mathcal{C}^*$ is sequentially compact for the sequential convergence defined by:*

$$d^n \to d \textit{ if and only if } \forall t \geq 0, i \in \{0, \ldots, N\}, d_t^{i,n}(\omega) \to d_t^i(\omega), \quad \mathbb{P}\textit{-a.s.}.$$

Proof of Lemma 20. By Theorem 2.2 and the first Corollary in Becker and Mandrekar (1969), the set of promotion decision $\mathcal{C}$ is sequentially compact for the sequential convergence defined by: for all K compact subset of $\mathbb{R}_+$ and all continuous function $f : \mathbb{R}_+ \times \Delta^{N+1} \to \mathbb{R}$

$$d^n \to d \text{ if and only if } \int_K \sum_{i=0}^N f(t,i) d_t^{i,n}(\omega) dt \to \int_K \sum_{i=0}^N f(t,i) d_t^i(\omega) dt \text{ } \mathbb{P}\text{-a.s.},$$

uniformly in K. We first show that $\mathcal{C}^*$ is closed in $\mathcal{C}$ for the above convergence, and, hence, that $\mathcal{C}^*$ is (sequentially) compact in the above sense.

Consider a sequence $(d^n)_{n \in \mathbb{N}}$ in $\mathcal{C}^*$ such that $d^n \to d$ for some $d \in \mathcal{C}$. Let $i \in \{0, \ldots, N\}$.

Then, for all continuous function $f : \mathbb{R}_+ \to \mathbb{R}$ and all compact set K,

$$\int_K f(t) d_t^{i,n}(\omega) dt \to \int_K f(t) d_t^i(\omega) dt, \quad \mathbb{P}\text{-a.s.},$$

uniformly in K. But $\left(d_t^{i,n}(\omega)\right)_{n\in\mathbb{N}}$ is a sequence of bounded (by 1) càdlàg monotone function, since $d^n \in \mathcal{C}^*$. By Helly's selection theorem, there exists a subsequence $\left(d_t^{i,n_k}(\omega)\right)_{k\in\mathbb{N}} \subseteq \left(d_t^{i,n}(\omega)\right)_{n\in\mathbb{N}}$ such that d_t^{i,n_k} converges to some nondecreasing càdlàg function $\bar{d}_t^i(\omega)$ pointwise almost everywhere on $\mathbb{R}_+$. But then, by the dominated convergence theorem, for all K compact,

$$\int_K f(t) d_t^{i,n_k}(\omega) dt \to \int_K f(t) \bar{d}_t^i(\omega) dt.$$

So, by uniqueness of the limit, $d_t^i = \bar{d}_t^i$ (in the sense of the topology of Becker, i.e., $\mathbb{P}\times\ell$-a.e.), and therefore d_t^i is nondecreasing $\mathbb{P}$-a.s..

There remains to show that it implies that, for all $t \geq 0$, $d_t^{i,n}(\omega) \to d_t^i(\omega)$, $\mathbb{P}$-a.s.. This follows from Lebesgue differentiation theorem, the right continuity of both $d^{i,n}$ and d^i, the fact that the convergence is uniform in K, and the Moore-Osgood theorem. ■

Lemma 21 *Let* (T, S, d) *be a promotion contest.* (T, S, d) *satisfies* (DPC) *if and only if* (T, S, d) *satisfies, for all* $i \in \{1, \ldots, N\}$*, for all* $\mathcal{F}^i$*-stopping times* $\tilde{\tau}$*,*

$$\begin{aligned}\mathbb{E}&\left[\int_0^\infty \left(e^{-r\tau} g^i d_\tau^i - \int_0^\tau e^{-rt} c^i\left(X^i_{T^i(t)}\right) dT^i(t)\right) dS(\tau)\right] \\ &\qquad \geq \mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d_\tau^i \mathbb{1}_{\{\tau<\tilde{\tau}\}} - \int_0^{\tau\wedge\tilde{\tau}} e^{-rt} c^i\left(X^i_{T^i(t)}\right) dt\right) dS(\tau)\right].\end{aligned}$$

Proof of Lemma 21. ($\Rightarrow$) This follows from lemma 29. (DPC) implies that, for all

$\tau' \leq \tau \in \mathcal{T}\left(\mathcal{F}^i_{T^i(\cdot)}\right)$,

$$\mathbb{E}\left[\int_0^\infty \left(e^{-r(\tau-\tau')} g^i d^i_\tau - \int_{\tau'}^\tau e^{-r(s-\tau')} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right) dS(\tau) \mid \mathcal{F}^i_{T(\tau')}\right] \geq 0;$$

which, by lemma 29 implies that

$$+\infty \in \arg\max_{\tilde{\tau} \in \mathcal{T}^i} \mathbb{E}\left[\int_0^\infty \left(e^{-r\tau\wedge\tilde{\tau}} g^i \tilde{d}^i_{\tau\wedge\tilde{\tau}} \mathbb{1}_{\{\tau<\tilde{\tau}\}} - \int_0^{\tau\wedge\tau'} e^{-rt} c^i \left(X^i_{T^i(t)}\right) dT^i(t)\right) dS(\tau)\right].$$

($\Leftarrow$) The other direction follows directly from Lemma 29. ■

Lemma 22 *Let T be a delegation rule and d be a $\mathcal{G}^T$-optional promotion decision. Then there exists a $\mathcal{G}^T$-(randomized) stopping time S^* that solves*

$$\Pi(T,d) := \sup_S \mathbb{E}\left[\int_0^\infty \left(\sum_{i=1}^N \int_0^\tau e^{-rt} \pi^i (X^i_{T^i(t)}) dT^i(t) + e^{-r\tau} \bar{\pi}\left(X_{T(\tau)}, d_\tau\right)\right) dS(\tau)\right] \tag{RP(T,d)}$$

subject to, for all $i \in \{1,\ldots,N\}$, for all $t \geq 0$, $\mathbb{P}$-a.s.,

$$\mathbb{E}\left[\int_0^\infty \left(e^{-r(\tau-\tau\wedge t)} g^i d^i_\tau - \int_{\tau\wedge t}^\tau e^{-r(s-\tau\wedge t)} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right) dS(\tau) \mid \mathcal{F}^i_{T^i(t)}\right] \geq 0. \tag{DPC}$$

Proof of Lemma 22. The set of $\mathcal{G}^T$-randomized stopping times $\mathcal{S}\left(\mathcal{G}^T\right)$ is $\sigma\left(\mathcal{M}^\infty(\mathcal{G}^T), \mathcal{R}(\mathcal{G}^T)\right)$-compact by Lemma 2 in Pennanen and Perkkiö (2018), where $\mathcal{R}(\mathcal{G}^T)$ is the set of regular processes equipped with the norm $\|y\| = \sup_\tau \mathbb{E}\left[y_\tau\right]$ and $\mathcal{M}^\infty(\mathcal{G}^T)$ is the space of random stopping time. So the set of feasible $\mathcal{G}^T$-randomized stopping times is $\sigma\left(\mathcal{M}^\infty(\mathcal{G}^T), \mathcal{R}(\mathcal{G}^T)\right)$-compact as a closed subset of a $\sigma\left(\mathcal{M}^\infty(\mathcal{G}^T), \mathcal{R}(\mathcal{G}^T)\right)$-compact set. To see this, consider $\left(S^n\right)_{n\in\mathbb{N}}$ a sequence of feasible $\mathcal{G}^T$-randomized stopping time that converges to some S.[6]

[6]It is enough to prove sequential closeness since the dual of a normed space is a Banach space by Theorem

Let $\left(\mathbb{1}^a_{\{\cdot<0\}}\right)_{a\in\mathbb{N}}$ be a sequence of continuous functions such that, for all a, $\mathbb{1}^a_{\{\cdot<0\}} \leq \mathbb{1}_{\{\cdot<0\}}$ and $\mathbb{1}^a_{\{\cdot<0\}} \to \mathbb{1}_{\{\cdot<0\}}$ pointwise.[7] Lemma 21 implies that, for all $a \in \mathbb{N}$, for all $i \in \{1,\ldots,N\}$ and all $\mathcal{F}^i_{T^i(\cdot)}$-stopping time $\tilde{\tau}$,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^i_\tau - \int_0^\tau e^{-rt} c^i\left(X^i_{T^i(t)}\right) dT^i(t)\right) dS^n(\tau)\right] \\
&\quad \geq \lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^i_\tau \mathbb{1}^a_{\{\tau-\tilde{\tau}<0\}} - \int_0^{\tau\wedge\tilde{\tau}} e^{-rt} c^i\left(X^i_{T^i(t)}\right) dT^i(t)\right) dS^n(\tau)\right].
\end{aligned}$$

But, for all $(T,d) \in \mathcal{D}\times\mathcal{C}^*$ with d $\mathcal{G}^T$-optional and all $a \in \mathbb{N}$, both the processes

$$\begin{aligned}
e^{-rt} g^i d^i_t - \int_0^t e^{-rs} c^i\left(X^i_{T^i(s)}\right) dT^i(s), \\
\text{and } e^{-rt} g^i d^i_t \mathbb{1}^a_{\{t-\tilde{\tau}<0\}} - \int_0^{t\wedge\tilde{\tau}} e^{-rs} c^i\left(X^i_{T^i(s)}\right) dT^i(s)
\end{aligned}$$

have continuous paths $\mathbb{P}$-a.s. and are $\mathcal{G}^T$-optional. Therefore they belong to $\mathcal{R}\left(\mathcal{G}^T\right)$. Theorem 1 in Pennanen and Perkkiö (2018) and Theorem 6.39 in Aliprantis and Border (2006) implies that the bilinear form

$$\begin{aligned}
\mathcal{R}\left(\mathcal{G}^T\right) \times \mathcal{M}^\infty_+\left(\mathcal{G}^T\right) &\to \mathbb{R} \\
(Y,S) &\to \mathbb{E}\left[\int_0^\infty Y_\tau dS(\tau)\right]
\end{aligned}$$

is continuous. So, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^i_\tau - \int_0^\tau e^{-rt} c^i\left(X^i_{T^i(s)}\right) dT^i(s)\right) dS(\tau)\right] \\
&\quad \geq \mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^i_\tau \mathbb{1}^a_{\{\tau-\tilde{\tau}<0\}} - \int_0^{\tau\wedge\tilde{\tau}} e^{-rt} c^i\left(X^i_{T^i(s)}\right) dT^i(s)\right) dS(\tau)\right].
\end{aligned}$$

6.8 in Aliprantis and Border (2006) and the Eberlein-Šmulian Theorem (Theorem 6.34 in Aliprantis and Border (2006)) then implies that the set of randomized stopping time is also sequentially compact.

[7]It is easily seen that such a sequence exists.

Taking the limit of the right-hand side as $a \to \infty$, by Lebesgue dominated convergence theorem (applied twice), we obtain

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^i_\tau - \int_0^\tau e^{-rt} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right) dS(\tau)\right] \\ &\geq \mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^i_\tau \mathbb{1}_{\{\tau<\tilde{\tau}\}} - \int_0^{\tau\wedge\tilde{\tau}} e^{-rt} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right) dS(\tau)\right].\end{aligned}$$

Lemma 21 again implies that (T, S, d) is a feasible promotion contest; and thus the set of all feasible $\mathcal{G}^T$-randomized stopping time is closed, hence compact.

To conclude, there remains to show that the objective function is continuous in S. This follows from the same argument as above by Theorem 1 in Pennanen and Perkkiö (2018) and Theorem 6.39 in Aliprantis and Border (2006).

Thus, by Weierstrass's maximum theorem, a solution to (RP) exists (since the feasible set is nonempty). ■

Lemma 23 (Theorem 5 in Coquet and Toldo (2007)) *Let $(T^n, d^n)_{n\in\mathbb{N}} \subseteq \mathcal{P}^{T^*}$ be a sequence of pairs of delegation rules and promotion decision such that $T^n \to T$ in the sense of Lemma 19 and $d^n \to d$ in the sense of Lemma 20. Suppose that d $\mathcal{G}^T$-optional. Then $\Pi(T^n, d^n) \to \Pi(T, d)$.*

Proof of Lemma 23. The proof follows from the proof of the second case of Theorem 5 in Coquet and Toldo (2007). To see this, observe that the process $Y^n \coloneqq \left(X^1_{T^{1,n}(t)}, \ldots, X^N_{T^{N,n}(t)}\right)$ is quasi-left continuous for all $n \in \mathbb{N}$, and, therefore, Aldous' criterion for tightness[8] holds by Proposition 3 in Coquet and Toldo (2007). Moreover, for all $t \geq 0$, $T^n(t) \to T(t)$ $\mathbb{P}$-a.s.. So Proposition 1 in Coquet et al. (2001) implies that $\mathcal{G}^{T^n} \to^w \mathcal{G}^T$. Finally, since each X^i is continuous in probability, we have $Y^n \to Y \coloneqq \left(X^1_{T^1(t)}, \ldots, X^N_{T^N(t)}\right)$ in probability.

[8]See equation (1) in Coquet and Toldo (2007) for a definition.

Therefore we obtain the desired result by Theorem 5 in Coquet and Toldo (2007), upon noting that the proof applies to our constrained stopping problem (RP(T,d)) provided that:

(i) If $S^n \to S$ for the $\sigma\left(\mathcal{M}^\infty\left(\mathcal{G}^T \vee \bigvee_n \mathcal{G}^{T^n}\right), \mathcal{R}\left(\mathcal{G}^T \vee \bigvee_n \mathcal{G}^{T^n}\right)\right)$-convergence, then S is feasible in the (RP(T,d)), i.e., (T, S, d) satisfies all the constraints (DPC).

(ii) The objective function $\sum_{i=1}^N \int_0^\tau e^{-rt}\pi^i(X^i_{T^i(t)})dT^i_n(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)}, d_n\right)$ converges to $\sum_{i=1}^N \int_0^\tau e^{-rt}\pi^i(X^i_{T^i(t)})dT^i(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)}, d\right)$ in $\mathcal{R}\left(\mathcal{G}^T \vee \bigvee_n \mathcal{G}^{T^n}\right)$.

Start with **(i)**. Consider a sequence of feasible promotion contests $(T^n, S^n, d^n)_{n\in\mathbb{N}}$ that converges to some (T, S, d), in the senses defined above. Suppose first that c^i is continuous. The result for general c^i's then follows by an approximation argument. Let $T \geq 0$. For all $i \in \{1, \ldots, N\}$ and all stopping times $\tau \leq T$,

$$e^{-r\tau}gd^{i,n}_\tau - \int_0^\tau e^{-rs}c^i\left(X^i_{T^i_n(s)}\right)dT^i_n(s) \to e^{-r\tau}g^i d^i_\tau - \int_0^\tau e^{-rs}c^i\left(X^i_{T^i(s)}\right)dT^i(s),$$

uniformly over $\tau \leq T$ $\mathbb{P}$-a.s.. Let $\epsilon > 0$. By Egorov theorem (Theorem 10.39 in Aliprantis and Border (2006)), the convergence is uniform on a set $\mathcal{O} \subseteq \Omega$ with $\mathbb{P}(\omega \in \mathcal{O}) \geq 1 - \frac{\epsilon}{C_1}$, where $C_1 > 2\left(g + \frac{\sup c^i}{r}\right)$. So, there exists N such that for all $n \geq N$ and all $\tau \leq T$,

$$\left|e^{-r\tau}g^i d^{i,n}_\tau - \int_0^\tau e^{-rt}c^i\left(X^i_{T^i_n(t)}\right)dT^i_n(t) - \left(e^{-r\tau}g^i d^i_\tau - \int_0^\tau e^{-rt}c^i\left(X^i_{T^i(t)}\right)dT^i(t)\right)\right| < \epsilon$$

on $\mathcal{O}$. But then, for all $n \geq N$ and all $\tau \leq T$,

$$\begin{aligned}
&\mathbb{E}\Bigg[\left|e^{-r\tau}g^i d^{i,n}_\tau - \int_0^\tau e^{-rt}c^i\left(X^i_{T^i_n(t)}\right)dT^i_n(t) - \left(e^{-r\tau}gd^i_\tau - \int_0^\tau e^{-rt}c^i\left(X^i_{T^i(s)}\right)dT^i(t)\right)\right|\mathbb{1}_O(\omega) \\
&+\left|e^{-r\tau}g^i d^{i,n}_\tau - \int_0^\tau e^{-rt}c^i\left(X^i_{T^i_n(t)}\right)dT^i_n(t) - \left(e^{-r\tau}g^i d^i_\tau - \int_0^\tau e^{-rt}c^i\left(X^i_{T^i(t)}\right)dT^i(t)\right)\right|\mathbb{1}_{O^c}(\omega)\Bigg] \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

So

$$\sup_{\tau \le T} \mathbb{E}\left[\left| e^{-r\tau} g^i d_\tau^{i,n} - \int_0^\tau e^{-rt} c^i \left(X^i_{T_n^i(t)}\right) dT_n^i(t) - \left(e^{-r\tau} g^i d_\tau^i - \int_0^\tau e^{-rt} c^i \left(X^i_{T^i(t)}\right) dT^i(t)\right)\right|\right] \to 0,$$

as $n \to \infty$. But, for all $T \ge 0$,

$$\begin{aligned}
&\sup_\tau \mathbb{E}\left[\left| e^{-r\tau} g^i d_\tau^{i,n} - \int_0^\tau e^{-rs} c^i \left(X^i_{T_n^i(s)}\right) dT_n^i(s) - \left(e^{-r\tau} g^i d_\tau^i - \int_0^\tau e^{-rs} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right)\right|\right] \\
&\quad \le \sup_\tau \mathbb{E}\Bigg[\Bigg| e^{-r\tau\wedge T} g^i \left(d^{i,n}_{\tau\wedge T} - d^i_{\tau\wedge T}\right) \\
&\qquad\qquad - \left(\int_0^{\tau\wedge T} e^{-rs} c^i \left(X^i_{T_n^i(s)}\right) dT_n^i(s) - \int_0^{\tau\wedge T} e^{-rs} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right)\Bigg| \\
&\qquad\qquad + \left| e^{-r\tau\wedge T} g^i \left(d_\tau^{i,n} - d_\tau^i + d^i_{\tau\wedge T} - d^{i,n}_{\tau\wedge T}\right)\right| \\
&\qquad\qquad + \left|\int_{\tau\wedge T}^\tau e^{-rs} c^i \left(X^i_{T_n^i(s)}\right) dT_n^i(s) - \int_{\tau\wedge T}^\tau e^{-rs} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right|\Bigg] \\
&\quad \le \sup_{\tau \le T} \mathbb{E}\Bigg[\Bigg| e^{-r\tau\wedge T} g^i d^{i,n}_{\tau\wedge T} - e^{-r\tau\wedge T} g^i d^i_{\tau\wedge T} \\
&\qquad - \left(\int_0^{\tau\wedge T} e^{-rt} c^i \left(X^i_{T_n^i(t)}\right) dT_n^i(t) - \int_0^{\tau\wedge T} e^{-rt} c^i \left(X^i_{T_n^i(t)}\right) dT^i(t)\right)\Bigg|\Bigg] + e^{-rT} C_1.
\end{aligned}$$

Therefore,

$$\sup_\tau \mathbb{E}\left[\left| e^{-r\tau} g^i d_\tau^{i,n} - \int_0^\tau e^{-rs} c^i \left(X^i_{T^i(s)}\right) dT_n^i(s) - \left(e^{-r\tau} g^i d_\tau^i - \int_0^\tau e^{-rs} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right)\right|\right] \to 0.$$

So,[9]

$$e^{-rt} g^i d_t^{i,n} - \int_0^t e^{-rs} c^i \left(X^i_{T^i(s)}\right) dT_n^i(s) \to e^{-rt} g^i d_t^i - \int_0^t e^{-rs} c^i \left(X^i_{T^i(s)}\right) dT^i(s) \text{ in } \mathcal{R}\left(\mathcal{G}^T \vee \bigvee_{n=0}^\infty \mathcal{G}^{T^n}\right)$$

[9]All the processes have continuous paths and are adapted to $\mathcal{G}^T \vee \bigvee_{n=0}^\infty \mathcal{G}^{T^n}$, so they belong to $\mathcal{R}\left(\mathcal{G}^T \vee \bigvee_{n=0}^\infty \mathcal{G}^{T^n}\right)$.

Letting $\left(\mathbb{1}^a_{\{\cdot<0\}}\right)_{a\in\mathbb{N}}$ be a sequence of continuous functions such that, for all a, $\mathbb{1}^a_{\{\cdot<0\}} \leq \mathbb{1}_{\{\cdot<0\}}$ and $\mathbb{1}^a_{\{\cdot<0\}} \to \mathbb{1}_{\{\cdot<0\}}$ pointwise as in the proof of 22, we obtain, by Theorem 1 in Pennanen and Perkkiö (2018) and Theorem 6.39 in Aliprantis and Border (2006), for all $a \in \mathbb{N}$ and all $\mathcal{F}^i$-stopping time $\tilde{\tau}$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^i_\tau - \int_0^\tau e^{-rs} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right) dS(\tau)\right] \\
&= \lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^{i,n}_\tau - \int_0^\tau e^{-rs} c^i \left(X^i_{T^{i,n}(s)}\right) dT^{i,n}(s)\right) dS^n(\tau)\right] \\
&\geq \lim_{n\to\infty} \mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^{i,n}_\tau \mathbb{1}^a_{\{\tau-\tilde{\tau}<0\}} - \int_0^{\tau\wedge\tilde{\tau}} e^{-rs} c^i \left(X^i_{T^{i,n}(s)}\right) dT^{i,n}(s)\right) dS^n(\tau)\right] \\
&= \mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^i_\tau \mathbb{1}^a_{\{\tau-\tilde{\tau}<0\}} - \int_0^{\tau\wedge\tilde{\tau}} e^{-rs} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right) dS(\tau)\right].
\end{aligned}$$

where the inequality follows from Lemma 21. Taking the limit of the right-hand side as $a \to \infty$, by Lebesgue dominated convergence theorem (applied twice), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^i_\tau - \int_0^\tau e^{-rt} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right) dS(\tau)\right] \\
&\geq \mathbb{E}\left[\int_0^\infty \left(e^{-r\tau} g^i d^i_\tau \mathbb{1}_{\{\tau<\tilde{\tau}\}} - \int_0^{\tau\wedge\tilde{\tau}} e^{-rt} c^i \left(X^i_{T^i(s)}\right) dT^i(s)\right) dS(\tau)\right].
\end{aligned}$$

If c^i is not continuous, we can find a sequence of continuous function $c^{i,n}$ that converge to c^i pointwise such that $\int_0^\tau c^i(X^i_{T^i(s)})dT^i(s) \to \int_0^\tau c^{i,n}(X^i_{T^i(s)})dT^i(s)$ for all τ $\mathbb{P}$-a.s.. We then obtain the same inequality by the Lebesgue dominated convergence theorem. Then Lemma 21 implies that (T, S, d) satisfies (DPC).

To conclude, there remains to show **(ii)**. This is done exactly as in the first part of the proof of **(i)**. ■

Second part of Proposition 10: Characterization of the promotion time

Lemma 24 *Suppose that the promotion contest* (T, τ, d) *solves* (RP). *If worker* i *is promoted at time* t*, then* $\{\tau = t\} \in \mathcal{F}^i_{T^i(t)}$.

Proof. I will prove the contrapositive. So let (T, τ, d) be an implementable promotion contest with $\{\tau = t\}$ such that i is promoted. Suppose that $\{\tau = t\} \notin \mathcal{F}^i_{T^i(t)}$. I want to show that (T, τ, d) is not optimal.

Since $\{\tau = t\} \notin \mathcal{F}^i_{T^i(t)}$, $\mathbb{P}\left(\{\tau = t\} \mid \mathcal{F}^i_{T^i(t)}\right) \in (0,1)$. But then

$$\tilde{U}^i_t := \mathbb{E}\left[e^{-r(\tau-t)} g^i d^i_\tau - \int_t^\tau e^{-r(s-t)} c^i\left(X^i_{T^i(s)}\right) dT^i(s) \mid \mathcal{F}^i_{T^i(t)}\right] < g^i.$$

To see this, suppose not, i.e., $\tilde{U}^i_t \geq g$. Then

$$\begin{aligned}
g^i \leq \tilde{U}^i_t &= \mathbb{E}\left[e^{-r(\tau-t)} g^i d^i_\tau - \int_t^\tau e^{-r(s-t)} c^i\left(X^i_{T^i(s)}\right) dT^i(s) \mid \mathcal{F}^i_{T^i(t)}\right] \\
&\leq \mathbb{E}\left[e^{-r(\tau-t)} g^i d^i_\tau \mid \mathcal{F}^i_{T^i(t)}\right] \\
\Leftrightarrow & \mathbb{E}\left[e^{-r(\tau-t)} d^i_\tau \mid \mathcal{F}^i_{T^i(t)}\right] = 1 \\
\Rightarrow & \mathbb{P}\left(\{\tau = t\} \cap \{d^i_\tau = 1\} \mid \mathcal{F}^i_{T^i(t)}\right) = 1.
\end{aligned}$$

This is a contradiction. So $\tilde{U}^i_t < g^i$.

Consider then the promotion contest $\left(\tilde{T}, \tilde{\tau}, \tilde{d}\right)$ with $\tilde{T} := T$ on $[0, \tau)$ and $\tilde{T}^j(t) = T^j(\tau)$ if $j \neq i$ and $\tilde{T}^j(t) = t - \tau$ if $j = i$ on $[\tau, \tilde{\tau}]$, $\tilde{d} \in \arg\max_{d \in \Delta^{N+1}} \mathbb{E}\left[\bar{\pi}^i\left(X^i_{T^i(\tilde{\tau})}\right) d^i_{\tilde{\tau}} + d^0_{\tilde{\tau}} W \mid \mathcal{G}^{\tilde{T}}_{\tilde{\tau}}\right]$ subject to $d^j_\tau = 0$ for all $j \notin \{0, i\}$, and $\tilde{\tau}$ is chosen to be the optimal continuation promotion contest in the single i-arm problem after time t starting at time t with

$$\mathbb{E}\left[e^{-r(\tilde{\tau}-t)} g^i d^i_{\tilde{\tau}} - \int_t^{\tilde{\tau}} e^{-r(s-t)} c^i\left(X^i_{T^i(s)}\right) dT^i(s) \mid \mathcal{F}^i_{T^i(\tau)}\right] = \tilde{U}^i_\tau,$$

given by Theorem 2.[10]

Finally observe that $\left(\tilde{T}, \tilde{\tau}, \tilde{d}\right)$ is feasible. The payoffs of all workers $j \neq i$ are the same after any history of the game. The payoff of player i is unchanged before τ (by the law of iterated expectations and the definition of τ) and nonnegative on the random interval $(\tau, \tilde{\tau}]$ as $\tilde{\tau}$ is chosen i-arm feasible.

Theorem 2 then guarantees that the alternative promotion contest $\left(\tilde{T}, \tilde{\tau}, \tilde{d}\right)$ yields a strictly higher payoff for the principal than the original promotion contest (T, τ, d). This concludes the proof. ■

A consequence of the above lemma is that we can think of the principal as choosing N promotion times, one for each agent, and one retiring time τ^0, instead of just one τ. Formally,

Corollary 9 *Let* (T, τ, d) *be a promotion contest solving* (RP). *Then* $\tau = \left(\bigwedge_{i=1}^{N} \tau^i\right) \wedge \tau^0$, *where* τ^i *is a* $\mathcal{F}^i$*-stopping time,* τ^0 *is a* $\mathcal{G}^T$*-stopping time, and* i *is promoted only if* $\tau^i \leq \tau = \left(\bigwedge_{i=1}^{N} \tau^i\right) \wedge \tau^0$.

Proof. Define, for all $i \in \{1, \ldots, N\}$,

$$\tau^i(\omega) := \begin{cases} \tau(\omega) \text{ if } \omega \in \{\omega \in \Omega \,:\, d^i_\tau = 1\} \\ +\infty \text{ otherwise} \end{cases} ;$$

and

$$\tau^0(\omega) := \begin{cases} \tau(\omega) \text{ if } \omega \in \{\omega \in \Omega \,:\, d^0_\tau = 1\} \\ +\infty \text{ otherwise} \end{cases} .$$

By lemma 24, each τ^i is a $\mathcal{F}^i$-stopping time. The result follows. ■

[10] I.e., set the value of the running minimum such that the above equation holds.

A.1.8 Omitted Proofs for Section 1.5.4

Proof of Proposition 11

From Proposition 10, we can assume without loss of optimality that $\tau = \left(\bigwedge_{i=1}^{N} \bar{\tau}^i\right) \wedge \tau^0$, where τ^i is a $\mathcal{F}^i_{T^i(\cdot)}$-stopping time, τ^0 is a $\mathcal{G}^T_t$-stopping time, and that $d^i_t = \mathbb{1}_{\{\tau^i \leq \tau \leq t\}}$, $i \in \{0, \ldots, N\}$. Then (T, τ, d) generates the following $\mathcal{G}^T_t$-adapted reward processes

$$h^i_t = \pi^i(X^i_t)\mathbb{1}_{\{t<T^i(\tau)\}} + r\bar{\pi}^i\left(X^i_{T^i(\tau)}\right)\mathbb{1}_{\{t\geq T^i(\tau)\}}, \quad i = 1, \ldots, N,$$

Moreover let $h^0_t = rW$, $t \geq 0$, i.e., I consider an alternative problem in which the outside option is an $N + 1^{th}$ arm that can be pulled at any instant and gives a flow payoff of rW to the principal. Let (T, τ, d) be a feasible promotion contest. This relaxes the principal problems. Observe that

$$\mathbb{E}\left[\sum_{i=1}^{N}\int_0^{\tau}\pi^i\left(X^i_{T^i(t)}\right)dT^i(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)}, d\right)\right] = \mathbb{E}\left[\sum_{i=0}^{N}\int_0^{\infty}h^i_{T^i(t)}d\tilde{T}^i(t)\right];$$

with $\tilde{T}(t) = T(t)$ if $t \leq \tau$ and, for all $i \in \{0, \ldots, N\}$ and $t \geq \tau$, $\tilde{T}^i(t) = d^i(t - \tau) + T^i(\tau)$.

Define $\bar{\tau}^i := T^i(\tau^i)$; and let τ^i_0 be the solution of

$$\sup_{\tau_0}\mathbb{E}\left[e^{-r\bar{\tau}^i\wedge\tau_0}g^i d^i\mathbb{1}_{\{\bar{\tau}^i\leq\tau^0\}} - \int_0^{\bar{\tau}^i\wedge\tau_0}e^{-rt}c^i\left(X^i_t\right)dt\right],$$

where the supremum is taken over all $\mathcal{F}^i$-stopping times. Then $\bar{\tau}^i \wedge \tau^i_0 \geq T^i(\tau)$ $\mathbb{P}$-a.s.. To see this, note that, by definition of τ^i_0 and Lemma 29, for all $\tau^i_0 < \tilde{\tau} \leq \bar{\tau}^i$,

$$\mathbb{E}\left[e^{-r\bar{\tau}^i\wedge\tilde{\tau}}g^i\mathbb{1}_{\{\bar{\tau}^i\leq\tilde{\tau}\}} - \int_{\bar{\tau}^i_0}^{\bar{\tau}^i\wedge\tilde{\tau}}e^{-rt}c^i\left(X^i_t\right)dt \mid \mathcal{F}^i_{\tau^i_0}\right] < 0$$
$$\Rightarrow \mathbb{E}\left[e^{-r\tau^i\wedge T^{i,-1}(\tilde{\tau})}g^i\mathbb{1}_{\{\tau^i\leq T^{i,-1}(\tilde{\tau})\}} - \int_{T^{i,-1}(\tau^i_0)}^{\tau^i\wedge T^{i,-1}(\tilde{\tau})}e^{-rt}c^i\left(X^i_{T^i(t)}\right)dT^i(t) \mid \mathcal{F}^i_{\tau^i_0}\right] < 0,$$

where $T^{i,-1}(t) := \inf\{s : T^i(s) > t\}$. Therefore $\bar{\tau}^i \wedge \tau_0^i \geq T^i(\tau)$ $\mathbb{P}$-a.s..

So, for all $i = 1, \ldots, N$, $\mathbb{P}$-a.s.,

$$h_t^i \leq \tilde{h}_t^i := \pi^i\left(X_t^i\right)\mathbb{1}_{\{t<\bar{\tau}^i\wedge\tau_0^i)\}} + r\bar{\pi}^i\left(X^i_{\bar{\tau}^i\wedge\tau_0^i}, \tilde{d}^i\right)\mathbb{1}_{\{t\geq\bar{\tau}^i\wedge\tau_0^i\}};$$

where $\tilde{d}_t^i = \mathbb{1}_{\{\bar{\tau}^i\leq\tau_0^i\wedge t\}}$. Then

$$\begin{aligned}\mathbb{E}\left[\sum_{i=1}^N\int_0^\tau \pi^i\left(X^i_{T^i(t)}\right)dT^i(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)}, d\right)\right] &= \mathbb{E}\left[\sum_{i=0}^N\int_0^\infty h^i_{\tilde{T}^i(t)}d\tilde{T}^i(t)\right]\\ &\leq \mathbb{E}\left[\sum_{i=0}^N\int_0^\infty \tilde{h}^i_{\tilde{T}^i(t)}d\tilde{T}^i(t)\right].\end{aligned}$$

Moreover, $(\tau_0^i\wedge\bar{\tau}^i, \tilde{d}^i)$ is feasible in the single i-arm problem. To see this, note that

$$\mathbb{E}\left[e^{-r\tau}g^i\mathbb{1}_{\{\tau^i\leq\tau\}} - \int_0^\tau e^{-rt}c^i\left(X^i_{T^i(t)}\right)dT^i(t)\right] \geq 0$$

implies that

$$\begin{aligned}&\mathbb{E}\left[e^{-r\tau_0^i\wedge\bar{\tau}^i}g^i\mathbb{1}_{\{\bar{\tau}^i\leq\tau_0^i\}} - \int_0^{\tau_0^i\wedge\bar{\tau}^i} e^{-rt}c^i\left(X_t^i\right)dt\right]\\ &\quad= \sup_S \mathbb{E}\left[\int_0^\infty e^{-r\tilde{\tau}}g^i\mathbb{1}_{\{\tau^i\leq\tilde{\tau}\}}dS(\tilde{\tau}) - \int_0^\infty\int_0^{\tilde{\tau}} e^{-rt}c^i\left(X_t^i\right)dtdS(\tilde{\tau})\right] \geq 0\end{aligned}$$

since

$$\begin{aligned}&\mathbb{E}\left[\int_0^\infty e^{-r\tilde{\tau}}g^i\mathbb{1}_{\{\bar{\tau}^i\leq\tilde{\tau}\}}dS(\tilde{\tau}) - \int_0^\infty\int_0^{\tilde{\tau}} e^{-rt}c^i\left(X_t^i\right)dtdS(\tilde{\tau})\right]\\ &\quad= \mathbb{E}\left[e^{-r\bar{\tau}^i}g^i\mathbb{1}_{\{\bar{\tau}^i\leq\tilde{\tau}\}} - \int_0^\infty e^{-rs}c^i\left(X_t^i\right)(1-S(t))\,dt\right]\\ &\quad= \mathbb{E}\left[e^{-r\tau}g^i\mathbb{1}_{\{\tau^i\leq\tau\}} - \int_0^\tau e^{-rt}c^i\left(X^i_{T^i(t)}\right)dT^i(t)\right]\end{aligned}$$

for $S(t) = 1 - \tilde{q}^i(t) + \mathbb{1}_{\{t=\tau^i\}}\tilde{q}^i(\tau)$ and $\tilde{q}^i(t) = e^{-r\left(T^{i,-1}(t)-t\right)}\mathbb{1}_{\{t\leq\tau\}}$ where $T^{i,-1}(t) \coloneqq \inf\left\{s : T^i(s) > t\right\}$. Then, by definition of τ_0^i and Lemma 29, for all $\tilde{\tau} \leq \tau_0^i \wedge \bar{\tau}^i$,

$$\mathbb{E}\left[e^{-r\bar{\tau}^i\wedge\tau_0^i}g^i\mathbb{1}_{\{\bar{\tau}^i\leq\tau_0^i\}} - \int_{\tilde{\tau}}^{\bar{\tau}^i\wedge\tau_0^i} e^{-rt}c^i\left(X_t^i\right)dt \mid \mathcal{F}_{\tilde{\tau}}^i\right] \geq 0.$$

Thus $(\bar{\tau}^i \wedge \tau_0^i, \tilde{d}^i)$ is feasible in the single i-arm problem.

Therefore, by Proposition 30 in Appendix A.1.8 and Theorem 3.7 in El Karoui and Karatzas (1997), since $\underline{\Gamma}^i_\cdot\left(\tilde{h}^i\right)$ is $\mathcal{F}^i$-adapted,

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^N\int_0^\tau \pi^i\left(X^i_{T^i(t)}\right)dT^i(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)},d\right)\right] &\leq \mathbb{E}\left[\sum_{i=0}^N\int_0^\infty e^{-rt}r\underline{\Gamma}^i_{\tilde{T}^{*,i}(t)}\left(\tilde{h}^i\right)d\tilde{T}^{*,i}(t)\right]\\
&= \mathbb{E}\left[\int_0^\infty e^{-rt}r\bigvee_{i=0}^N\underline{\Gamma}^i_{\tilde{T}^{*,i}(t)}\left(\tilde{h}^i\right)dt\right]
\end{aligned}$$

where $\tilde{T}^*$ is any delegation strategy satisfying the synchronization identity and $\underline{\Gamma}^0_t = W$ for all $t \geq 0$.

Then Lemma 27 and Corollary 1 implies that, for all $\bar{W} \geq W$,

$$\mathbb{E}\left[\int_0^\infty e^{-rt}r\underline{\Gamma}^i_t\left(\tilde{h}^i\right)\vee\bar{W}dt\right] = \mathbb{E}\left[\int_0^{\rho(\bar{W};\tilde{h}^i)}e^{-rt}\tilde{h}^i_tdt + e^{-r\rho(W;h^i)}\bar{W}\right] \leq \mathbb{E}\left[\int_0^\infty re^{-rt}\underline{\Gamma}^{s,i}_t\vee\bar{W}dt\right], \tag{A.5}$$

where $\rho(\bar{W};h^i) \coloneqq \inf\left\{t \geq 0 : \underline{\Gamma}^i_t\left(\tilde{h}^i\right) \leq W\right\}$.

We conclude following the proof of indexability for superprocesses in Durandard (2022a). Let

$$\underline{\Gamma}^{-i}_t(\tilde{h}^{-i}) = \vee_{j\neq i}\underline{\Gamma}^j_{T^{*,j}(t+T^{*,i}(t))}(\tilde{h}^j) \text{ and } \bar{\underline{\Gamma}}^{-i,K}_t(\tilde{h}^{-i}) = \sum_{k=0}^\infty\underline{\Gamma}^{-i}_{\sigma^k}(\tilde{h}^{-i})\mathbb{1}_{\{t\in[\sigma^k,\sigma^{k+1})\}},$$

with $\sigma^k = \inf\left\{t \geq 0 \,:\, \Gamma_t^{-i}(\tilde{h}^{-i}) \leq \Gamma_0^{-i} - \frac{k}{K}\Gamma_0^{-i}\right\}$ for some K large. Then

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{N}\int_0^\infty e^{-rt} r \underline{\Gamma}^i_{\tilde{T}^{*,i}(t)}\left(\tilde{h}^i\right) d\tilde{T}^{*,i}(t)\right] \\
&\leq \mathbb{E}\left[\int_0^\infty e^{-rt} r \underline{\Gamma}^i_{\tilde{T}^{*,i}(t)}\left(\tilde{h}^i\right) d\tilde{T}^{*,i}(t) + \int_0^\infty e^{-rt} r \underline{\bar{\Gamma}}^{-i,K}_{\tilde{T}^{*,-i}(t)}(\tilde{h}^{-i}) d\tilde{T}^{*,-i}(t)\right] \\
&\leq \mathbb{E}\left[\int_0^\infty e^{-rt} r \underline{\Gamma}^i_{\bar{T}^{K,i}(t)}\left(\tilde{h}^i\right) d\bar{T}^{K,i}(t) + \int_0^\infty e^{-rt} r \underline{\bar{\Gamma}}^{-i,K}_{\bar{T}^{K,-i}(t)}(\tilde{h}^{-i}) d\bar{T}^{K,-i}(t)\right];
\end{aligned}$$

where $\bar{T}^K$ is an optimal index strategy for the two arms bandits with rewards $\underline{\Gamma}^i_t(h^i)$ and $\underline{\bar{\Gamma}}^{-i,K}_t(h^{-i})$ giving priority to arm $-i$, using Theorem 3.7 in El Karoui and Karatzas (1997) again. Letting $\tau^k = \inf\left\{t \geq 0 \,:\, \Gamma_t^i(\tilde{h}^i) \leq \Gamma_0^{-i} - \frac{k-1}{K}\Gamma_0^{-i}\right\}$ and $\tau^0 = 0$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-rt} r \underline{\Gamma}^i_{\bar{T}^{K,i}(t)}\left(\tilde{h}^i\right) d\bar{T}^{K,i}(t) + \int_0^\infty e^{-rt} r \underline{\bar{\Gamma}}^{-i,K}_{\bar{T}^{*,-i}(t)}(\tilde{h}^{-i}) d\bar{T}^{K,-i}(t)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^K e^{-r\sigma^k}\int_{\tau^k}^{\tau^{k+1}} e^{-rt} r \underline{\Gamma}^i_t(\tilde{h}^i)dt + e^{-r\tau^{k+1}}\int_{\sigma^k}^{\sigma^{k+1}} e^{-rt} r \underline{\bar{\Gamma}}^{-i,K}_t(\tilde{h}^{-i})dt\right] \\
&= \mathbb{E}\Bigg[\sum_{k=0}^K e^{-r\sigma^k}\left(\int_0^{\tau^{k+1}} e^{-rt} r \underline{\Gamma}^i_t(\tilde{h}^i)dt - \int_0^{\tau^k} e^{-rt} r \underline{\Gamma}^i_t(\tilde{h}^i)dt\right) \\
&\qquad\qquad + e^{-r\tau^{k+1}} r \underline{\bar{\Gamma}}^{-i,K}_{\sigma^k}(\tilde{h}^{-i})\left(e^{-r\sigma^k} - e^{-r\sigma^{k+1}}\right)\Bigg] \\
&= \mathbb{E}\Bigg[\sum_{k=0}^K \left(e^{-r\sigma^k} - e^{-r\sigma^{k+1}}\right)\left(\int_0^{\tau^{k+1}} e^{-rt} r \underline{\Gamma}^i_t(\tilde{h}^i)dt\right) \\
&\qquad\qquad + + e^{-r\tau^{k+1}} r \underline{\bar{\Gamma}}^{-i,K}_{\sigma^k}(\tilde{h}^{-i})\left(e^{-r\sigma^k} - e^{-r\sigma^{k+1}}\right)\Bigg] \\
&\leq \mathbb{E}\Bigg[\sum_{k=0}^K \left(e^{-r\sigma^k} - e^{-r\sigma^{k+1}}\right)\left(\int_0^{\lambda^{k+1}} e^{-rt} r \underline{\Gamma}^{s,i}_t dt + \int_{\lambda^{k+1}}^{\tau^{k+1}} e^{-rt} r \underline{\bar{\Gamma}}^{-i,K}_{\sigma^k}(\tilde{h}^{-i})dt\right) \\
&\qquad\qquad + e^{-r\tau^{k+1}} r \underline{\bar{\Gamma}}^{-i,K}_{\sigma^k}(\tilde{h}^{-i})\left(e^{-r\sigma^k} - e^{-r\sigma^{k+1}}\right)\Bigg] \\
&= \mathbb{E}\left[\sum_{k=0}^\infty \left(e^{-r\sigma^k} - e^{-r\sigma^{k+1}}\right)\int_0^{\lambda^{k+1}} e^{-rt} r \underline{\Gamma}^{s,i}_t dt + e^{-r\lambda^{k+1}} r \underline{\bar{\Gamma}}^{-i,K}_{\sigma^k}(\tilde{h}^{-i})\left(e^{-r\sigma^k} - e^{-r\sigma^{k+1}}\right)\right] \\
&\leq \mathbb{E}\left[\int_0^\infty e^{-rt} r \underline{\Gamma}^{s,i}_{T^{K,i}(t)} dT^{K,i}(t) + \int_0^\infty e^{-rt} r \underline{\bar{\Gamma}}^{-i,K}_{T^{K,-i}(t)}(\tilde{h}^{-i}) dT^{K,-i}(t)\right],
\end{aligned}
$$

where $\lambda^k = \inf\left\{t \geq 0 \,:\, \Gamma_t^{s,i} \leq \Gamma_0^{-i} - \frac{k-1}{K}\Gamma_0^{-i}\right\}$, $\lambda^0 = 0$, and T^K is an optimal strategy for the two arms bandits with rewards $\underline{\Gamma}_t^{s,i}$ and $\underline{\bar{\Gamma}}_t^{-i,K}(h^{-i})$. The first inequality follows from (A.5) and the independence of $\Gamma^{s,i}$ and $\Gamma^{-i,K}$ (as Γ^i is $\mathcal{F}^i$-adapted and $\Gamma^{-i,K}$ is $\mathcal{F}^{-i}$-adapted and the filtrations are independent).

As K goes to infinity, $\underline{\Gamma}_t^{-i,K}(\tilde{h^{-i}}) \to \underline{\Gamma}_t^{-i}(\tilde{h}^{-i})$ for all $t \geq 0$, $\mathbb{P}$-a.s.. By Lemma 19, T^K converges to some T along a subsequence. By Lebesgue dominated convergence theorem (applied twice) and Theorem 3.7 in El Karoui and Karatzas (1997), we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=0}^{N}\int_0^\infty e^{-rt} r \underline{\Gamma}^{s,i}_{\tilde{T}^{*,i}(t)}\left(\tilde{h}^i\right) d\tilde{T}^{*,i}(t)\right] \\
&\leq \mathbb{E}\left[\int_0^\infty e^{-rt} r \underline{\Gamma}^{s,i}_{T^{*,i}(t)} dT^{*,i}(t) + \int_0^\infty e^{-rt} r \bar{\underline{\Gamma}}^{-i}_{T^{*,-i}(t)}(\tilde{h}^{-i}) dT^{*,-i}(t)\right],
\end{aligned}$$

where T^* is an optimal index strategy for the two arms bandits with rewards $\underline{\Gamma}^i_t$ and $\bar{\underline{\Gamma}}^{-i}_t(\tilde{h}^{-i})$.

Reproducing the same argument for all $j \neq i$, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{N}\int_0^\tau \pi^i\left(X^i_{T^i(t)}\right) dT^i(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)}, d\right)\right] &\leq \mathbb{E}\left[\sum_{0=1}^{N}\int_0^\infty e^{-rt} r \underline{\Gamma}^{s,i}_{T^{s,i}(t)} dT^{s,i}(t)\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-rt} r \bigvee_{i=1}^{N} \underline{\Gamma}^{s,i}_{T^{s,i}(t)} dt\right].
\end{aligned}$$

This concludes the proof.

Supporting Lemmas for the proof of Proposition 11

Let $(\tau^i, d^i) \in \mathcal{P}^{I,r,i}$. As above, define the process h^i as

$$h^i_t := \pi^i(X^i_t)\mathbb{1}_{\{t<\tau^i\}} + r\bar{\pi}^i\left(X^i_{\tau^i}, d^i\right)\mathbb{1}_{\{t\geq\tau^i\}}, \quad t \geq 0$$

Consider the family of stopping problems: for all $t \geq 0$,

$$\tilde{V}^i\left(t, W; \tau^i, d^i\right) := ess \sup_{\rho\in\mathcal{T}^s(t;\tau^i,d^i)} \mathbb{E}\left[\int_t^\rho e^{-r(s-t)} h^i_s ds + e^{-r(s-t)} W \mid \mathcal{F}^i_t\right] \tag{A.6}$$

where

$$\mathcal{T}^s(t;\tau^i,d^i) = \left\{s \geq t \,:\, U_s^i(\tau^i,d^i) = 0\right\},$$

and

$$U_s^i(\tau^i,d^i) = \mathbb{E}\left[e^{-r\tau^i} g^i d^i - \int_s^{\tau^i} e^{-rt} c^i\left(X_t^i\right) dt \mid \mathcal{F}_s^i\right]$$

is the continuation value of worker i. Let $\rho(t, W; \tau^i, d^i)$ be the smallest optimal stopping time in the problem (A.6) with outside option W.[11] In what follows, we will abuse a little notation and denote $\rho(0, W; \tau^i, d^i)$ as $\rho(W; \tau^i, d^i)$.

Lemma 25 *The mapping*

$$W \to \tilde{V}^i\left(0, W; \tau^i, d^i\right) := \sup_{\rho \in \mathcal{T}^s(0;\tau^i,d^i)} \mathbb{E}\left[\int_0^\rho e^{-r(s-t))} h_s^i ds + e^{-r\rho} W\right]$$

is convex, nondecreasing, and locally Lipschitz, with

$$\lim_{W\to\infty} \tilde{V}^i\left(0, W; \tau^i, d^i\right) - W = 0,$$

Proof of Lemma 25. We first show that the $W \to \tilde{V}^i\left(0, W; \tau^i, d^i\right)$ is nondecreasing. Observe that for all $W' \geq W \geq 0$ and all $\rho \in \mathcal{T}^s(0;\tau^i,d^i)$, we have

$$\mathbb{E}\left[\int_0^\rho e^{-rs} h_s^i ds + e^{-r\rho} W'\right] \geq \mathbb{E}\left[\int_0^\rho e^{-rs} h_s^i ds + e^{-r\rho} W\right].$$

It follows that $W \to \tilde{V}^i\left(0, W; \tau^i, d^i\right)$ is nondecreasing.

Next we show that $W \to \tilde{V}^i\left(0, W; \tau^i, d^i\right)$ is convex. Let $\alpha \in (0,1)$ and $W, W' \geq 0$. We

[11] Existence of an optimal stopping time follows from the standard Snell envelope argument.

have

$$\begin{aligned}
&\tilde{V}^i\left(0,\alpha W+(1-\alpha)W';\tau^i,d^i\right)\\
&= ess\sup_{\rho\in\mathcal{T}^s(0;\tau^i,d^i)}\mathbb{E}\left[\int_0^\rho e^{-rs}h_s^i ds+e^{-r\rho}\left(\alpha W+(1-\alpha)W'\right)\right]\\
&\leq \alpha ess\sup_{\rho\in\mathcal{T}^s(0;\tau^i,d^i)}\mathbb{E}\left[\int_0^\rho e^{-rs}h_s^i ds+e^{-r\rho}W\right]\\
&\quad +(1-\alpha)ess\sup_{\rho\in\mathcal{T}^s(t;\tau^i,d^i)}\mathbb{E}\left[\int_0^\rho e^{-rs}h_s^i ds+e^{-r\rho}W'\right]\\
&= \alpha\tilde{V}^i\left(0,W;\tau^i,d^i\right)+(1-\alpha)\tilde{V}^i\left(0,W';\tau^i,d^i\right);
\end{aligned}$$

where we used that the supremum of the sum is less than the sum of the supremum. So $W\to\tilde{V}^i\left(0,W;\tau^i,d^i\right)$ is convex on $[0,\infty)$.

Taken together, these first two results implies that $W\to\tilde{V}^i\left(0,W;\tau^i,d^i\right)$ is locally Lipschitz, as a convex function is locally Lipschitz in the interior of its domain and $W\to\tilde{V}^i\left(0,W;\tau^i,d^i\right)$ being nondecreasing implies that it is continuous at zero.

There remains to show that $\tilde{V}^i(0,W;\tau^i,d^i)-W\to 0$. Note that

$$\begin{aligned}
\tilde{V}^i(0,W;\tau^i,d^i)-W &\leq \mathbb{E}\left[\int_0^\infty e^{-rt}\left(\pi^i\left(X_t^i\right)-W\right)^+dt\right]\\
&\to 0,
\end{aligned}$$

as $W\to\infty$ by the monotone convergence theorem. Since $\tilde{V}^i(0,W;\tau^i,d^i)\geq 0$ (as $U_{0^-}^i(\tau^i,d^i)=0$ by convention), we have the desired result. ■

Lemma 26 *The mapping* $W\to\tilde{V}^i\left(0,W;\tau^i,d^i\right)$ *is differentiable almost everywhere* $\mathbb{P}$*-a.s. with*

$$\frac{\partial\tilde{V}^i\left(0,W;\tau^i,d^i\right)}{\partial W}=\mathbb{E}\left[e^{-r\rho\left(W;\tau^i,d^i\right)}\right],\quad a.e.. \tag{A.7}$$

Proof of Lemma 26. Let $\delta > 0$. Observe that

$$\begin{aligned}\tilde{V}(0, W+\delta; \tau^i, d^i) &= \mathbb{E}\left[\int_0^{\rho(W+\delta;\tau^i,d^i)} e^{-rt} h_s^i ds + e^{-\rho(W+\delta;\tau^i,d^i)}(W+\delta)\right] \\ &\geq \mathbb{E}\left[\int_0^{\rho(W;\tau^i,d^i)} e^{-rt} h_s^i ds + e^{-\rho(W;\tau^i,d^i)}(W+\delta)\right] \\ &= \tilde{V}(0, W; \tau^i, d^i) + \delta \mathbb{E}\left[e^{-r\rho(W;\tau^i,d^i)}\right].\end{aligned}$$

Similarly,

$$\begin{aligned}\tilde{V}(0, W-\delta; \tau^i, d^i) &\geq \mathbb{E}\left[\int_0^{\rho(W;\tau^i,d^i)} e^{-rt} h_s^i ds + e^{-\rho(W;\tau^i,d^i)}(W-\delta)\right] \\ &= \tilde{V}(0, W; \tau^i, d^i) - \delta \mathbb{E}\left[e^{-r\rho(W;\tau^i,d^i)}\right].\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\tilde{V}^i(0, W; \tau^i, d^i) - \tilde{V}^i(0, W-\delta; \tau^i, d^i)}{\delta} &\leq \mathbb{E}\left[e^{-r\rho(W;\tau^i,d^i)}\right] \\ &\leq \frac{\tilde{V}^i(0, W+\delta; \tau^i, d^i) - \tilde{V}^i(0, W; \tau^i, d^i)}{\delta}.\end{aligned}$$

By Alexandrov's Theorem (Theorem 7.28 in Aliprantis and Border (2006)), $W \to \tilde{V}^i(0, W + \delta; \tau^i, d^i)$ is differentiable almost everywhere and it derivative is continuous almost everywhere. Letting $\delta \to 0$, for almost every W, we then have

$$\frac{\partial \tilde{V}^i(0, W; \tau^i, d^i)}{\partial W} \leq \mathbb{E}\left[e^{-r\rho(W;\tau^i,d^i)}\right] \leq \frac{\partial \tilde{V}^i(0, W; \tau^i, d^i)}{\partial W}.$$

This concludes the proof. ■

For all $i \in \{1, \ldots, N\}$, define the h^i-index processes

$$\Gamma_t^i\left(h^i\right) \coloneqq \inf\left\{W \geq 0 \,:\, \tilde{V}^i\left(u, W; \tau^i, d^i\right) = W \text{ for } u = \sup\left\{y \leq t \,:\, U_y^i(\tau^i, d^i) = 0\right\}\right\},$$

and their lower envelope $\underline{\Gamma}_t^i\left(h^i\right) \coloneqq \inf_{0 \leq s \leq t} \Gamma_s^i(h^i)$.

Lemma 27 *For all W,*

$$\begin{aligned}\tilde{V}^i\left(0, W; \tau^i, d^i\right) &= \mathbb{E}\left[\int_0^{\rho(W;\tau^i,d^i)} h_s^i ds + e^{-r\rho(W;\tau^i,d^i)}W\right]\\ &= \mathbb{E}\left[\int_0^{\rho(W;\tau^i,d^i)} e^{-rt} r \underline{\Gamma}_t^i\left(h^i\right) dt + e^{-r\rho(W;\tau^i,d^i)}W\right].\end{aligned}$$

Proof of Lemma 27. By Lemma A.7,

$$\frac{\partial \tilde{V}^i\left(0, W; \tau^i, d^i\right)}{\partial W} = \mathbb{E}\left[e^{-r\rho\left(W;\tau^i,d^i\right)}\right] \text{ a.e..}$$

By Lemma 25, $\tilde{V}^i\left(0, W; \tau^i, d^i\right)$ is locally Lipschitz, hence absolutely continuous. Therefore

$$\tilde{V}^i\left(0, \bar{W}; \tau^i, d^i\right) - \bar{W} - \tilde{V}^i\left(0, W; \tau^i, d^i\right) + W = \int_W^{\bar{W}} \left(\mathbb{E}\left[e^{-r\rho\left(\tilde{W};\tau^i,d^i\right)}\right] - 1\right) d\tilde{W}.$$

Letting $\bar{W} \to \infty$ and using Lemma 25, we get

$$\tilde{V}^i\left(0, W; \tau^i, d^i\right) - W = \int_W^{\infty} \left(1 - \mathbb{E}\left[e^{-r\rho\left(\tilde{W};\tau^i,d^i\right)}\right]\right) d\tilde{W}.$$

Then

$$\begin{aligned}
\tilde{V}^i\left(0, W; \tau^i, d^i\right) - W &= \mathbb{E}\left[\int_W^\infty \left(1 - e^{-r\rho(\tilde{W};\tau^i,d^i)}\right) d\tilde{W}\right] \\
&= \mathbb{E}\left[\int_W^\infty r \int_0^\infty e^{-rt} \mathbb{1}_{\{t \le \rho\left(\tilde{W};\tau^i,d^i\right)\}} dt d\tilde{W}\right] \\
&= \mathbb{E}\left[\int_0^\infty r e^{-rt} \int_W^\infty \mathbb{1}_{\{t \le \rho\left(\tilde{W},\tau^i,d^i\right)\}} d\tilde{W} dt\right],
\end{aligned}$$

by Fubini's theorem (twice). But,

$$\begin{aligned}
t \le \rho(\tilde{W}; \tau^i, d^i) &\Leftrightarrow \forall s \le t \text{ such that } U_s^i(\tau^i, d^i) = 0,\ \tilde{V}^i\left(s, \tilde{W}; \tau^i, d^i\right) > \tilde{W} \\
&\Leftrightarrow \forall s \le t,\ \Gamma_s^i\left(h^i\right) > \tilde{W} \\
&\Leftrightarrow \underline{\Gamma}_t^i\left(h^i\right) > \tilde{W}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\tilde{V}^i\left(0, W; \tau^i, d^i\right) - W &= \mathbb{E}\left[\int_0^\infty r e^{-rt} \int_W^\infty \mathbb{1}_{\{\underline{\Gamma}_t^i(h^i) > \tilde{W}\}} d\tilde{W} dt\right] \\
&= \mathbb{E}\left[\int_0^\infty r e^{-rt} \left(\underline{\Gamma}_t^i(h^i) - W\right)^+ dt\right];
\end{aligned}$$

and the results follows. ■

Finally,

Proposition 30 (Whittle Computation)

$$\begin{aligned}
\Phi(0, W) &:= \sup_{(T,\bar{\tau}^1,\ldots,\bar{\tau}^N), T^i(\bar{\tau}^i) \in \mathcal{T}^s(0;\tau^i,d^i)} \mathbb{E}\left[\sum_{i=1}^N \int_0^{\bar{\tau}^i} e^{-rt} h_{T^i(t)}^i dT^i(t) + e^{-r \bigvee \bar{\tau}^i} W\right] \\
&\le \sup_T \mathbb{E}\left[\sum_{i=1}^N \int_0^\infty e^{-rt} r \underline{\Gamma}_{T^i(t)}^i\left(h^i\right) \vee W dT^i(t)\right].
\end{aligned}$$

Proof of Proposition 30. This follows from the double inequality: on $\{U_t^i(\tau^i, d^i) = 0\}$,

$$\Phi^{-i}(\bar{t}^{-i}, W) \vee \tilde{V}^i\left(t^i, W; \tau^i, d^i\right) \leq \Phi(\bar{t}, W) \leq \Phi^{-i}(\bar{t}^{-i}, W) + \tilde{V}^i\left(t^i, W; \tau^i, d^i\right) - W.$$

But then, on $\left\{\tilde{V}^i(t, W; \tau^i, d^i) = W\right\} \cap \{U_t^i(\tau^i, d^i) = 0\}$, $\Phi^{-i}(\bar{t}^{-i}, W) = \Phi(\bar{t}, W)$, and it is optimal to retire arm i. It follows that the optimal stopping time $\tau(W)$ is weakly smaller than $\sum_i^N \rho^i(W; \tau^i, d^i)$. Then, by the same argument as in the proof of Lemma 26,

$$\frac{\partial \Phi(0, W)}{\partial W} = \mathbb{E}\left[e^{-r\tau(W)}\right].$$

Integrating, we get

$$\begin{aligned} \Phi(0, W) - W &= \int_W^\infty \left(1 - \mathbb{E}\left[e^{-r\tau(\tilde{W})}\right]\right) d\tilde{W} \\ &\leq \mathbb{E}\left[\int_W^\infty \left(1 - e^{-r\sum_i^N \rho^i\left(\tilde{W}; \tau^i, d^i\right)}\right) d\tilde{W}\right] \end{aligned}$$

where the inequality follows from $\tau(W) \leq \sum_i^N \rho^i(W; \tau^i, d^i)$. By Theorem 3.7 in El Karoui and Karatzas (1997),

$$\mathbb{E}\left[\int_W^\infty \left(1 - e^{-r\sum_i^N \rho^i\left(\tilde{W}; \tau^i, d^i\right)}\right) d\tilde{W}\right] + W$$

is the value of the bandit problem with decreasing rewards $\Gamma_t^i(h_t^i)$, and the result follows. ■

A.2 Appendix B

A.2.1 The strategic index policy

We construct the index delegation rule T^{Γ} associated with the index processes $\left(\Gamma^1, \ldots, \Gamma^N\right)$ following El Karoui and Karatzas (1997). We define T^{Γ} pointwise on Ω. Let $\omega \in \Omega$, and define

- $\sigma^i(W) = \inf\{t \geq 0 \,:\, \Gamma^i_t \leq W\}$.
- D^i is the set of discontinuities of the function $W \to \sigma^i(W)$. $D \coloneqq \bigcup_{i=1}^N D^i$.
- $\mathcal{D}^i = \left\{t \geq 0 \,:\, \sigma^i\left(\underline{\Gamma}^i_{t^-}\right) > t\right\} = \bigcup_{W \in D^i}[\sigma^i(W), \sigma^i(W^-))$. The intervals in $\mathcal{D}^i$ are the flat stretches of the function $\underline{\Gamma}^i_t$. $\mathcal{D} \coloneqq \bigcup_{i=1}^N \mathcal{D}^i$.
- B^i is the set of discontinuities of the function $t \to \underline{\Gamma}^i_t$. $B \coloneqq \bigcup_{i=1}^N B^i$.
- $\mathcal{B}^i \coloneqq \left\{W > 0 \,:\, \underline{\Gamma}^i_{\sigma^i(W^-)} < W\right\} = \bigcup_{t \in B^i}(\underline{\Gamma}^i_t, \underline{\Gamma}^i_{t^-}]$. The intervals in $\mathcal{B}^i$ are the flat stretches of the function $W \to \sigma^i(W)$. $\mathcal{B} \coloneqq \bigcup_{i=1}^N \mathcal{B}^i$.
- $\tau^0(W) = \sum_{i=1}^d \sigma^i(W)$, $0 \leq m < \infty$.
- $N(t) = \inf\{W \geq 0 \,:\, \tau^0(W) \leq t\}$, $0 \leq t, W < \infty$.

For all $i = 1, \ldots, N$, and all $t \notin \mathcal{D}$, define

$$T^{\Gamma,i}(t) \coloneqq \sigma^i(N(t)^-).$$

For $t \in \mathcal{D}$, we still need to decide which arm to pull in the case that more than one arm achieves the highest index. In that case, we specify a priority rule: if the indices of two or more workers are the same at a time point of discontinuity, the principal delegates to the worker with smallest i. Formally, for $t \in \mathcal{D}$, observe that $t \in [\tau^0(W), \tau^0(W^-))$ with $W = N(t) \in D$. Define then

$$y^0 = y^0(W) \coloneqq \tau^0(W),$$

and, recursively,

$$y^i \coloneqq y^i(W) \coloneqq y^{i-1}(W) - \Delta\sigma^i(W) = \sum_{j=1}^{i} \sigma^j(W^-) + \sum_{j=i+1}^{N} \sigma^j(W),$$

where $\Delta\sigma^i(W) \coloneqq \sigma^i(W) - \sigma^i(W^-)$. Set $L^i(W) \coloneqq [y^{i-1}(W), y^i(W))$, so that $L(m) = \bigcup_{i=1}^{N} L^i(m)$. In particular, $y^N = \tau^0(W^-)$, and $L^i(W) = \emptyset$ if σ^i is continuous at W. Now find the unique $k = k(t) \in \{1, \ldots, N\}$ for which $t \in L^k(m)$, and write

$$\sum_{i=1}^{N} T^{\Gamma,i}(t) = (t - y^{k-1}) + y^{k-1} = \sum_{j=1}^{k-1} \sigma^j(W^-) + (t - y^{k-1} + \sigma^k(W)) \sum_{j=k+1}^{N} \sigma^j(W).$$

We then take

$$T^{\Gamma,i}(t) \coloneqq \begin{cases} \sigma^i(W^-) \text{ if } i = 1, \ldots, k(t) - 1 \\ \sigma^i(W) + t - y^{k(t)-1} \text{ if } i = k(t) \\ \sigma^i(W) \text{ if } i = k(t) + 1, \ldots, N \end{cases} \tag{A.8}$$

for $t \in \mathcal{D}$.

Proposition 31 *The vector T^Γ is an index delegation rule associated with the index processes $(\Gamma^1, \ldots, \Gamma^N)$.*

Proof of proposition 31. We show that $T^{\Gamma}(t)$ is flat off the set

$$\left\{t \geq 0 \,:\, \underline{\Gamma}^{i}_{T^{\Gamma,i}(t)} = \bigvee_{j=1}^{N} \underline{\Gamma}^{j}_{T^{\Gamma,j}(t)}\right\} \ \mathbb{P}\text{-a.s..}$$

Observe that, by construction, for all $W \geq 0$,

$$\bigvee_{j=1}^{N} \underline{\Gamma}^{j}_{T^{\Gamma,j}(t)} \leq W \Leftrightarrow \underline{\Gamma}^{j}_{T^{\Gamma,j}(t)} \leq W \text{ for all } i \Leftrightarrow \sigma^{i}(W) \leq T^{\Gamma,i}(t) \text{ for all } i$$
$$\Rightarrow \tau^{0}(W) \leq t \Leftrightarrow N(t) \leq W \text{ for all } 0 \leq t.$$

Moreover, by construction, we also have, for all $t \geq 0$,

$$\underline{\Gamma}^{j}_{\sigma^{i}(N(t)^{-})} \leq \underline{\Gamma}^{j}_{T^{\Gamma,i}(t)} \leq \underline{\Gamma}^{j}_{\sigma^{i}(N(t))} \leq N(t);$$

using that $T^{\Gamma,i}$ is nondecreasing, since $T^{\Gamma,i}(\tau^{0}(W)) = \sigma^{i}(W)$ for all $W \geq 0$ implies that $T^{\Gamma,i}(\tau^{0}(N(t))) \leq T^{\Gamma,i}(t) \leq T^{\Gamma,i}(\tau^{0}(N(t)^{-}))$.

So $\bigvee_{j=1}^{N} \underline{\Gamma}^{j}_{T^{\Gamma,j}(t)} \leq N(t)$, and, thus, $\bigvee_{j=1}^{N} \underline{\Gamma}^{j}_{T^{\Gamma,j}(t)} = N(t)$.

Then $N(t) \notin \mathcal{B}^{i} \Rightarrow \underline{\Gamma}^{i}_{T^{\Gamma,i}(t)} = N(t)$, and, therefore,

$$\begin{aligned} 0 \leq \int_{0}^{\infty} \mathbb{1}_{\{\underline{\Gamma}^{i}_{T^{\Gamma,i}(t)} < \bigvee_{j=1}^{N} \underline{\Gamma}^{j}_{T^{\Gamma,j}(t)}\}} dT^{\Gamma,i}(t) &= \int_{0}^{\infty} \mathbb{1}_{\{\underline{\Gamma}^{i}_{T^{\Gamma,i}(t)} < N(t)\}} dT^{\Gamma,i}(t) \\ &\leq \int_{0}^{\infty} \mathbb{1}_{\{N(t) \in \mathcal{B}^{i}\}} dT^{\Gamma,i}(t) = 0, \end{aligned}$$

where the last equality holds as $N(t) \in \mathcal{B}^{i}$ implies that $T^{\Gamma,i}(t^{-}) = \sigma^{i}(N(t)^{-}) = \sigma^{i}(N(t)) = T^{\Gamma,i}(t)$ is flat at t. Then

$$\sum_{i=1}^{N} \int_{0}^{\infty} \mathbb{1}_{\{\underline{\Gamma}^{i}_{T^{\Gamma,i}(t)} < \bigvee_{j=1}^{N} \underline{\Gamma}^{j}_{T^{\Gamma,j}(t)}\}} dT^{\Gamma,i}(t) = 0,$$

and the result follows. ■

Next we recall one classic results from the study of bandit problems, which is used in the proof of Proposition 2.

Lemma 28 *For all $i = 1, \ldots, N$,*

$$\tilde{\mathbb{E}}\left[\int_0^\infty e^{-rt} h^{s,i}_{T^{s,i}(t)} dT^{s,i}(t)\right] = \mathbb{E}\left[\int_0^\infty e^{-rt} r \underline{\Gamma}^{s,i}_{T^{s,i}(t)} dT^{s,i}(t)\right]. \tag{A.9}$$

A similar statement is used in the proof of Theorem 8.1 in El Karoui and Karatzas (1994); and a proof follows from the arguments there and from Lemma 7.5 in El Karoui and Karatzas (1997). We reproduce it below for completeness.

Proof of lemma 28. By proposition 3.2 in El Karoui and Karatzas (1994),

$$U^i_t = e^{-rt}\left[V^i\left(t; \underline{\Gamma}^{s,i}_t\right) - \underline{\Gamma}^{s,i}_t\right] + \int_0^t e^{-ru}\left(h^{s,i}_u - r\underline{\Gamma}^{s,i}_u\right) du$$

is a $\mathcal{F}^i$-martingale with càdlàg paths, and, hence, by lemma 4.6 in El Karoui and Karatzas (1997) an $\tilde{\mathcal{F}}^i$-martingale. Then

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\int_0^\infty e^{-r(t-T^{s,i}(t))} dU^i_{T^{s,i}(t)}\right] &= \tilde{\mathbb{E}}\Bigg[\int_0^\infty e^{-rt}\left(h^{s,i}_{T^{s,i}(t)} - r\underline{\Gamma}^i_{T^{s,i}(t)}\right) dT^{s,i}(t) \\
&\quad + \int_0^\infty e^{-r(t-T^{s,i}(t))} d\left(e^{-rT^{s,i}(t)}\left(V^i\left(T^{s,i}(t); \underline{\Gamma}^i_{T^{s,i}(t)}\right) - \underline{\Gamma}^i_{T^{s,i}(t)}\right)\right)\Bigg] \\
&= 0.
\end{aligned}$$

Observe that, by lemma 7.5 in El Karoui and Karatzas (1997) and the definition of the strategic index policy,

$$\begin{aligned}
&\int_0^\infty e^{-r(t-T^{s,i}(t))} d\left(e^{-rT^{s,i}(t)}\left(V^i\left(T^{s,i}(t);\underline{\Gamma}^i_{T^{s,i}(t)}\right)-\underline{\Gamma}^i_{T^{s,i}(t)}\right)\right)\\
&\qquad=\sum_{m\in D^i}\int_{y_{i-1}(m)}^{y_i(m)} e^{-r(t-T^{s,i}(t))} d\left(e^{-rT^{s,i}(t)}\left(V^i\left(T^{s,i}(t);\underline{\Gamma}^i_{T^{s,i}(t)}\right)-\underline{\Gamma}^i_{T^{s,i}(t)}\right)\right)\\
&\qquad=\sum_{m\in D^i} e^{-r(y_{i-1}(m)-\sigma^i(m))}\left(e^{-rT^{s,i}(t)}\left(V^i\left(T^{s,i}(t);\underline{\Gamma}^i_{T^{s,i}(t)}\right)-\underline{\Gamma}^i_{T^{s,i}(t)}\right)\right)\Bigg|_{t=y_{i-1}(m)}^{t=y_i(m)}\\
&\qquad=\sum_{m\in D^i} e^{-ry_{i-1}(m)}\Bigg(e^{-r\Delta\sigma^i(m)}\left(V^i\left(T^{s,i}(y_i(m));\underline{\Gamma}^i_{T^{s,i}(y_i(m))}\right)-\underline{\Gamma}^i_{T^{s,i}(y_i(m))}\right)\\
&\qquad\qquad-\left(V^i\left(T^{s,i}(y_{i-1}(m));\underline{\Gamma}^i_{T^{s,i}(y_{i-1}(m))}\right)-\underline{\Gamma}^i_{T^{s,i}(y_{i-1}(m))}\right)\Bigg).
\end{aligned}$$

Lemma 7.5 in El Karoui and Karatzas (1997) again implies that $\mathbb{P}$-a.s.

$$\begin{aligned}
V^i\left(T^{s,i}(y_{k-1}(m));\underline{\Gamma}^i_{T^{s,i}(y_{k-1}(m))}\right)-\underline{\Gamma}^i_{T^{s,i}(y_{k-1}(m))}&=0\\
&=V^i\left(T^{s,i}(y_k(m));\underline{\Gamma}^i_{T^{s,i}(y_k(m))}\right)-\underline{\Gamma}^i_{T^{s,i}(y_k(m))}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\int_0^\infty e^{-r(t-T^{s,i}(t))} dU^i_{T^{s,i}(t)}\right]&=\tilde{\mathbb{E}}\left[\int_0^\infty e^{-rt}\left(h^{*,i}_{T^{s,i}(t)}-r\underline{\Gamma}^i_{T^{s,i}(t)}\right)dT^{s,i}(t)\right]\\
&=0;
\end{aligned}$$

and (A.9) holds. ■

A.2.2 Useful Results on Optimal Stopping

I will be interested in the following problem. Let $Y_t \in D_r$. Consider

$$\sup_{\tau \in \mathcal{T}} \mathbb{E}\left[Y_\tau\right]. \tag{A.10}$$

Lemma 29 *τ solves (A.10) if and only if, for all $\tau' \leq \tau$,*

$$\mathbb{E}[Y_\tau \mid \mathcal{F}_{\tau'}] \geq \mathbb{E}[Y_{\tau'} \mid \mathcal{F}'_\tau],$$

and, for all $\tau' \geq \tau$,

$$\mathbb{E}[Y_\tau \mid \mathcal{F}_\tau] \geq \mathbb{E}[Y_{\tau'} \mid \mathcal{F}_\tau].$$

Proof of lemma 29. ($\Rightarrow$) This is immediate. To see this, observe that the contrapositive is the following. Suppose that there exists $\tau' \leq \tau$ such that

$$\mathbb{E}[Y_\tau \mid \mathcal{F}_{\tau'}] < \mathbb{E}[Y_{\tau'} \mid \mathcal{F}'_\tau],$$

or $\tau' \geq \tau$ such that

$$\mathbb{E}[Y_\tau \mid \mathcal{F}_\tau] < \mathbb{E}[Y_{\tau'} \mid \mathcal{F}_\tau],$$

then τ does not solve (A.10), which is obviously true.

($\Leftarrow$) Let τ be a Markov time and suppose that, for all $\tau' \leq \tau$,

$$\mathbb{E}[Y_\tau \mid \mathcal{F}_{\tau'}] \geq \mathbb{E}[Y_{\tau'} \mid \mathcal{F}_{\tau'}],$$

and, for all $\tau' \geq \tau$,

$$\mathbb{E}[Y_\tau \mid \mathcal{F}_\tau] \geq \mathbb{E}[Y_{\tau'} \mid \mathcal{F}_\tau].$$

Let $\tilde{\tau}$ be any Markov time. I then have

$$\begin{aligned}
\mathbb{E}[Y_\tau] &= \mathbb{E}\left[Y_\tau \mathbb{1}_{\{\tau \leq \tilde{\tau}\}} + Y_\tau \mathbb{1}_{\{\tau > \tilde{\tau}\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[Y_\tau \mid \mathcal{F}_\tau\right] \mathbb{1}_{\{\tau \leq \tilde{\tau}\}} + \mathbb{E}\left[Y_\tau \mid \mathcal{F}_{\tilde{\tau}}\right] \mathbb{1}_{\{\tau > \tilde{\tau}\}}\right] \\
&\geq \mathbb{E}\left[\mathbb{E}\left[Y_{\tau \vee \tilde{\tau}} \mid \mathcal{F}_\tau\right] \mathbb{1}_{\{\tau \leq \tilde{\tau}\}} + \mathbb{E}\left[Y_{\tau \wedge \tilde{\tau}} \mid \mathcal{F}_{\tilde{\tau}}\right] \mathbb{1}_{\{\tau > \tilde{\tau}\}}\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[Y_{\tilde{\tau}} \mathbb{1}_{\{\tau \leq \tilde{\tau}\}} \mid \mathcal{F}_\tau\right] + \mathbb{E}\left[Y_{\tilde{\tau}} \mathbb{1}_{\{\tau > \tilde{\tau}\}} \mid \mathcal{F}_{\tilde{\tau}}\right]\right] \\
&= \mathbb{E}\left[Y_{\tilde{\tau}}\right]
\end{aligned}$$

where the second equality follows from the law of iterated expectations and the $\mathbb{F}_\tau \wedge \mathcal{F}_{\tilde{\tau}}$-measurability of $\mathbb{1}_{\{\tau > \tilde{\tau}\}}$ and $\mathbb{1}_{\{\tau \leq \tilde{\tau}\}}$, the inequality follows by assumption, the third equality follows from the same measurability conditions, and the last equality from the law of iterated expectation again. Since $\tilde{\tau}$ was arbitrary, τ solves (A.10). ■

Finally, I need the following extension of Theorem 2.4 in Peskir and Shiryaev (2006), which relaxes the assumption on G and V.

Theorem 11 (Theorem 2.4 in Peskir and Shiryaev (2006)) *Let X be a càdlàg Feller process with values in $\mathcal{X}$ defined on the filtered probability space $\left(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, \mathbb{P}\right)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions. Let π, G be measurable function from $\mathcal{X}$ to $\mathbb{R}$ such that*

$$\mathbb{E}\left[\sup_{0\leq t\leq\infty} \int_0^t e^{-rt}\pi(X_t)dt + e^{-rt}G(X_t)\right] < \infty.$$

Consider the family of optimal stopping problem

$$V_t := \operatorname*{esssup}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau e^{-r(s-t)} p(X_s) ds + e^{-r(\tau - t)} G(X_\tau) \mid \mathcal{F}_t \right], \tag{A.11}$$

where the sup is taken over all $\mathcal{F}_t$-Markov time $\tau \geq t$. Then $V_t = V(X_t)$ $\mathbb{P}$-a.s., where

$$V(X_t) := \operatorname*{esssup}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau e^{-r(s-t)} p(X_s) ds + e^{-r(\tau - t)} G(X_\tau) \mid X_t \right],$$

and, for all Markov time $\theta \geq t$,

$$\begin{aligned} V_t &= V(X_t) \\ &= \operatorname*{esssup}_{\tau \geq t} \mathbb{E}\left[\int_t^{\tau\wedge\theta} e^{-r(s-t)} p(X_s) ds + e^{-r(\tau\wedge\theta - t)} \left(V(X_\theta) \mathbb{1}_{\{\theta \leq \tau\}} + G(X_\tau) \mathbb{1}_{\{\tau < \theta\}} \right) \mid X_t \right]. \end{aligned}$$

Finally the Markov time

$$\tau_t^* := \inf \left\{ s \geq t \,:\, V(X_s) = G(X_s) \right\},$$

is the smallest optimal Markov time for V_t.

Proof of theorem 11. Consider the family of optimal stopping problems

$$V_t := \operatorname*{esssup}_{\tau \geq t} \mathbb{E}\left[\int_t^\tau e^{-r(s-t)} p(X_s) ds + e^{-r(\tau - t)} G(X_\tau) \mid \mathcal{F}_t \right], \tag{A.12}$$

Note that V_t is a well-defined $\mathcal{F}_t$-measurable random variable by lemma 1.3 in Peskir and Shiryaev (2006), so that the process $\{V_t\}_{t\geq 0}$ is adapted.

Define

$$Z_t := \int_0^t e^{-rs} p(X_s) ds + e^{-rt} V_t.$$

Z_t is the Snell envelope of (i.e., the smallest supermartingale with càdlàg paths that dominates) the process

$$Y_t := \int_0^t e^{-rs} p(X_s^i) ds + e^{-rt} G(X_t).$$

By theorem 2.2 in Peskir and Shiryaev (2006),

$$\begin{aligned}\tau_t^* &:= \inf\{s \geq t \,:\, Y_s = Z_s\} \\ &= \inf\{s \geq t \,:\, V_s = G(X_s)\}\end{aligned}$$

is the smallest optimal Markov time for the problem (A.12). Furthermore,

$$\left\{Z_{s\wedge\tau_t^*},\, \mathcal{F}_s,\, t \leq s \leq \infty\right\}$$

is a martingale. The processes Y and Z are càdlàg, progressively measurable, nonnegative, and agrees at $t = \infty$. Furthermore $\mathbb{E}\left[\sup_{0\leq t\leq\infty} Y_t\right] < \infty$. It follows that Z is class D, hence uniformly integrable.

By the optional sampling theorem for uniformly integrable càdlàg martingale (Theorem 7.29 in Kallenberg (2006)), for all Markov time $\theta \geq t$,

$$V_t = \underset{\tau\geq 0}{ess\sup}\, \mathbb{E}\left[\int_t^{\tau\wedge\theta} e^{-r(s-t)} p(X_s) ds + e^{-r(\tau\wedge\theta-t)} \left(V_\theta \mathbb{1}_{\{\theta\leq\tau\}} + G(X_s)\mathbb{1}_{\{\tau<\theta\}}\right) \mid \mathcal{F}_t\right]. \quad \text{(A.13)}$$

But, by the strong Markov property for Feller processes (Theorem 19.17 in Kallenberg (2006)), for all stopping time $\tau < \infty$,

$$V_\tau = V(X_\tau)\, \mathbb{P}\text{-a.s.}.$$

In particular, V_τ does not depend on the history prior to τ, and equation (A.13) becomes, for all Markov time $\theta \geq t$, $\theta \mathcal{F}^1_{t^-}$,

$$V(X_t) = \underset{\tau \geq t}{esssup}\, \mathbb{E}\left[\int_t^{\tau \wedge \theta} e^{-r(s-t)} p(X_s) ds + e^{-r(\tau \wedge \theta - t)} \left(V(X_\theta) \mathbb{1}_{\{\theta \leq \tau\}} + G(X_\tau) \mathbb{1}_{\{\tau < \theta\}} \right) \mid X_t \right].$$

Finally τ^*_t becomes

$$\tau^*_t = \inf \left\{ s \geq t \,:\, V(X_s) = G(X_\tau) \right\}.$$

This concludes the proof. ■

A.3 Supplemental Appendix

A.3.1 Omitted Proofs for Section 1.6.1

Proof of Corollary 2. Suppose that Assumption 7 is violated. Then there exists $i \in \{1, \dots, N\}$ such that

$$\begin{aligned} \sup_{(\tau,d) \in \mathcal{P}^{I,r,i}} & \mathbb{E}\left[\int_0^\tau e^{-rt} \pi^i \left(X^i_t \right) dt + e^{-r\tau} \left((1 - d^0_\tau) \int_\tau^\infty e^{-r(t-\tau)} \pi^i \left(X^i_t \right) dt + d^0_\tau W \right) \right] \\ & \leq \mathbb{E}\left[\int_0^\infty e^{-rt} \pi^i \left(X^i_t \right) dt \right]. \end{aligned}$$

Let $\epsilon > 0$. Define W^i as

$$W^i \coloneqq \inf \left\{ W \,:\, \text{Assumption 7 holds with } W \right\},$$

and let $\tilde{W} = \bigvee_{i=1}^N W^i + \epsilon > W$. For $\tilde{W}$, Assumption 7 holds. So, by Theorem 1, the optimal implementable promotion contest is the index contest. Letting $\epsilon \to 0$, we see that the index contest is also optimal with outside option $\tilde{W}$. Finally observe that for i such that $W^i = \bigvee_{i=1}^N W^i$, it must be that, for all x such that $\mathbb{P}\left(\tau^i_{(-\infty,x]} < \infty\right) > 0$, with $\tau^i_{(-\infty,x]} := \inf\{t \geq 0 \,:\, X^i_t \leq x\}$,

$$\mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i\left(X^i_t\right)dt \mid X^i_0 = x\right] \geq W^i.$$

But $\Gamma^{s,i}_t \geq \mathbb{E}\left[\int_0^\infty e^{-rt}\pi^i\left(X^i_t\right)dt \mid X^i_t\right]$ for all $t \geq 0$, $\mathbb{P}$-a.s.. Therefore, the principal never takes the outside option $\tilde{W}$, and thus the index contest is optimal in the original problem too. ■

Proof of Corollary 3. For simplicity, suppose that Assumption 5 holds for all $j \neq i$. Suppose that, for all $n \in \mathbb{N}$, $X^{i,n}$ satisfies Assumption 5, and that $X^i = \lim_{n\to\infty} X^{i,n}$ uniformly on compact sets $\mathbb{P}$-a.s.. Let $\Gamma^{s,i,n}$ be the strategic index process associated with worker i and $\tau^{s,i,n}$ his promotion time when his type process is given by $X^{i,n}$. Define also $\tau^{s,i} := \inf\left\{t \geq 0 \,:\, X^i_t \geq \bar{P}^i\left(\underline{X}^i_t\right)\right\}$. Observe then that, for almost all $t \geq 0$,

$$\underline{\Gamma}^{s,i,n}_t \to \Gamma^{s,i}_t, \quad \mathbb{P}-a.s.,$$

where

$$r\Gamma^{s,i}_t := \sup_{\tau>0}\frac{\mathbb{E}\left[\int_t^\tau e^{-r(s-t)}\pi^i\left(X^i_s\right)\mathbb{1}_{\{s\leq\tau^{s,i}\}} + \bar{\pi}^i\left(X^i_{\tau^{s,i}}\right)\mathbb{1} \mid \mathcal{F}^i_t\right]}{\mathbb{E}\left[\int_t^\tau e^{-r(s-t)}ds \mid \mathcal{F}^i_t\right]}.$$

For all $n \in \mathbb{N}$, by Theorem 1, the principal's value is given by

$$\mathbb{E}\left[\int_0^{\tau^n} e^{-rt}\pi^i\left(X^i_{T^i_n(t)}\right)dT^i_n(t) + \sum_{j\neq i}\int_0^{\tau^n} e^{-rt}\pi^j\left(X^j_{T^j_n(t)}\right)dT^j_n(t) + e^{-r\tau^n}\bar{\pi}\left(X^n_{T_n(\tau^n)}, d^n\right)\right],$$

where T_n is the index rule associated with the indices $\Gamma^{s,j}$'s, $j \neq i$, and $\Gamma^{s,i,n}$,
$\tau^n := \inf\{t \geq 0 \,:\, T^j(t) \geq \tau^{s,j} \text{ or } T^i(t) \geq \tau^{s,i,n}\}$, and d^n is the optimal promotion rule.

Next, observe that if there exists x such that $\bar{P}^i(x) = x$, $\tau^{s,i,n} \not\to \tau^{s,i}$. To see this, simply note that, for all $n \in \mathbb{N}$, $\mathbb{P}\left(\tau^i = 0 \mid \underline{X}_t^i = x\right) = 0$. But, by Lemma 2 in Pennanen and Perkkiö (2018), $\tau^{s,i,n} \to \tau^i \in \mathcal{S}\left(\mathcal{F}^i\right)$, at least along a subsequence.

So, as $n \to \infty$, passing to a subsequence if necessary, $T_n \to T$ where T is the index rule associated with the strategic indices $\Gamma^{s,j}$'s, $\tau^n \to \tau^*$ with $\tau^* = \tau^i \bigwedge_{j \neq i} \tau^{s,j}$, and $d^n \to d$ with $d_t^i = 1$ only if $T^i(t) \geq \tau^i$ and $d_t^j = 1$ only if $T^j(t) \geq \tau^{s,j}$, $j \neq i$. By Theorem 6.39 in Aliprantis and Border (2006) and the Lebesgue dominated convergence theorem,

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^{\tau^n} e^{-rt} \pi^i \left(X^i_{T^i_n(t)}\right) dT^i_n(t) + \sum_{j \neq i} \int_0^{\tau^n} e^{-rt} \pi^j \left(X^j_{T^j_n(t)}\right) dT^j_n(t) + e^{-r\tau^n} \bar{\pi}\left(X^n_{T^n(\tau^n)}, d^n\right)\right] \\
&\quad \to \mathbb{E}\left[\sum_{j=1}^N \int_0^{\tau^*} e^{-rt} \pi^j \left(X^j_{T^j(t)}\right) dT^j(t) + e^{-r\tau} \bar{\pi}\left(X_{T(\tau^*)}, d\right)\right].
\end{aligned}
$$

Note that the principal's value is continuous in X^i. This is easily deduced as the difference in values is bounded by

$$
\mathbb{E}\left[\int_0^\infty e^{-rt} \left|\pi^i\left(X_t^i\right) - \pi^i\left(X_t^{i,n}\right)\right| dt\right].
$$

Therefore, the randomized promotion contest (T, τ^*, d) is optimal. Finally, one easily deduce from both the optimality and the limit characterization that $\tau^i = \inf\{t \geq 0 \,:\, X_t^i > \bar{P}^i(\underline{X}_t^i)\} \wedge \tau^{p,i}$, where $\tau^{p,i}$ is the first tick of a Poisson clock that runs only when $X_t^i = \bar{P}^i(\underline{X}_t^i)\}$ with the intensity that leaves i indifferent between exerting effort or not if promoted at time τ^i.

■

A.3.2 Proof of Theorem 3

As in the proof of 1, consider the relaxed program:

$$\Pi := \sup_{(T,\{\tau_k\}_{k=1}^{N},d)\in\mathcal{P}^r} \mathbb{E}\left[\sum_{i=1}^{N}\int_0^{\tau} e^{-rt}\pi^i\left(X^i_{T^i(t)}\right)dT^i(t) + e^{-r\tau}\bar{\pi}\left(X_{T(\tau)},d\right)\right] \qquad \text{(RP(P-d))}$$

subject to, for all $i\in\{1,\ldots,N\}$, for all $t\geq 0$, $\mathbb{P}$-a.s.,

$$\mathbb{E}\left[\sum_{k=1}^{K} e^{-r(\tau_k-t)}gd^i_{\tau_k}\mathbb{1}_{\{t\leq\tau_k\}} - \int_t^{\infty} e^{-rt}(1-\sum_{k=1}^{K}d^i_{\tau_k}\mathbb{1}_{\{t\geq\tau_k\}})c^i dT^i(t)\mid\mathcal{F}^i_{T^i(t)}\right]\geq 0. \qquad \text{(DPC)}$$

By the same arguments as in the proof of Proposition 9,

Proposition 32 *The value of* (Prize design) *is weakly lower than the value of* (RP)*:* $\Pi^M\leq\Pi$.

Next, a straightforward adaptation of the proof of Theorem 10 in Appendix A.1.7 yields:

Theorem 12 *A solution to* (RP(P-d)) *exists.*

Finally, I show that any promotion contest that allocate the entire prize upon at once can be improved upon. This follows from Proposition 33 below.

Proposition 33 (RP(P-d)) *admits a solution* $(T,\{\tau_k\}_{k=1}^{K},d)$ *such that* $K=1$ $\mathbb{P}$*-a.s..*

Theorem 3 then follows from Theorem 1.

Proof of Proposition 33. Let $\left(T,\{\tau_k\}_{k=1}^{K},d\right)$ be a solution of (RP(P-d)), which exists by Theorem 12. τ_1 is the smallest promotion time. The continuation value of the principal at τ_1 is

$$\begin{aligned} e^{-r\tau_i}\Pi^M_{\tau_1} :=& e^{-r\tau_1}\bar{\pi}\left(X_{T(\tau_1)},d_{\tau_1}\right) \\ &+\mathbb{E}\left[\sum_{k=2}^{K}\sum_{i=1}^{N}\left(\int_{\tau_{k-1}}^{\tau_k} e^{-rt}\pi^i\left(X^i_{T^i(t)}\right)dT^i(t)+e^{-r\tau_k}\bar{\pi}\left(X_{T(\tau_k)},d_{\tau_k}\right)\right)\mid\mathcal{G}^T_{\tau_1}\right]. \end{aligned}$$

By Assumption 9(i),

$$\bar{\pi}\left(X_{T(\tau_1)}, d_{\tau_1}\right) \leq d^0_{\tau_1} W + \sum_{i=1}^{N} d^i_{\tau_1} \bar{\pi}^i \left(X^i_{T^i(\tau_1)}\right).$$

If $\sum_{i=0}^{N} d^i_\tau = 1$, we are done. So suppose not. Observe then that

$$\mathbb{E}\left[\sum_{k=2}^{K}\sum_{i=1}^{N}\left(\int_{\tau_{k-1}}^{\tau_k} e^{-rt}\pi^i\left(X^i_{T^i(t)}\right)dT^i(t) + e^{-r\tau_k}\bar{\pi}\left(X_{T(\tau_k)}, d_{\tau_k}\right)\right) \mid \mathcal{G}^T_{\tau_1}\right] \tag{A.14}$$

$$\leq (1-\sum_{i=0}^{N} d^i_{\tau_1}) \sup_{(T,\{\tau_k\},d)\in\mathcal{P}^{I,r}(\tau_1)} \mathbb{E}\Bigg[\sum_{k=1}^{K}\sum_{i=1}^{N}\Bigg(\int_{\tau_{k-1}}^{\tau_k} e^{-rt}\pi^i\left(X^i_{T^i(t)}\right)dT^i(t)$$

$$+ e^{-r\tau_k}\bar{\pi}\left(X_{T(\tau_k)}, d_{\tau_k}\right)\Bigg) \mid \mathcal{G}^T_{\tau_1}\Bigg], \tag{A.15}$$

where $\mathcal{P}^{I,r}(\tau_1)$ is the set of implementable continuation contest that coincides with (T, τ, d) up to time τ_1. To see this, let T_* be the continuation delegation process generated by $\left(T, \{\tau_k\}_{k=1}^K, d\right)$ after time τ^1 so that :

$$\sum_{k=2}^{K}\sum_{i=1}^{N}\left(\int_{\tau_{k-1}}^{\tau_k} e^{-rt}\pi^i\left(X^i_{T^i(t)}\right)dT^i(t) + e^{-r\tau_k}\bar{\pi}\left(X_{T(\tau_k)}, d_{\tau_k}\right)\right)$$
$$= \sum_{i=1}^{N}\int_0^\infty e^{-rt}\pi^i\left(X^i_{T^i(\tau_1)+T^i_*(t)}\right)dT^i_*(t) \quad \mathbb{P}\text{-a.s..}$$

Then, letting X^i_0 be any process taking value in $\mathcal{X}^0$ and $\pi^0(x) = W$ for all $x \in \mathcal{X}^0$,

$$\sup_{(T,\{\tau_k\},d)\in\mathcal{P}^{I,r}(t)} \mathbb{E}\left[\sum_{k=1}^{K}\sum_{i=1}^{N}\left(\int_{\tau_{k-1}}^{\tau_k} e^{-rt}\pi^i\left(X^i_{T^i(t)}\right)dT^i(t) + e^{-r\tau_k}\bar{\pi}\left(X_{T(\tau_k)}, d_{\tau_k}\right)\right) \mid \mathcal{G}^T_{\tau_1}\right]$$
$$\geq E^{-r\tau_1}\mathbb{E}\left[\sum_{i=0}^{N}\int_0^\infty e^{-rt}\pi^i\left(X^i_{T^i(\tau_1)+\frac{T^i_*(t)}{1-\sum_{i=0}^{N} d^i_{\tau_1}}}\right) d\left(\frac{T^i_*(t)}{1-\sum_{i=0}^{N} d^i_{\tau_1}}\right) \mid \mathcal{G}^T_{\tau_1}\right],$$

as $\frac{T_*^i(t)}{1-\sum_{i=0}^N d_{\tau_1}^i}$ is implementable by a promotion contest when the total information in the game is restricted to $\mathcal{G}^{T_*}$ and, more information benefits the principal. So

$$\begin{aligned}(1-\sum_{i=0}^N d_{\tau_1}^i) \sup_{(T,\{\tau_k\},d)\in\mathcal{P}^{I,r}(t)} \mathbb{E}\Bigg[\sum_{k=1}^K\sum_{i=1}^N \Bigg(\int_{\tau_{k-1}}^{\tau_k} e^{-rt}\pi^i\left(X^i_{T^i(t)}\right) dT^i(t) \\ + e^{-r\tau_k}\bar{\pi}\left(X_{T(\tau_k)}, d_{\tau_k}\right)\Bigg) \mid \mathcal{G}^T_{\tau_1}\Bigg] \\ \geq e^{-r\tau_1}\mathbb{E}\left[\sum_{i=0}^N \int_0^\infty e^{-rt}\pi^i\left(X^i_{T^i(\tau_1)+\frac{T_*^i(t)}{1-\sum_{i=0}^N d_{\tau_1}^i}}\right) dT_*^i(t) \mid \mathcal{G}^T_{\tau_1}\right]\end{aligned}$$

But, by a time-change argument, for $q_*^i(t) := e^{-r\left(T_*^{i-1}(t)-t\right)}$, where $T_*^{i-1}(\cdot)$ is the generalized inverse of $T_*^i(\cdot)$,

$$\mathbb{E}\left[\sum_{i=0}^N \int_0^\infty e^{-rt}\pi^i\left(X^i_{T^i(\tau_1)+T_*^i(t)}\right) dT_*^i(t) \mid \mathcal{G}^T_{\tau_1}\right] = \mathbb{E}\left[\sum_{i=0}^N \int_0^\infty e^{-rt}q_*^i(t)\pi^i\left(X^i_{T^i(\tau_1)+t}\right) dt \mid \mathcal{G}^T_{\tau_1}\right],$$

and

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^N \int_0^\infty e^{-rt}\pi^i\left(X^i_{T^i(\tau_1)+\frac{T_*^i(t)}{1-\sum_{i=0}^N d_{\tau_1}^i}}\right) dT_*^i(t) \mid \mathcal{G}^T_{\tau_1}\right] \\ = \mathbb{E}\left[\sum_{i=0}^N \int_0^\infty e^{-rt}q_*^i(t)\pi^i\left(X^i_{T^i(\tau_1)+\frac{t}{1-\sum_{i=0}^N d_{\tau_1}^i}}\right) dt \mid \mathcal{G}^T_{\tau_1}\right].\end{aligned}$$

By definition q^i is $\mathcal{G}^{T_*}$-adapted. Furthermore, for all $i \in \{1,\ldots,N\}$,

$$\sup_{\tau\in\mathcal{T}(\mathcal{F}^i)} \mathbb{E}\left[\int_0^\tau e^{-rt}\left(\pi^i\left(X^i_t\right) - \pi^i\left(X^i_{\frac{t}{1-\sum_{i=0}^N d_{\tau_1}^i}}\right)\right) dt \mid x_0^i = X^i_{T^i(\tau)}\right] = 0. \tag{A.16}$$

Hence, I claim that $\tau^* = 0$ is optimal in the above problem. To see this, argue by contradiction, i.e., suppose not. Then the smallest optimal stopping time, which exists by Snell's theorem, is $\tilde{\tau} > 0$. By Lemma 29, there exists $t \geq 0$ such that

$$\mathbb{E}\left[\int_t^{\tilde{\tau}} e^{-rs}\left(\pi^i\left(X_s^i\right) - \pi^i\left(X^i_{\frac{s}{1-\sum_{i=0}^N d^i_{\tau_1}}}\right)\right) ds \mid \mathcal{F}_t^i\right] > 0.$$

The above inequality is equivalent to

$$\begin{aligned} &\int_t^\infty e^{-rs}\mathbb{E}\left[\left(\pi^i\left(X_s^i\right) - \pi^i\left(X^i_{\frac{s}{1-\sum_{i=0}^N d^i_{\tau_1}}}\right)\right)\mathbb{1}_{\{s\leq\tilde{\tau}\}} \mid \mathcal{F}_t^i\right] ds > 0 \\ \Leftrightarrow &\int_t^\infty e^{-rs}\mathbb{E}\left[\mathbb{E}\left[\left(\pi^i\left(X_s^i\right) - \pi^i\left(X^i_{\frac{s}{1-\sum_{i=0}^N d^i_{\tau_1}}}\right)\right) \mid \mathcal{F}_s^i\right]\mathbb{1}_{\{s\leq\tilde{\tau}\}} \mid \mathcal{F}_t^i\right] ds > 0 \end{aligned}$$

by Fubini's theorem, the law of iterated expectations, and the fact that $\tilde{\tau}$ is a $\mathcal{F}^i$-stopping time. But, for all $s \in [t, \infty)$,

$$\mathbb{E}\left[\pi^i\left(X_s^i\right) \mid \mathcal{F}_s^i\right] \leq \mathbb{E}\left[\pi^i\left(X^i_{\frac{s}{1-\sum_{i=0}^N d^i_{\tau_1}}}\right) \mid \mathcal{F}_s^i\right]$$

by Assumption 9 (i): a contradiction. So, for all $i \in \{1, \ldots, N\}$, (A.16) holds. Similarly, (A.16) is easily seen to hold for $i = 0$. Thus by Lemma 5 in Kaspi and Mandelbaum (1998),

$$\begin{aligned} &\mathbb{E}\left[\sum_{i=0}^N \int_0^\infty e^{-rt} q_*^i(t)\left(\pi^i\left(X^i_{T^i(\tau_1)+t}\right) - \pi^i\left(X^i_{\frac{t}{1-\sum_{i=0}^N d^i_{\tau_1}}}\right)\right) dt\right] \leq 0 \\ \Leftrightarrow\ &\mathbb{E}\left[\sum_{i=0}^N \int_0^\infty e^{-rt}\pi^i\left(X^i_{T^i(\tau_1)+T^i_*(t)}\right) dT^i_*(t) \mid \mathcal{G}^T_{\tau_1}\right] \\ &\leq \mathbb{E}\left[\sum_{i=0}^N \int_0^\infty e^{-rt}\pi^i\left(X^i_{T^i(\tau_1)+\frac{T^i_*(t)}{1-\sum_{i=0}^N d^i_{\tau_1}}}\right) dT^i_*(t) \mid \mathcal{G}^T_{\tau_1}\right]. \end{aligned}$$

Thus (A.14) holds. But then, the randomized promotion contest that promotes worker i at time τ_1 with probability $d^i_{\tau_1}$ and otherwise play the optimal continuation contest yields a higher payoffs to the principal than $(T, \{\tau_k\}_{k=1}^K, d)$. This concludes the proof. ■

Appendix B

Appendix to Chapter Two

B.1 Definitions

We define properties of domains and functional spaces that are used for our main results. Let $x \in \mathbb{R}$ and $y = (t, x) \in \mathbb{R}_+ \times \mathbb{R}$. For $z \in \{x, y\}$ and $R > 0$, let $B_R(z) = \{z' : |z' - z| < R\}$ be the open ball of radius centered at z. Define also $C_R(t, x)$ the R-cylindrical neighborhood of (t, x): $C_R(t, x) := [t, t + R) \times B_R(x)$.

B.1.1 Function spaces, norms, and regularity properties

Let $\mathcal{Y}$ be an open subset of $\mathbb{R}_+ \times \mathbb{R}$. We will write $\mathcal{Y}' \subset\subset \mathcal{Y}$ to indicate that (i) $\mathcal{Y}'$ is precompact[1] and (ii) $\bar{\mathcal{Y}}' \subset \mathcal{Y}$. On $\mathcal{Y}$, we define $\mathcal{C}^0(\mathcal{Y})$, $L^p(Y)$, $\mathcal{H}^{\alpha,\beta}(\mathcal{Y})$, and $W^{1,2,p}(\mathcal{Y})$.[2]

The space $\mathcal{C}^0(\mathcal{Y})$ is the space of continuous functions on $\mathcal{Y}$ endowed with the sup-norm topology. $L^p(\mathcal{Y})$ is the space of Lebesgue integrable functions with finite L^p-norm.

Next, we give the definitions of $\mathcal{H}^{\alpha,\beta}(Y)$ and $W^{1,2,p}(\mathcal{Y})$.

[1]Recall that a set is precompact if its closure is compact.
[2]If $\alpha = \beta$, we will simply write $\mathcal{H}^{\alpha}(\mathcal{Y})$.

Definition 15 *The Hölder space* $\mathcal{H}^{\alpha,\beta}(\mathcal{Y})$, $0<\alpha,\beta<1$, *is the space of functions* $u:\mathcal{Y}\to\mathbb{R}$ *such that*

$$\|u\|_{\mathcal{H}^{\alpha,\beta}(Y)} = \|u\|_{L^\infty(\mathcal{Y})} + \sup_{(t,x),(t',x')\in\mathcal{Y}} \frac{|u(t,x)-u(t',x')|}{|t-t'|^\alpha + |x-x'|^\beta} < \infty.$$

Remark 6 *All the functions in the Hölder space* $\mathcal{H}^{\alpha,\beta}(\mathcal{Y})$ *are continuous. This follows from the norm being finite.*

To define the Sobolev space $W^{1,2,p}(\mathcal{Y})$, we first need to define *weak derivatives.*

Definition 16 *A function* $f \in L^1(\mathcal{Y})$ *is* ***weakly differentiable*** *with respect to* $Y = \{y_1,\ldots,y_n\}$, *with* $y_i \in \{t,x_1,\ldots,x_d\}$ *for all* $i=1,\ldots,n$, *if there exists a function* $\Delta^Y \in L^1(\mathcal{Y})$ *such that*

$$\int_{\mathcal{Y}} f(y)\partial_{y_1,\ldots,y_n}\phi(y)dy = (-1)^n \int_{\mathcal{Y}} \Delta^Y(y)\phi(y)dy,$$

for all smooth test functions ϕ *with compact support.*

The function Δ^Y *is called the* Y *weak partial derivative of* f *and denoted* $f_{y_1\cdots y_n}$.

Remark 7 *Weak derivatives are well-defined as one can show that they are unique in* L^1: *if* Δ^Y *and and* $\tilde{\Delta}^Y$ *are weak derivatives with respect to* $\{y_1,\ldots,y_n\}$, *then* $\Delta^Y=\tilde{\Delta}^Y$ *almost everywhere. Moreover, from the definition of weak derivatives and Schwarz's theorem, one sees that, if* Y *and* $\tilde{Y}$ *are permutations of one another, then* $\Delta^Y = \Delta^{\tilde{Y}}$ *almost everywhere.*

Definition 17 *For* $p\in[1,\infty]$, *the Sobolev space* $W^{1,2,p}(\mathcal{Y})$ *is the space of functions* $u:\mathcal{Y}\to\mathbb{R}$, *such that* u *is in* $L^p(\mathcal{Y})$ *and its weak derivatives* u_t, u_{x_i}, *and* $u_{x_ix_j}$ *exists, for all* $i,j=1,\ldots,d$, *and are also in* $L^p(\mathcal{Y})$. *It is normed by*

$$\|u\|_{W^{1,2,p}(Y)} = \|u\|_{L^p(\mathcal{Y})} + \|u_t\|_{L^p(\mathcal{Y})} + \|u_x\|_{L^p(\mathcal{Y})} + \|u_{xx}\|_{L^p(\mathcal{Y})}.$$

$u_{xx} = \left(u_{x_ix_j}\right)_{i,j=1}^d$ *is the symmetric matrix of weak second order derivatives.*

We also define the local version of these spaces. A function u belong to $\mathcal{C}^0_{loc}(\mathcal{Y})$ if for all $\mathcal{Y}' \subset\subset \mathcal{Y}$, $u \in \mathcal{C}^0(\mathcal{Y}')$. The spaces $\mathcal{H}^{\alpha,\beta}_{loc}(\mathcal{Y})$,$L^p_{loc}(\mathcal{Y})$, and $W^{1,2,p}_{loc}(\mathcal{Y})$ are defined similarly.

Finally, we recall the notions of weak convergence in $L^p(\mathcal{Y})$ and $W^{1,2,p}(\mathcal{Y})$.

Definition 18 *Let $p \in [1, \infty)$. A sequence of functions $(f^n)_{n\in\mathbb{N}} \subseteq L^p(\mathcal{Y})$ converges weakly in $L^p(\mathcal{Y})$ to some function f if and only if, for all $g \in L^q(\mathcal{Y})$ with $\frac{1}{p} + \frac{1}{q} = 1$,*

$$\int_{\mathcal{Y}} f^n g \to \int_{\mathcal{Y}} fg.$$

B.2 Relation between classic, L^p- and viscosity solutions

In the proof of Proposition 13, we need the concept of viscosity solutions. We recall both the definition of viscosity solution and the relation between and viscosity and L^p-solutions below.

Definition 19 *An continuous function $v : \bar{[}0, T) \times \mathcal{X} \to \mathbb{R}$ is a viscosity subsolution (respectively, supersolution) of* (HJB) *if (i) $v(t, x) \leq g(x)$ (respectively, $\geq g(x)$) on $\partial[0, T) \times \mathcal{X}$, and (ii) for all $(t, x) \in \mathcal{Y}_T$ and all $\varphi \in \mathcal{C}^{1,2}(\mathcal{Y}_T)$ such that $v \leq \varphi$ (respectively $v \geq \varphi$) in $\mathcal{Y}_T$ and $v(t, x) = \varphi(t, x)$,*

$$\max\left\{g(x) - v(t, x), \left(\partial_t + \mathcal{L}^{(t,x)} - r(t, x)\right)\varphi(t, x) + f(t, x)\right\} \geq 0$$
$$\left(\textit{respectively, } \max\left\{g(x) - v(t, x), \left(\partial_t + \mathcal{L}^{(t,x)} - r(t, x)\right)\varphi(t, x) + f(t, x)\right\} \leq 0\right).$$

u is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

Lemma 30 *Let u be an L^p-solution of* (HJB). *Then u is a viscosity solution of* (HJB).

Proof. This follows from the proof of Proposition 2.10 in Crandall et al. (2000). The proof in that paper is local and applies to operators that satisfy degenerate ellipticity. See also Lemma 5 in Durandard and Strulovici (2022). ■

B.3 Omitted proofs for Section 2.3.1

Proof of Theorem 4. By Assumption 10 and the definition of $V(t,x)$, we have $V \in \mathcal{C}\left(\bar{[0,T)} \times \mathcal{X}\right)$ and $v(t,x) = g(x)$ on $\partial\mathcal{Y}_T$. There remains to show that V is an L^p-solution of (HJB) in $\mathcal{Y}_T$. Let $(\tilde{t},\tilde{x}) \in \mathcal{Y}_T$. Distinguish two cases:

- Either g is $\mathcal{C}^2$ in a neighborhood of $\tilde{x}$. Let $\epsilon > 0$ be such that $C_\epsilon(\tilde{t},\tilde{x}) \subset \mathcal{Y}_T$ and and $g \in \mathcal{C}^2\left(B_\epsilon(\tilde{x})\right)$. Let $\tau_{C_\epsilon} := \inf\left\{t \geq 0 \,:\, (t,X_t) \notin C_\epsilon(\tilde{t},\tilde{x})\right\}$. By the Snell envelope theorem, for all $(t,x) \in C_\epsilon(\tilde{t},\tilde{x})$,

$$
\begin{aligned}
V(t,x) = \sup_{\tau \in \mathcal{T}(t)} \mathbb{E}_{(t,x)}\Bigg[&\int_0^{\tau\wedge\tau_{C_\epsilon}} e^{-\int_t^s r(u,X_u)du} f(s,X_s)ds \\
&+ e^{-\int_t^{\tau\wedge\tau_{C_\epsilon}} r(u,X_u)du}\left(V\left(\tau_{C_\epsilon}, X_{\tau_{C_\epsilon}}\right)\mathbb{1}_{\{\tau \geq \tau^D\}} + g\left(X_\tau\right)\mathbb{1}_{\{\tau < \tau^D\}}\right)\Bigg].
\end{aligned}
$$

  By Assumption 10, V is continuous. So there exists $\lambda > 0$ such that $\sigma(t,x) > \lambda$ on $C_\epsilon(\tilde{t},\tilde{x})$ by the continuity of σ and Weierstrass theorem. Theorem 1 in Durandard and Strulovici (2022)[3] then guarantees that V is the unique L^p-strong solution of

$$
\begin{cases}
\max\left\{g(x) - v(t,x), \left(\partial_t + \mathcal{L}^{(t,x)} - r(t,x)\right)v(t,x) + f(t,x)\right\} = 0 \text{ if } (t,x) \in C_\epsilon(\tilde{t},\tilde{x}), \\
v(t,x) = V(t,x) \text{ if } (t,x) \in \partial C_\epsilon(\tilde{t},\tilde{x}).
\end{cases}
$$

[3]Theorem 1 in Durandard and Strulovici (2022) is still valid for $r = r(t,x)$ under our assumption. Moreover, it holds also for $r = 0$. Hence $r > \bar{\mu}_2 > 0$ is only used in the proof of the comparison principle (Proposition 5). When $\mathcal{X}$ is bounded, which we assume when $r = 0$, the proof is seen to hold. Alternatively, the same result follows from Proposition 4 and Theorem 2 of Durandard and Strulovici (2022), which holds as their proofs do not use that r is strictly positive.

- or there is no neighborhood $B_\delta(\tilde{x})$ of $\tilde{x}$ such that g is $\mathcal{C}^2$ on $B_\epsilon(\tilde{x})$. Then $x = x^c$. Since the continuation region $\mathcal{C}$ is open (since g is continuous and V is too by Assumption 10), there exists $\epsilon > 0$ small, such that $C_\epsilon(\tilde{t}, \tilde{x}) \subset \mathcal{Y}_T \subseteq \mathcal{C}$. Let $\tau_{C_\epsilon} := \inf\left\{t \geq 0 : (t, X_t) \notin C_\epsilon(\tilde{t}, \tilde{x})\right\}$. By the Snell envelope theorem, for all $(t, x) \in C_\epsilon(\tilde{t}, \tilde{x})$,

$$V(t,x) = \mathbb{E}_{(t,x)}\left[\int_0^{\tau_{C_\epsilon}} e^{-\int_t^s r(u,X_u)du} f(s, X_s)ds + e^{-r\int_t^{\tau_{C_\epsilon}} r(u,X_u)du} V\left(\tau_{C_\epsilon}, X_{\tau_{C_\epsilon}}\right)\right].$$

  Since V is continuous by Assumption 10 and $\sigma(t,x) > \lambda$ for some $\lambda > 0$ on $C_\epsilon(\tilde{t}, \tilde{x})$ by the continuity of σ and Weierstrass theorem, Theorem 1 in Durandard and Strulovici (2022) then guarantees that V is the unique L^p-strong solution of

$$\begin{cases} \left(\partial_t + \mathcal{L}^{(t,x)} - r(t,x)\right) v(t,x) + f(t,x) = 0 \text{ if } (t,x) \in C_\epsilon(\tilde{t}, \tilde{x}), \\ v(t,x) = V(t,x) \text{ if } (t,x) \in \partial C_\epsilon(\tilde{t}, \tilde{x}). \end{cases}$$

Since $\left(\tilde{t}, \tilde{x}\right)$ was arbitrary, V is the unique L^p-strong solution of (HJB).

Finally, let $(\tilde{t}, \tilde{x}) \in \mathcal{C}$. Since $\mathcal{C}$ is open (since g is continuous and V is too by Assumption 10), there exists $\delta > 0$ such that $C_\delta(\tilde{t}, \tilde{x}) \subset \mathcal{C}$. By the first part of the proof, V is the unique L^p-solution of

$$\begin{cases} \left(\partial_t + \mathcal{L}^{(t,x)} - r(t,x)\right) v(t,x) + f(t,x) = 0 \text{ if } (t,x) \in C_\delta(\tilde{t}, \tilde{x}), \\ v(t,x) = V(t,x) \text{ if } (t,x) \in \partial C_\delta(\tilde{t}, \tilde{x}). \end{cases}$$

Under our maintained assumptions, any solution of the above equation is in $\mathcal{C}^{1,2,\alpha}\left(C_\delta(\tilde{t}, \tilde{x})\right)$ by standard results in the theory of partial differential equations, see, e.g., Theorem 3.5.10 in Friedman (2008). So $V \in \mathcal{C}^{1,2,\alpha}(\mathcal{C})$. ■

Proof of Proposition 12. Let $t^c \geq 0$ such that $(t^c, x^c) \in \mathcal{Y}_T$ and consider the process X

starting at (t^c, x^c). Define $\theta \coloneqq g''(\{x^c\})$.[4] Since g has a convex kink at x^c, $\theta > 0$. For $\epsilon > 0$, let $\tau_\epsilon \coloneqq \inf\{t \geq 0 \,:\, X_t \notin (x^c - \epsilon, x^c + \epsilon)\}$ and $\delta > 0$. Since $\tau_\epsilon \wedge \delta$ is an admissible stopping time,

$$V(t^c, x^c) - g(x^c) \geq \mathbb{E}_{(t^c,x^c)}\left[e^{-\int_t^{\tau_\epsilon\wedge\delta} r(u,X_u)du} g(X_{\tau_\epsilon\wedge\delta}) - \int_0^{\tau_\epsilon\wedge\delta} e^{-\int_t^s r(u,X_u)du} f(s, X_s)ds\right]. \tag{B.1}$$

Next, observe that, by Itô's formula, the processes $\{g^i(X_t)\}_{t\geq 0}$, $i = 1, 2$, are continuous local semi-martingales. Applying Itô-Tanaka-Meyer formula (Theorem 22.5 in Kallenberg (2006)) yields

$$g(X_t) = g(x) + \int_0^t \mu(s, X_s)g'(X_s)ds + \int_0^t \sigma(s, X_s)g'(X_s)dB_s + \frac{1}{2}\int_{\mathcal{X}} L_t^x dg''(x),$$

where L_t^x is the local time process, which we can choose continuous almost surely. We used that $L_s^x = 0$ for $x \notin \mathcal{X}$. Therefore $g(X_t)$ is a continuous semi-martingale and by Itô's product rule (and Fubini's theorem),

$$\begin{aligned} e^{-r\int_0^t r(u,X_u)du} g(X_t) - g(x) = &\int_0^t e^{-\int_0^s r(u,X_u)du}\left(\mu(s, X_s)g'(X_s) - r(s, X_s)g(X_s)\right)ds \\ &+ \int_0^t \sigma(s, X_s)g'(X_s)dB_s + \frac{1}{2}\int_{\mathcal{X}}\int_0^t e^{-\int_0^s r(u,X_u)du} dL_s^x dg''(x). \end{aligned}$$

Taking expectations and by the optional sampling theorem, the right-hand side of (B.1) is

[4] The second derivative of g is interpreted here as a measure, since g is convex.

equal to

$$\mathbb{E}_{(t^c,x^c)}\Bigg[\int_0^{\tau_\epsilon\wedge\delta} e^{-\int_0^s r(u,X_u)du}\left(f(s,X_s)+\mu(s,X_s)g'(X_s)-r(s,X_s)g(X_s)\right)ds + \frac{1}{2}\int_{\mathcal{X}}\int_0^{\tau_\epsilon\wedge\delta} e^{-\int_0^s r(u,X_u)du} dL_s^x dg''(x)\Bigg].$$

Since $dg''(x)$ is absolutely continuous everywhere except at x^c, the above simplify to

$$\mathbb{E}_{(t^c,x^c)}\Bigg[\int_0^{\tau_\epsilon\wedge\delta} e^{-\int_0^s r(u,X_u)du}\left(f(s,X_s)+\mu(s,X_s)g'(X_s)+\frac{\sigma(s,X_s)^2}{2}\tilde{g}''(X_s)-r(s,X_s)g(X_s)\right)ds + \frac{1}{2}\theta\int_0^{\tau_\epsilon\wedge\delta} e^{-\int_0^s r(u,X_u)du} dL_s^{x^c}\Bigg],$$

where $\tilde{g}''(x)=g''(x)\mathbb{1}_{\{x\neq x^c\}}$. Since $g^i\in W^{2,\infty}_{loc}(\mathcal{X})$, $i=1,2$, the first integrand is bounded below by some constant $-K<0$. Therefore

$$V(t^c,x^c)-g(x^c)\geq\theta\mathbb{E}_{(t^c,x^c)}\left[\int_0^{\tau_\epsilon\wedge\delta} e^{-\int_0^s r(u,X_u)du} dL_s^{x^c}\right]-K\delta.$$

By Lemma 4.1 in De Angelis (2022),[5] there exists a constant $C>0$ such that

$$\mathbb{E}_{(t^c,x^c)}\left[\int_0^{\tau_\epsilon\wedge\delta} e^{-\int_0^s r(u,X_u)du} dL_s^{x^c}\right]>C\sqrt{\delta}.$$

Thus

$$V(t^c,x^c)-g(x^c)\geq\theta C\sqrt{\delta}-K\delta,$$

and the right-hand side is strictly positive for $\delta>0$ small enough. This concludes the proof.

[5]The proof is done in the case that $\mu(t,x)=0$ and $\sigma(t,x)$ is independent of time, but the argument easily extends to our case. The only requirement is that $\sigma(t,x)>\bar{\lambda}$ in a neighborhood of (t^c,x^c).

■

B.4 Omitted proofs for Section 2.3.2

Proof of proposition 13. By Theorem 4, the value function is the unique L^p-solution of the HJB equation

$$\begin{cases} \max\left\{g(x) - v(t,x), \left(\partial_t + \mathcal{L}^{(t,x)} - r(t,x)\right) v(t,x) + f(t,x)\right\} = 0 \text{ in } \mathcal{Y}_T \\ v(t,x) = g(x) \text{ on } \partial\mathcal{Y}_T. \end{cases} \tag{HJB}$$

Moreover, it is $\mathcal{C}^{1,2}$ in the continuation region. Let $\bar{b}(t) = \inf\{x \in [x^c, \bar{x}) : (t,x) \in \mathcal{S}\}$ and

$$\tilde{v}(t,x) = \begin{cases} v(t,x) \text{ if } t < b(t) \\ g(x) \text{ if } t \geq b(t). \end{cases}$$

Then $\tilde{v}(t,x)$ is a viscosity solution of the HJB equation by Assumption 11. Moreover, by the comparison principle in Durandard and Strulovici (2022) (Proposition 5), there is a unique viscosity solution.[6] Therefore $v(t,x) = \tilde{v}(t,x)$ and $\mathcal{S} \cap \{(t,x) \in \mathbb{R}_+ \times \mathcal{X} : x \geq x^c\} = \left\{(t,x) \in \mathbb{R}_+ \times \mathcal{X} : x \geq \bar{b}(t)\right\}$.

By the same argument, we obtain $\mathcal{S} \cap \{(t,x) \in \mathbb{R}_+ \times \mathcal{X} : x \leq x^c\} = \{(t,x) \in \mathbb{R}_+ \times \mathcal{X} : x \leq \underline{b}(t)\}$, with $\underline{b}(t) = \sup\{x \in (\underline{x}, x^c] : (t,x) \in \mathcal{S}\}$. ■

Proof of Lemma 2. If $\{(t,x) \in \mathcal{Y}_T : x \in (x^-(t), x^+(t))\} = \emptyset$, we are done. So suppose not and let $(\tilde{t}, \tilde{x}) \in \{(t,x) \in \mathcal{Y}_T : x \in (x^-(t), x^+(t))\}$. Define

$$\bar{\tau} := \inf\left\{t \geq 0 : X_t^{(\tilde{t},\tilde{x})} \notin \left(x^-(t+\tilde{t}), x^+(t+\tilde{t})\right)\right\}.$$

[6] Again, if $r = 0$, it is seen from the proof of the comparison principle in Durandard and Strulovici (2022) that it holds when $\mathcal{X}$ is bounded, which we assume when $r = 0$.

Then, as in the proof of Proposition 12,

$$V(\tilde{t},\tilde{x}) - g(\tilde{x}) \geq \mathbb{E}_{(\tilde{t},\tilde{x})}\Bigg[\int_0^{\bar{\tau}} e^{-\int_0^s r(u,X_u)du}\Bigg(f(s,X_s) + \mu(s,X_s)g'(X_s)$$
$$\frac{\sigma(s,X_s)^2}{2}\tilde{g}''(X_s) - r(s,X_s)g(X_s)\Bigg)ds$$
$$+\frac{1}{2}\theta\int_0^{\bar{\tau}} e^{-\int_0^s r(u,X_u)du} dL_s^{x^c}\Bigg],$$

where $\theta \coloneqq g''(\{x^c\}) > 0$. Since $\mathbb{P}(\bar{\tau} > 0) = 1$,

$$V(\tilde{t},\tilde{x}) - g(\tilde{x}) > 0,$$

and it is not optimal to stop at $(\tilde{t},\tilde{x})$. ■

Proof of Proposition 14. Since the optimal stopping problem is monotone decreasing, by Theorem 5, the stopping region is given by

$$\mathcal{S} \coloneqq \left\{(t,x) \in \mathbb{R}_+ \times \mathcal{X} \,:\, x \notin \left(\underline{b}(t), \bar{b}(t)\right)\right\},$$

where $\underline{b} : \mathbb{R}_+ \to \mathcal{X}$ is càdlàg nondecreasing and $\bar{b} :: \mathbb{R}_+ \to \mathcal{X}$ is càdlàg nonincreasing, with $\underline{b}(T) \leq x^c \leq \bar{b}(T)$. So we only need to prove that the boundaries are left-continuous too.

The proof is by contradiction. Suppose that there exists $\bar{t} \in [0,T)$ such that $\bar{b}$ or $\underline{b}$ is discontinuous at t. For the rest of the proof, we will assume that $\bar{b}$ is discontinuous at $\bar{t}$. The proof for $\underline{b}$ is identical. Since $\bar{b}$ is càdlàg nonincreasing, $\bar{b}(\bar{t}^-) > \bar{b}(\bar{t})$. Let $a,b \in \mathcal{X}$ be such that $(a,b) \subset (\bar{b}(\bar{t}), \bar{b}(\bar{t}^-))$ and consider the rectangular domain $\mathcal{Y} \coloneqq [\underline{t},\bar{t}) \times (a,b)$ with parabolic boundary $\partial\mathcal{Y} \coloneqq ([\underline{t},\bar{t}] \times (\{a\} \cup \{b\})) \cup (\{T\} \times (a,b))$. Using Lemma 2 and (2.5), we can choose $\underline{t}$ and (a,b) such that

$$f(s,x)+\left(\mathcal{L}^{(t,x)}-r(t,x)\right)g(x)<-\delta \text{ on } \mathcal{Y} \tag{B.2}$$

for some $\delta>0$ small.

Moreover, by Theorem 4, V is the unique L^p-solution of the boundary value problem

$$\begin{cases} \left(\partial_t+\mathcal{L}^{(t,x)}-r(t,x)\right)v(t,x)+f(t,x)=0 \text{ if } (t,x)\in\mathcal{Y}, \\ v(t,x)=V(t,x) \text{ if } (t,x)\in\partial\mathcal{Y}. \end{cases}$$

Let $\phi\in\mathcal{C}_c^\infty([a,b])$[7] such that $\phi(x)\geq 0$ and $\int_a^b\phi(x)dx=1$. Then, multiplying the above equality by ϕ and integrating, we have

$$\int_t^{\bar{t}}\int_a^b\left(V_t(s,x)+f(s,x)+\left(\mathcal{L}^{(s,x)}-r(s,x)\right)V(s,x)\right)\phi(x)dxds=0.$$

for all $t\in[\underline{t},\bar{t})$. By Fubini's theorem,

$$\begin{aligned}\int_t^{\bar{t}}\int_a^b V_t(t,x)\phi(x)dxds &= \int_a^b\int_t^{\bar{t}}V_t(t,x)ds\phi(x)dx \\ &= \int_a^b\left(V(\bar{t},x)-V(t,x)\right)\phi(x)dx \\ &< \int_a^b\left(g(x)-g(x)\right)\phi(x)dx \\ &= 0.\end{aligned}$$

where the inequality follows from the fact that $V(t,x)>g(x)$ for all $(t,x)\in\mathcal{Y}$, as it is a

[7] $\mathcal{C}_c^\infty([a,b])$ is the set of infinitely many times differentiable function supported in the interior of $[a,b]$.

subset of the continuation region. Therefore, for all $t \in [\underline{t}, \bar{t})$,

$$\int_t^{\bar{t}} \int_a^b \left(f(s,x) + \left(\mathcal{L}^{(s,x)} - r(s,x)\right) V(s,x)\right) \phi(x) dx ds > 0. \tag{B.3}$$

Moreover, integrating by parts, for all $t \in [\underline{t}, \bar{t})$,

$$\begin{aligned}
&\int_t^{\bar{t}} \int_a^b \left(f(s,x) + \left(\mathcal{L}^{(s,x)} - r(s,x)\right) V(s,x)\right) \phi(x) dx ds \\
&\qquad = \int_t^{\bar{t}} \int_a^b \left(\left(f(s,x) - r(s,x)V(s,x)\right) \phi(x) + V(s,x)\mathcal{L}_*^{(s,x)} \phi(x)\right) dx ds;
\end{aligned}$$

where $\mathcal{L}_*$ is the adjoint of $\mathcal{L}$: for all $\psi \in \mathcal{C}^\infty\left(\mathcal{Y}_T\right)$,[8]

$$\mathcal{L}_*^{(t,x)} \psi(t,x) \coloneqq \frac{\partial^2}{\partial x^2} \left(\frac{\sigma^2(t,x)}{2} \psi(t,x)\right) - \frac{\partial}{\partial x} \left(\mu(t,x)\psi(t,x)\right).$$

For $s \in [\underline{t}, \bar{t})$,

$$\begin{aligned}
&\int_a^b V(s,x)\mathcal{L}_*^{(s,x)} \phi(x) dx \\
&\qquad = \int_a^b \mathbb{1}_{\{\mathcal{L}_*^{(s,x)}\phi(x)<0\}} V(s,x)\mathcal{L}_*^{(s,x)} \phi(x) dx + \int_a^b \mathbb{1}_{\{\mathcal{L}_*^{(s,x)}\phi(x)\geq 0\}} V(s,x)\mathcal{L}_*^{(s,x)} \phi(x) dx \\
&\qquad \leq C\left|s - \bar{t}\right| \int_a^b \mathbb{1}_{\{\mathcal{L}_*^{(s,x)}\phi(x)\geq 0\}} \mathcal{L}_*^{(s,x)} \phi(x) dx + \int_a^b g(x)\mathcal{L}_*^{(s,x)} \phi(x) dx \\
&\qquad = C\left|s - \bar{t}\right| \int_a^b \mathbb{1}_{\{\mathcal{L}_*^{(s,x)}\phi(x)\geq 0\}} \mathcal{L}_*^{(s,x)} \phi(x) dx + \int_a^b \phi(x)\mathcal{L}^{(s,x)} g(x) dx.
\end{aligned}$$

The inequality follows from the two inequalities $V(t,x) \geq g(x)$ and $V(s,x) \leq g(x) + C\left|s - \bar{t}\right|$ for some $C > 0$ (since $t \to V(t,x)$ is Lipschitz, as V belongs to $W^{1,2,p}\left(\mathcal{Y}\right)$). The last equality

[8] $\mathcal{L}$ is well defined as $\sigma(t,\cdot)$ and $\mu(t,\cdot)$ are piecewise $\mathcal{C}^2(\mathcal{X})$ for all $t \in [0,T)$.

is obtained by integration by parts. Let

$$\bar{\phi}(t) := \int_t^{\bar{t}} \int_a^b \mathbb{1}_{\{\mathcal{L}_*^{(s,x)}\phi(x) \geq 0\}} \mathcal{L}_*^{(s,x)}\phi(x) dx ds \geq 0, \quad t \in [\underline{t}, \bar{t}).$$

Then, for all $t \in [\underline{t}, \bar{t})$,

$$\begin{aligned}
&\int_t^{\bar{t}} \int_a^b \left(f(s,x) + \left(\mathcal{L}^{(s,x)} - r(s,x)\right) V(s,x)\right) \phi(x) dx ds \\
&\qquad \leq \int_t^{\bar{t}} \int_a^b \left(f(s,x) + \left(\mathcal{L}^{(s,x)} - r(s,x)\right) g(x)\right) \phi(x) dx ds + \bar{\phi} C \left|t - \bar{t}\right|^2 \\
&\qquad \leq -\delta(\bar{t} - t) + \bar{\phi} C \left|t - \bar{t}\right|^2,
\end{aligned}$$

where we used (B.2) and that ϕ integrates to 1 to obtain the second inequality. But then there exists $t \in [\underline{t}, \bar{t})$ such that

$$\int_t^{\bar{t}} \int_a^b \left(f(s,x) + \left(\mathcal{L}^{(s,x)} - r(s,x)\right) V(s,x)\right) \phi(x) dx ds \leq 0,$$

which contradicts (B.3).

This concludes the proof. ■

B.5 Omitted proofs for Section 2.4

B.5.1 Omitted proofs for Section 2.4.1

Proof of Proposition 15. For all $n \in \mathcal{N}$, let $\mathcal{T}_n(t)$ be the set of stopping times in $\mathcal{T}(t)$ taking value in

$$\mathcal{T}_n(t) := \left\{s_k \,:\, s_0 = 0 \text{ and } s_{k+1} = s_k + 2^{-n} \text{ for all } k \in \mathbb{N}\right\} \cap [0, T \wedge n].$$

Observe that, for all $n \in \mathbb{N}$, $\mathcal{T}_n(t) \subseteq \mathcal{T}_{n+1}(t)$. Define

$$V_n(t,x) \coloneqq \sup_{\tau \in \mathcal{T}_n(t)} \mathbb{E}_{(t,x)} \left[\int_t^\tau e^{-r(s-t)} f(s, X_s^{(t,x)}) ds + e^{-r(\tau-t)} g\left(X_\tau\right) \right].$$

Then $V_n(T \vee n, x) = g(x)$ is convex in x. By Theorem 2 in Bergman et al. (1996), convexity is preserved for one dimensional diffusion. Therefore, for all $t \in (T \wedge n - 2^{-n}, T \wedge n)$,

$$x \to \mathbb{E}_{(t,x)} \left[e^{-r(T\wedge n - t)} g\left(X_{T\wedge n}\right) \right]$$

is convex in x. Moreover, by Fubini's theorem,

$$\begin{aligned} &\mathbb{E}_{(t,x)} \left[\int_t^{T\wedge n} e^{-r(s-t)} f(s, X_s) ds \right] \\ &\qquad = \int_t^{T\wedge n} \mathbb{E}_{(t,x)} \left[e^{-r(s-t)} f(s, X_s) \right] ds. \end{aligned}$$

Then, using Theorem 2 in Bergman et al. (1996) again, for all $t \in (T \wedge n - 2^{-n}, T \wedge n)$ and all $s \in (t, T \wedge n)$,

$$x \to \mathbb{E}_{(t,x)} \left[e^{-r(s-t)} f(s, X_s) \right]$$

is convex. Therefore, for all $t \in (T \wedge n - 2^{-n}, T \wedge n)$,

$$x \to \mathbb{E}_{(t,x)} \left[\int_t^{T\wedge n} e^{-r(s-t)} f\left(s, X_s^{(t,x)}\right) ds \right]$$

is convex, and thus

$$V_n(t,x) = \mathbb{E}_{(t,x)} \left[e^{-r(T\wedge n - t)} g\left(X_{T\wedge n}^{(t,x)} \right) \right] + \mathbb{E}_{(t,x)} \left[\int_t^{T\wedge n} e^{-r(s-t)} f\left(s, X_s\right) ds \right]$$

is also convex in x for all $t \in (T \wedge n - 2^{-n}, T \wedge n)$. At time $T \wedge n - 2^{-n}$, the value function is given by the dynamic programming equation:

$$\begin{aligned}
&V_n\left(T \wedge n - 2^{-n}, x\right) \\
&\quad = \max\left\{g(x), \mathbb{E}_{(T\wedge n-2^{-n},x)}\left[e^{-r2^{-n}}V_n\left(T \wedge n, X_{T\wedge n}\right) + \int_{T\wedge n-2^{-n}}^{T\wedge n} e^{-r\left(s-\left(T\wedge n-2^{-n}\right)\right)}f(s,X_s)ds\right]\right\},
\end{aligned}$$

which is convex in x as the maximum of two convex functions. Proceeding recursively shows that $x \to V_n(t,x)$ is convex (in x) for all $t \in [0, T \wedge n]$.

To conclude the proof, note that V_n converges pointwise to V as $n \to \infty$, and, hence, $V(t,x)$ is convex in x.

To see this, let $M > 0$ and consider the alternative stopping problems

$$V^M(t,x) := \sup_{\tau\in\mathcal{T}(t)} \mathbb{E}\left[\int_t^\tau e^{-r(s-t)}f(s,X_s)ds + e^{-r(\tau-t)}g\left(X_\tau\right) \wedge M\right];$$

and

$$V_n^M(t,x) := \sup_{\tau\in\mathcal{T}_n(t)} \mathbb{E}\left[\int_t^\tau e^{-r(s-t)}f(s,X_s)ds + e^{-r(\tau-t)}g\left(X_\tau\right) \wedge M\right];$$

By the monotone convergence theorem, $V_n^M(t,x) \uparrow V_n(t,x)$ and $V^M(t,x) \uparrow V(t,x)$ as $M \to \infty$. Consider then $\tau \in \mathcal{T}(t)$ and let $\tau_n := \inf\{\tilde{\tau} \in \mathcal{T}_n(t) : \tilde{\tau} \geq \tau \text{ } \mathbb{P}\text{-a.s.}\}$. Then $\tau_n \in \mathcal{T}_n(t)$ and $\tau_n \to \tau$ $\mathbb{P}$-a.s. as $n \to \infty$. By the dominated convergence theorem,

$$\begin{aligned}
&\Bigg|\mathbb{E}_{(t,x)}\left[\int_t^\tau e^{-r(s-t)}f(s,X_s)ds + e^{-r(\tau-t)}g\left(X_\tau\right) \wedge M\right] \\
&\qquad - \mathbb{E}_{(t,x)}\left[\int_t^{\tau_n} e^{-r(s-t)}f(s,X_s)ds + e^{-r(\tau_n-t)}g\left(X_{\tau_n}\right) \wedge M\right]\Bigg| \\
&\qquad\qquad \leq \mathbb{E}_{(t,x)}\left[\left|\int_\tau^{\tau_n} e^{-r(s-t)}f(s,X_s)ds + e^{-r(\tau-t)}g\left(X_\tau\right) \wedge M - e^{-r(\tau_n-t)}g\left(X_{\tau_n}\right) \wedge M\right|\right] \to 0
\end{aligned}$$

as $n \to \infty$ (since either $e^{-r\tau}g(X_\tau) \wedge M$ is continuous and uniformly integrable and $e^{-rs}f(s, X_s)$ is integrable if $r > 0$, or $g(\cdot) \wedge M$ is bounded and continuous, f is locally bounded, and we can focus on τ such that $\mathbb{E}[\tau] < K$ for some $K > 0$ by Lemma 32 below if $r = 0$). Therefore,

$$\liminf_{n\to\infty} V_n^M(t,x) \geq V^M(t,x),$$

and, since $V_n^M(t,x) \leq V^M(t,x)$ for all $(t,x) \in \mathcal{Y}_T$ and all $n \in \mathbb{N}$, it follows that V_n^M converges pointwise to V^M as $n \to \infty$. Moreover, since $\mathcal{T}_n(t) \subseteq \mathcal{T}_{n+1}(t)$, $V_n^M \uparrow V^M$. Therefore, we can interchange the order of the limits (using that $V(t,x)$ is locally bounded as a consequence of Lemma 2 in Durandard and Strulovici (2022) if $r > 0$ and of condition 4. in Definition 13 if $r = 0$). Thus, V_n converges pointwise to V as $n \to \infty$. ■

Proof of Lemma 3. By proposition 15, for all $t \in [0,T)$, $x \to V(t,x)$ is convex, hence, locally Lipschitz continuous on $\mathcal{X}$. Moreover, since $V(t,\cdot)$ is convex, it can only jump up on $\partial\mathcal{X}$. But $g(x) \leq V(t,x)$, and, therefore, $V(t,\cdot)$ is locally Lipschitz continuous on $\bar{\mathcal{X}}$. So to show that V is continuous, it is enough to show that $t \to V(t,x)$ is continuous for all x. This follows from lemma 31 below. ■

Lemma 31 *Suppose that condition 4. of Definition 13 and that the functions $f : \mathcal{Y}_T \to \mathbb{R}$ and $g : \bar{[}0,T) \times \mathcal{X} \to \mathbb{R}$ are Lipschitz continuous. Then, for all $x \in \bar{\mathcal{X}}$, $t \to V(t,x)$ is continuous.*

Proof of Lemma 31. Let $x \in \bar{\mathcal{X}}$. If $x \in \{\underline{x}, \bar{x}\}$, $V(t,x) = g(x)$ for all $t \geq 0$, and we are done. So suppose that $x \in \mathcal{X}$ and let $t, t' \in [0,T)$ with $t' \geq t$. Let

$$V^M(t,x) \coloneqq \sup_{\tau\in\mathcal{T}(t)} \mathbb{E}\left[\int_t^\tau e^{-r(s-t)}f(s,X_s)ds + e^{-r(\tau-t)}g(X_\tau) \wedge M\right];$$

and observe that $V^M \uparrow V$ pointwise by the monotone convergence theorem (since V is locally bounded by condition 4. in Definition 13 if $r = 0$ and by Lemma 2 in Durandard and

Strulovici (2022) otherwise). So, for all $\epsilon > 0$, there exists M such that

$$\begin{aligned}|V(t',x) - V(t,x)| &\leq \left|V(t',x) - V^M(t',x)\right| + \left|V^M(t',x) - V^M(t,x)\right| + \left|V^M(t,x) - V(t,x)\right| \\ &\leq \left|V^M(t',x) - V^M(t,x)\right| + 2\epsilon.\end{aligned}$$

Thus, it is enough to show that $t \to V(t,x)$ is continuous for g bounded. So suppose that g is bounded for the remaining of the proof.

Then, by Snell's envelope theorem, for all $\bar{t} \geq 0$,

$$\begin{aligned}|V(t',x) - V(t,x)| \leq \sup_{\tau} \mathbb{E}\Bigg[&\int_0^{\tau\wedge\bar{t}} e^{-rs} \left| f\left(s + t', X_s^{t',x}\right) - f\left(s+t, X_s^{t,x}\right)\right| ds \\ &+ e^{-r\tau}\mathbb{1}_{\{\tau\leq\bar{t}\}} \left| g\left(X_\tau^{t',x}\right) - g\left(X_\tau^{t,x}\right)\right| \Bigg] \\ &+ \mathbb{E}_{(t',x)}\left[e^{-r\bar{t}}\mathbb{1}_{\{\tau\geq\bar{t}\}}\left|V\left(\bar{t}, X_{\bar{t}}\right)\right|\right] + \mathbb{E}_{(t,x)}\left[e^{-r\bar{t}}\mathbb{1}_{\{\tau\geq\bar{t}\}}\left|V\left(\bar{t}, X_{\bar{t}}\right)\right|\right],\end{aligned}$$

where the supremum is taken over all stopping time if $r > 0$ and over all stopping times satisfying the conditions of Lemma 32 below if $r = 0$.

If $r = 0$, this implies, using Lemma 32, that, for all $\epsilon > 0$, we can choose $\bar{t}$ such that

$$\begin{aligned}&\mathbb{E}_{(t',x)}\left[e^{-r\bar{t}}\mathbb{1}_{\{\tau\geq\bar{t}\}}\left|V\left(\bar{t}, X_{\bar{t}}\right)\right|\right] + \mathbb{E}_{(t,x)}\left[e^{-r\bar{t}}\mathbb{1}_{\{\tau\geq\bar{t}\}}\left|V\left(\bar{t}, X_{\bar{t}}\right)\right|\right] \\ &\leq \mathbb{E}_{(t,x)}\left[e^{-r\bar{t}}\mathbb{1}_{\{\tau\geq\bar{t}\}}2M\right] \vee \mathbb{E}_{(t',x)}\left[e^{-r\bar{t}}\mathbb{1}_{\{\tau\geq\bar{t}\}}2M\right] \\ &< \epsilon.\end{aligned}$$

If $r > 0$, by standard estimates (e.g Lemma 2 and Lemma 9 in Durandard and Strulovici

(2022)), for all $\epsilon > 0$, we can choose $\bar{t}$ such that

$$\begin{aligned}
&\mathbb{E}_{(t',x)}\left[e^{-r\bar{t}}\mathbb{1}_{\{\tau\geq\bar{t}\}}\left|V\left(\bar{t},X_{\bar{t}}\right)\right|\right]+\mathbb{E}_{(t,x)}\left[e^{-r\bar{t}}\mathbb{1}_{\{\tau\geq\bar{t}\}}\left|V\left(\bar{t},X_{\bar{t}}\right)\right|\right]\\
&\quad\leq 2\mathbb{E}_{(t,x)}\left[e^{-r\bar{t}}\mathbb{1}_{\{\tau\geq\bar{t}\}}K\left(1+\bar{t}+|X_{\bar{t}}|\right)\right]\vee\mathbb{E}_{(t',x)}\left[e^{-r\bar{t}}\mathbb{1}_{\{\tau\geq\bar{t}\}}K\left(1+\bar{t}+|X_{\bar{t}}|\right)\right]\\
&\quad<\epsilon,
\end{aligned}$$

where $K > 0$ is a constant.

Let $\epsilon > 0$. In the remaining of the proof, we then take $\bar{t}$ such that

$$\begin{aligned}
|V(t',x)-V(t,x)|\leq\sup_{\tau}\mathbb{E}\Bigg[&\int_0^{\tau\wedge\bar{t}}e^{-rs}\left|f\left(s+t',X_s^{t',x}\right)-f\left(s+t,X_s^{t,x}\right)\right|ds\\
&+e^{-r\tau}\mathbb{1}_{\{\tau\leq\bar{t}\}}\left|g\left(X_\tau^{t',x}\right)-g\left(X_\tau^{t,x}\right)\right|\Bigg]+\epsilon.
\end{aligned}$$

Since g and f are Lipschitz continuous, there exists $C > 0$ such that

$$|V(t',x)-V(t,x)|\leq\mathbb{E}\left[C\bar{t}\wedge 1\left(|t'-t|+\sup_{0\leq s\leq\bar{t}}\left|X_s^{t',x}-X_s^{t,x}\right|\right)\right]+\epsilon.$$

But, by standard estimates (see, e.g., Theorem 2.5.9 in Krylov (2008) and Hölder inequality),

$$\mathbb{E}\left[\sup_{0\leq s\leq\bar{t}}\left|X_s^{t',x}-X_s^{t,x}\right|\right]\leq K\left(1+|x|^2\right)\sqrt{t'-t}$$

where $K > 0$ depends on $\bar{t}$ and the bounds on σ and μ only. Thus $t\to V(t,x)$ is continuous in t for all $x\in\bar{\mathcal{X}}$. ■

Lemma 32 *Suppose that g is bounded, and that condition 4. in Definition 13 holds. Then, for all $\kappa > 0$, there exists T_κ such that $\mathcal{P}_{(t,x)}\left(\tau_{\mathcal{S}}<T_\kappa\right)<\kappa$ for all $(t,x)\in\mathcal{Y}_T$. Moreover, there exists a constant $K > 0$ such that, for all $(t,x)\in\mathcal{Y}_T$, $\mathbb{E}_{(t,x)}\left[\tau_{\mathcal{S}}\right]<K$.*

Proof of Lemma 32. If $T < \infty$, we are done, so suppose not. Without loss of generality, assume that $\bar{t} = 0$. Since g is bounded (say by $M > \epsilon$) and $\tau_{\mathcal{S}}$ is optimal, there exists $N > 0$ and $\delta \in (0,1)$ such that, for all $(t,x) \in \mathcal{Y}_T$,

$$\mathbb{P}_{(t,x)}\left(\tau_{\mathcal{S}} < N\right) > \delta. \tag{B.4}$$

To see this, observe that for all $(t,x) \in \mathcal{Y}_T$,

$$-M \leq V(t,x) \leq \mathbb{E}_{(t,x)}\left[M - \int_0^{\tau_{\mathcal{S}}} \epsilon ds\right].$$

Then

$$\mathbb{E}_{(t,x)}\left[\tau_{\mathcal{S}}\right] \leq \frac{2M}{\epsilon},$$

and, therefore,

$$\mathbb{P}_{(t,x)}\left(\tau_{\mathcal{S}} < \frac{4M}{\epsilon}\right) > \frac{1}{4}.$$

Importantly, observe that the bound on $\mathbb{P}_{(t,x)}\left(\tau_{\mathcal{S}} < N\right)$ holds uniformly over $(t,x) \in \mathcal{Y}_T$. Therefore, for all $t \geq 0$,

$$\mathbb{P}\left(\tau_{\mathcal{S}} - t < N \mid \mathcal{F}_t\right) > \delta.$$

Fix $(t,x) \in \mathcal{Y}_T$. Then

$$\mathbb{P}_{(t,x)}\left(kN \leq \tau_{\mathcal{S}} < kN + N\right) = \mathbb{P}_{(t,x)}\left(\tau_{\mathcal{S}} \geq kN\right) - \mathbb{P}_{(t,x)}\left(\tau_{\mathcal{S}} \geq kN + N\right) \geq \delta \mathbb{P}_{(t,x)}\left(\tau_{\mathcal{S}} \geq kN\right),$$

and, therefore,

$$\mathbb{P}_{(t,x)}\left(\tau_{\mathcal{S}} \geq kN + N\right) \leq (1-\delta)\mathbb{P}\left(\tau_{\mathcal{S}} \geq kN\right).$$

By induction,

$$\mathbb{P}_{(t,x)}\left(\tau_{\mathcal{S}} \geq kN\right) \leq (1-\delta)^k.$$

This proves the first claim of the lemma. To obtain the second claim, note that

$$\tau_{\mathcal{S}} \leq \sum_{k=0}^{\infty}(k+1)N\mathbb{1}_{\{\tau_{\mathcal{S}} \geq kN\}}.$$

Taking expectations, we get

$$\mathbb{E}_{(t,x)}\left[\tau_{\mathcal{S}}\right] \leq N\sum_{k=0}^{\infty}(k+1)(1-\delta)^k := K < \infty.$$

■

B.5.2 Omitted proofs for Section 2.4.2

Proof of Proposition 16. As noted before, Assumption 14 implies that 12 is satisfied. Moreover, by Lemma 3, V^s is continuous and, by Assumption 14 and Lemma 9 in Durandard and Strulovici (2022) if $r = 0$ or Lemma 2 in Durandard and Strulovici (2022) if $r > 0$, the value function grows ar most linearly. So Assumption 10 also holds. Since V^s is convex, there remains only to show that V^s is nonincreasing and strictly decreasing in the continuation region when $t \to \sigma(t,x)$ is strictly decreasing; and that V^s is nondecreasing and strictly increasing in the continuation region when $t \to \sigma(t,x)$ is strictly increasing.

We only prove the first statement as the proof of the second one follows the exact same steps with the obvious changes.

For all $t \in [0,T]$, define $A_t := \int_0^t a(s,X_t)ds$ with $a(s,x) := \frac{\sigma(t'-t+s,x)^2}{\sigma(s,x)^2}$ and $V_t := \inf\{s \geq 0 : A_s > t\}$, the (generalized) inverse of A_t. Let $Y := \{Y_t = X_{V_t}\}_{t\geq 0}$ be the strong Feller process defined by the infinitesimal generator $\mathcal{L}^Y = \frac{1}{a(t,x)}\mathcal{L}^X$. Observe that Y and $X^{(t',x)}$ are identically distributed by construction. Define also $\mathcal{T}^Y$, the set of Y-adapted stopping times. From proposition 7.9 in Kallenberg (2006), we have that for all τ $\mathcal{F}_t$-stopping time, A_τ is a $\mathcal{F}_t^Y$-stopping time, and $\tau = V_{A_\tau}$ $\mathbb{P}$-a.s.. It follows that

$$\begin{aligned}
V^s(t,x) &= \sup_{\tau\in\mathcal{T}} \mathbb{E}_{(t,x)}\left[e^{-r\tau}g(X_\tau) - \int_0^\tau e^{-rs}cds\right] \\
&= \sup_{\tau\in\mathcal{T}} \mathbb{E}_{(t,x)}\left[e^{-r\tau}g(X_\tau) - \int_0^\tau e^{-rs}c\frac{a(s,X_s)}{a(s,X_s)}ds\right] \\
&= \sup_{\tau} \mathbb{E}_{(t,x)}\left[e^{-rV_{A_\tau}}g(X_{V_{A_\tau}}) - \int_0^{V_{A_\tau}} ce^{-rV_{A_s}}\frac{1}{a(V_{A_s},X_{V_{A_s}})}dA_s\right] \\
&= \sup_{\tau\in\mathcal{T}^Y} \mathbb{E}_{(t,x)}\left[e^{-rV_\tau}g(X_{V_\tau}) - \int_0^\tau ce^{-rV_s}\frac{1}{a(V_s,X_{V_s})}ds\right] \\
&\geq \sup_{\tau\in\mathcal{T}^Y} \mathbb{E}_{(t,x)}\left[e^{-rV_\tau}g(Y_\tau) - \int_0^\tau ce^{-rV_s}ds\right] \\
&= \sup_{\tau\in\mathcal{T}^Y} \mathbb{E}_{(t,x)}\left[e^{-r\tau}q(\tau)g(Y_\tau) - \int_0^\tau ce^{-rs}q(s)ds\right] \\
&\geq \sup_{\tau\in\mathcal{T}^Y} \mathbb{E}_{(t,x)}\left[e^{-r\tau}g(Y_\tau) - \int_0^\tau ce^{-rs}ds\right] \\
&= V(t',x),
\end{aligned}$$

where the first inequality follows from the fact that $a(s,X_s) > 1$, the second inequality from the fact that $q(s) = e^{-r(V_s-s)}$ is nondecreasing so the later problem can be seen as a constrained version of the above problem (where τ has to be smaller than the random deadline whose (random) cumulative distribution function is given by $1 - \frac{1}{q(t)}$), and the last equality from the fact that Y and $X^{(t',x)}$ are identically distributed for $a(s,x) = \frac{\sigma(t'-t+s,x)^2}{\sigma(s,x)^2}$.

Finally, if $(t,x) \in \mathcal{C}$, the optimal stopping time is strictly positive, and, therefore, the first inequality is seen to be strict. This concludes the proof. ■

Proof of Proposition 17. The result follows immediately from Proposition 16 and Theorem 6 if the stopping boundaries are locally bounded away from $\{\underline{x}, \bar{x}\}$. This follows from Lemma 33 below. ■

Lemma 33 *Suppose that Assumptions 13 and 14 holds. Then the endpoints of the domain $\underline{x}$ and $\bar{x}$ are in the optimal stopping region of the sampling problem for all $t \in [0, T)$.*

Proof of Lemma 33. Observe that, for all $x \in \mathcal{X}$, the value function in the sampling problem V^s is weakly smaller than

$$\bar{V}(x) := \sup_{\tau \in \mathcal{T}} \mathbb{E}_x \left[e^{-r\tau} g\left(\bar{X}_\tau\right) - \int_0^\tau e^{-rt} c dt \right],$$

subject to

$$\bar{X}_t = x + \int_0^t \bar{\sigma}(X_s) dB_s.$$

This follows from the same argument as in the proof of Lemma 16. Then, from Snell envelope's theorem, the continuation region in the sampling problem is a subset of the continuation region associated with $\bar{V}$:

$$C^{\bar{V}} := \left\{ x \in \bar{\mathcal{X}} \, : \, \bar{V}(x) > g(x) \right\} = [\underline{b}, \bar{b}],$$

with $\underline{b} \leq x^c \leq \bar{b}$, where we used that $\bar{X}$ is strongly Markov and Proposition 13. So to prove the lemma, we can simply show that

$$\bar{x}, \underline{x} \notin C^{\bar{V}}.$$

Observe first that, by Assumption 13, either $\bar{b} \in \mathcal{X}$ or $\underline{b} \in \mathcal{X}$, for, otherwise, $\bar{V}(x) = -\infty$: a contradiction. Without loss of generality assume that $\bar{b} \in \mathcal{X}$. There remains to show that $\underline{b}$ in $\mathcal{X}$. Suppose not. Then

$$g(\underline{x}) = \lim_{x \to \underline{x}} g(x) \leq \lim_{x \to \underline{x}} \bar{V}(x) = \lim_{x \to \underline{x}} \mathbb{E}_x \left[e^{-r\bar{\tau}(\bar{b})} g(\bar{b}) - \int_0^{\bar{\tau}(\bar{b})} ce^{-rt} dt \right] < 0,$$

where the last inequality follows from Assumption 13. But g is nonnegative by Assumption 14: a contradiction. So $\underline{b} \in \mathcal{X}$. This concludes the proof. ■

B.5.3 Omitted proof for Section 2.4.2

Proof of Proposition 18. Assumptions 13 and 14 hold (with $\bar{\sigma}(x) = \frac{\sqrt{2}\sigma_0^2 \alpha}{\alpha^2}$). The result then follows from Proposition 17. Finally, the symmetry of the boundaries is obtained from the symmetry of the original problem. ■

B.5.4 More general deadline structure for Section 2.4.4

Formally, to define stochastic deadlines, we need to enlarge the filtered probability space to $(\Omega \times [0,1], \mathcal{F} \times \mathcal{B}([0,1]), \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P} \otimes \lambda)$, where λ is the Lebesgue measure on $[0,1]$. The extended probability space allows for the randomization device needed for deadlines to be stochastic. All the objects defined on Ω are extended to $\Omega \times [0,1]$ in the obvious way.

Definition 20 *A map $\delta : \Omega \times [0,1] \to [0,\infty]$ is a stochastic deadline if it is a $\mathcal{F}_t \otimes \mathcal{B}([0,1])$-stopping time. We will denote by $\mathcal{D}$ the set of all stochastic deadlines.*

The probability that the deadline d arrives before time t is given by the optional stochastic measure F^δ on $\mathbb{R}$ defined by:

$$F^\delta(t,\omega) := \int_0^1 \mathbb{1}_{\{\delta(\omega,u) \leq t\}} du, \quad \mathbb{P}\text{-a.s..}$$

In this sense, we can interpret a stochastic deadline as a distribution over stopping times.[9]

The DM's problem then consists of finding a decision time τ and a rule $d \in \{-1, 1\}$ to maximize the expected total gain given the presence of a stochastic deadline. The accuracy and urgency of a decision are determined by a gain function together with a stochastic deadline δ and a payoff at the deadline $f(\mu, d)$. The expected total gain is

$$\mathbb{E}\left[\mathbb{1}_{\{\tau\leq\delta\}}\left(\mathbb{1}_{\{d=1,\mu=1\}}+\mathbb{1}_{\{d=-1,\mu=-1\}}\right)+\mathbb{1}_{\{\delta<\tau\}}f(\delta,d,\mu)\right].$$

We assume that $f(\cdot, d)$ is an affine function of $\mathbb{1}_{\{\mu=1\}}$. Then, as in the previous section, the optimal information acquisition problem admits the equivalent optimal stopping formulation:

$$V^{\delta}(t,x) := \sup_{\tau\in\mathcal{T}}\mathbb{E}\left[X_\tau \vee (1-X_\tau)\chi_{\{\tau\leq\delta\}}+f(\delta,-1,X_\delta)\vee f(\delta,-1,X_\delta)\chi_{\{\tau>\delta\}}\right], \qquad (V^\delta)$$

subject to

$$X_t = X_0 + \int_0^t \sigma(s, X_s)dB_s,$$

where X_t is the decision maker belief that $\mu = 1$ at time t. We also assume that the volatility of the belief process is smooth (i.e., $\sigma \in \mathcal{C}^{2,\alpha}$).

Due to the presence of a stochastic deadline, this problem appears different from the stopping problems we considered above. However, in the appendix we show that we can reformulate it to fit our framework. First, we show that it is without loss of generality that every Markovian stochastic deadline can be seen as a deadline that arrives at a under Assumption 18 below . First, observe that

[9]We can also define a stochastic deadline directly as a distribution over stopping times. The proof of the equivalence between these two definitions is available upon request.

Proposition 34 *For all functions g and f, all $\delta \in \mathcal{D}$, and all $\tau \in \mathcal{T}$,*

$$\mathbb{E}\left[g(X_\tau)\chi_{\{\tau\leq\delta\}} + f(\delta, X_\delta)\chi_{\{\tau>\delta\}}\right] = \mathbb{E}\left[(1-F_\tau^\delta)g(X_\tau) + \int_0^\tau f(t, X_t)dF_t^\delta\right].$$

Proof of Proposition 34. Let $\delta \in \mathcal{D}$, $d \in \{-1, 1\}$ and $\tau \in \mathcal{T}$. Recall that $F_t^\delta = \int_0^1 \mathbb{1}_{\{\delta(\omega,u)\leq t\}}du$. Then

$$\begin{aligned}
&\mathbb{E}\left[g(X_\tau)\chi_{\{\tau\leq\delta\}} + f(\delta(\omega,u), d, X_\delta(\omega,u))\mathbb{1}_{\{\tau>\delta\}}\right] \\
&\quad= \mathbb{E}\left[g(X_\tau)\int_0^1 \mathbb{1}_{\{\delta(\omega,u)\geq\tau\}}du + \int_0^1 f(\delta(\omega,u), d, X_{\delta(\omega,u)})\mathbb{1}_{\{\tau>\delta(\omega,u)\}}du\right] \\
&\quad= \mathbb{E}\left[g(X_\tau)\int_0^1 \mathbb{1}_{\left\{F_\tau^{\delta(\omega,u)}\leq u\right\}}du + \int_0^1 f(\delta(\omega,u), d, X_{\delta(\omega,u)})\mathbb{1}_{\{\tau>\delta(\omega,u)\}}du\right] \\
&\quad= \mathbb{E}\left[(1-F_\tau^\delta)g(X_\tau) + \int_0^\tau f(\delta, d, X_\delta)dF_t^\delta\right],
\end{aligned}$$

where we used Fubini's theorem and the independence of τ and $\mathcal{B}([0,1])$ to get the first equality and proposition 4.9 of chapter 0 in Revuz and Yor (2013) to get the last one. ■

Next, for all δ, define A_t through the bijection

$$F_t^\delta = 1 - e^{-A_t}.$$

Assumption 18 *Suppose that $r > 0$. The process A_t is a nonnegative continuous additive functional with*

$$A_t := \int_0^t a(s, X_s)ds \,\mathbb{P}\text{-}a.s.,$$

where the function $a : \mathcal{Y}_T \to [r, \infty)$ is twice continuously differentiable with α-Hölder derivatives.

That is, we assume that the deadline arrives at rate $a(t, X_t)$. Then

$$V^\delta(t,x) = \sup_{\tau \in \mathcal{T}(t)} \mathbb{E}_{(t,x)} \left[e^{-\int_0^\tau a(s,X_s)ds} X_\tau \vee (1 - X_\tau) + \int_0^\tau e^{-\int_0^t a(s,X_s)ds} f(t, X_t) a(t, X_t) dt \right].$$

subject to

$$X_{t+s} = X_t + \int_t^{t+s} \sigma(u, X_u) dB_u.$$

Next, we present two useful lemmas.

Lemma 34 *For all $t \in [0, T)$, the value function (V^δ) is convex in x.*

Proof of Lemma 34. By the same argument as in the proof of Proposition 16, we see that, for all $(t, x) \in \mathcal{Y}_T$,

$$\begin{aligned} V^d(t,x) &= \sup_{\tau \in \mathcal{T}(t)} \mathbb{E}_{(t,x)} \left[e^{-\int_0^\tau a(s,X_s)ds} g(X_\tau) + \int_0^\tau e^{-\int_0^t a(s,X_s)ds} f(t, X_t) a(t, X_t) dt \right] \\ &= \sup_{\tau \in \mathcal{T}^Y(t)} \mathbb{E}_{(t,x)} \left[e^{-(\tau - t)} g(X_\tau) + \int_0^\tau e^{-t} f(V_t, Y_t) dt \right], \end{aligned}$$

where Y is the strong Feller process with generator $\frac{1}{a(t,x)} \mathcal{L}^{(t,x)}$, $\mathcal{T}^Y(t)$ is the set of stopping time adapted to the filtration generated by Y, and $V_t = \int_0^t \frac{1}{a(s,X_s)} ds$ is twice continuously differentiable with α-Hölder derivatives as a consequence of Assumption 18. As a result, Proposition 15 applies and the value function is convex in x for all $t \in [0, T)$. ■

Lemma 35 *The value function (V^δ) is continuous and bounded.*

Proof of Lemma 35. This follows immediately from Lemmas 34 and 31 as in the proof of Lemma 3. ■

Proof of Proposition 20. By Lemma 35, the value function V^δ is continuous. By Lemma 34, it is convex in x, and thus $sign\,(\sigma_t(t,x) V_{xx}(t,x)) = sign\,(\sigma_t(t,x))$. Moreover, by

Proposition 34, in this setting the flow payoff of the principal is $f(t,x) = a(t,x)$. So, when $a(t,x)$ is strictly decreasing, $a_t(t,x)\,(V(t,x)-1) \geq 0$, since $V(t,x) \leq 1$; and when $a(t,x)$ is strictly decreasing, $a_t(t,x)\,(V(t,x)-1) \geq 0$, since $V(t,x) \leq 1$. From Remark 3, we can then apply Corollary 4 if V is monotone in t, strictly so in the stopping region. This follows from the same argument as the proof of Proposition 16 and the fact that rate of arrival of news is monotone over time. So Corollary 4 applies and we obtain the desired result. ■

Appendix C

Appendix to Chapter Three

C.1 Proof of Theorem 8

Suppose $(1')$ and (3) hold for some $N^* \subseteq N$. Then,

$$\begin{aligned}
\sup_{d\in D}\inf_{n\in N}\ \pi(d,n) &\leq \sup_{d\in D}\inf_{n\in N^*}\ \pi(d,n) && (N^* \subseteq N)\\
&= \sup_{d\in D^*}\inf_{n\in N^*}\ \pi(d,n) && (1')\\
&= \sup_{d\in D^*}\inf_{n\in N}\ \pi(d,n) && (3)\\
&\leq \sup_{d\in D}\inf_{n\in N}\ \pi(d,n) && (D^* \subseteq D)
\end{aligned}$$

So $\sup_{d\in D}\inf_{n\in N}\ \pi(d,n) = \sup_{d\in D^*}\inf_{n\in N}\ \pi(d,n)$, as desired.

To show the converse, suppose that

$$\sup_{d\in D}\inf_{n\in N}\ \pi(d,n) = \sup_{d\in D^*}\inf_{n\in N}\ \pi(d,n)$$

Then this immediately implies conditions (1') and (3) for $N^* = N$.

C.2 Proof of Proposition 23

As in the main text, consider

- $D^* := \{d \in D \,:\, d = \{w : \mathbb{R}_+ \to \mathbb{R},\, y \to \alpha y\}\,,\, \alpha \geq 0\}$: the set of singleton menu that offers one linear contract, and
- $N^* := \left\{\mathcal{A} \,:\, \mathcal{A} = \mathcal{A}_0 \cup (0, \delta_{\{y\}}),\, y \in \left[0, \bigvee_{(c,F)\in\mathcal{A}_0} \left(\mathbb{E}_{\tilde{y}\sim F}\left[\tilde{y}\right] - c\right)\right]\right\}$: the set of technologies that includes the known sets $\mathcal{A}_0$ and a unique other option $(0, \delta_{\{y\}})$, where y is less than the maximum surplus of the known actions.

Then, for $d \in D^*$ and $n \in N^*$, π rewrites

$$\pi\left(d, n\right) = (1-\alpha)y + i\left(\alpha y \geq \bigvee_{(c,F)\in\mathcal{A}_0} \left(\alpha\mathbb{E}_{\tilde{y}\sim F}\left[\tilde{y}\right] - c\right)\right).$$

To conclude, there only remains to show that conditions (1), (2), and (3) of Theorem 7 hold. Note that, WLOG, we can focus on menus over random contracts such that, for any w in the support of the lottery, $\pi\left(w, n\right) < \infty$.

First, we show that for all $n \in N^*$, there exists $d \in D^*$ that maximizes $\pi(d, n)$. To see this, let d be any menus over randomized contracts and let w^n be the contract that maximizes $\pi(w, n)$ among those in the support of the lottery chosen by the agent whose technology is $n = \mathcal{A}_0 \cup \left\{\left(0, \delta_{\{y\}}\right)\right\}$ for some $y \in \left[0, \bigvee_{(c,F)\in\mathcal{A}_0} \left(\mathbb{E}_{\tilde{y}\sim F}\left[\tilde{y}\right] - c\right)\right]$. Consider then singleton menu that offers the linear contract with slope $\alpha = \frac{w^n(y)}{y}$. Distinguish two cases: (i) either $\pi\left(y \to \alpha y, n\right) = \infty$ and we are done, (ii) or $\pi\left(y \to \alpha y, n\right) < \infty$, in which case $\pi\left(\{y \to \alpha y\}, n\right) = \pi\left(\{w^n\}, n\right) \geq \pi\left(d, n\right)$. So, for all $n \in N^*$, the principal has a best response in D^*. Next, note that it is obvious that, for each $d \in D^*$, Nature has a best response in N^*: it can simply pick the y that minimizes $\pi(d,n)$. Finally, we show that

$$\sup_{d\in D^*}\inf_{n\in N^*}\pi\left(d,n\right)=\inf_{n\in N^*}\sup_{d\in D^*}\pi\left(d,n\right).$$

Letting

$$i(s)=\begin{cases}0 \text{ if } s \text{ is true} + \infty \text{ otherwise,}\end{cases}$$

this is equivalent to

$$\begin{aligned}&\sup_{\alpha\geq 0}\inf_{y\in\left[0,\bigvee_{(c,F)\in\mathcal{A}_0}\left(\mathbb{E}_{\tilde{y}\sim F}[\tilde{y}]-c\right)\right]}(1-\alpha)y+i\left(\alpha y\geq\bigvee_{(c,F)\in\mathcal{A}_0}\left(\alpha\mathbb{E}_{\tilde{y}\sim F}\left[\tilde{y}\right]-c\right)\right)\\&=\inf_{y\in\left[0,\bigvee_{(c,F)\in\mathcal{A}_0}\left(\mathbb{E}_{\tilde{y}\sim F}[\tilde{y}]-c\right)\right]}\sup_{\alpha\geq 0}(1-\alpha)y+i\left(\alpha y\geq\bigvee_{(c,F)\in\mathcal{A}_0}\left(\alpha\mathbb{E}_{\tilde{y}\sim F}\left[\tilde{y}\right]-c\right)\right),\end{aligned}$$

which holds by Theorem 2 in Geraghty and Lin (1984).

Then Theorem 7 guarantees that there exists a robustly ϵ-optimal singleton menu that offers a linear contract. To conclude, there remains to show existence. WLOG, we can assume that $\alpha\in[0,\bar{\alpha}]$ for some $\bar{\alpha}>0$, hence that the set of linear contract is compact. Existence then follows if $\alpha\to\underset{n\in N^*}{\pi(\alpha,n)}$ is upper-semicontinuous, which is easy to verify by Berge's theorem.

C.3 Proof of Proposition 24

Let N^* be the set of all information processes such that the associated demand curve, G, is a continuous c.d.f. with $\tilde{G}\left(\mathbb{E}_{v\sim F}\left[v\right]\right)=1$. Here, $G(p)$ is the buying probability at price p at time 1. Let D^* be the set of mechanism that randomizes over constant posted price $p\in\mathbb{R}_+$ in every period.

By the replacement lemma (Lemma 1) in Libgober and Mu (2021), we know that when the principal uses a constant posted price mechanism with $p_t = p \in \mathbb{R}_+$ for all t, Nature has a best-response among the class of information processes that reveals information in period $t = 1$ only. By Proposition 1 in Libgober and Mu (2021), Nature's best response is then the "pressed" distribution $\bar{G}$ associated with F, which is continuous on $[\underline{v}, \mathbb{E}_{\tilde{v}\sim F}[v]]$ with $\bar{G}(\underline{v}) = F(\underline{v})$ and $\bar{G}(\mathbb{E}_{\tilde{v}\sim F}[v]) = 1$. So condition (2) of Theorem 7 holds.

Furthermore, by classic results from the theory of durable good monopolist (see, e.g., Stokey (1979) or Riley and Zeckhauser (1983)), the principal optimal mechanism when facing a constant demand curve is a constant posted price mechanism, i.e., it can be found in D^*. To see this, simply note that the principal's problem is the same as the one in which the buyer's valuation are fixed and distributed according to G. So, condition (1) of Theorem 7 holds.

Finally, when Nature is restricted to choose information processes from N^* and the principal is restricted from choosing random constant posted price mechanisms, the principal's value is

$$\sup_{P\in D^*} \inf_{G\in N^*} \int_0^\infty p\,(1 - G(p))\, dP(p),$$

which, by Sion's minimax theorem, is equal to

$$\inf_{G\in N^*} \sup_{P\in D^*} \int_0^\infty p\,(1 - G(p))\, dP(p).$$

So condition (3) of Theorem 7 holds.

Theorem 7 then guarantees that an ϵ-optimal robust selling mechanism can be found among the class of randomized constant posted price mechanisms.

There only remains to show that the principal does not need to randomize, which is immediate from Proposition 1 in Libgober and Mu (2021), since

$$\sup_{P\in D^*}\inf_{G\in N^*}\int_0^\infty p\,(1-G(p))\,dP(p)=\sup_{P\in D^*}\int_0^\infty p\left(1-\bar{G}(p)\right)dP(p).$$

Moreover, existence also follows as the "pressed" distribution $\bar{G}$ is continuous on the convex closure of $supp(F)$, and $p=\inf\{v\,:\,v\in supp(F)\}$ is not optimal by Assumption 15.

C.4 Proof of Proposition 25

Points 1 and 2 follow by replication. Given that the agent only learns her total valuation for the bundle, the problem reduces to a one-dimensional maximization in which a posted-price for the grand bundle is optimal. This is the pointwise limit of grand-bundle mechanisms with continuous transfers. Conversely, given that a random bundling mechanism is measurable with respect to the buyer's total valuation, it is without loss for Nature to only reveal this information.

Finally, the minimax equality (3) holds once we restrict to (D^*, N^*). The objective function is a bilinear form $\int t\ dG$, and we have restricted t to be continuous. Additionally, D^* is convex and N^* is convex and compact. So Theorem 7 gives the existence of a robustly ϵ-optimal random bundling mechanism. Existence then follows from Helly's selection theorem, since the transfers must be nondecreasing, and the continuity of $\pi(t,G)$ for all $G\in\Delta([0,N])$ in the topology associated with pointwise convergence.

C.5 Proof of Proposition 26 and Corollary 8

Proof of Proposition 26: Let N^* consists in the set of randomization over deterministic type processes:

$$N^* := \{F \in N \,:\, \forall 2 \leq t \leq T,\, v_t = f_t(v_1)\, F\text{-a.s. for some function } f_t : V_1 \to V_t\}\,.$$

Let D^* be the set of (non-adaptive) sequence of posted prices in $[0, \bar{v}_t]$:

$$D^* := \Delta\left(\prod_{t=1}^{T}[0, \bar{v}_t]\right).$$

Then, for any $P \in D^*$ and $F \in N^*$, we have

$$\pi(P, F) = \sum_{t=1}^{T} \delta^t \int_0^{\bar{v}_t} (1 - F_t(p_t)) p_t dP_t(p_t)$$

Next observe that, by Theorem 4' (and the remark afterwards) in Baron and Besanko (1984), for all $F \in N^*$, the principal has a best response in D^*. It is also easy to see that, for all (non-adaptive) sequence of random posted prices in $[0, \bar{v}_t]$, Nature has a best response in N^*. Consider, for example, the set of probability distributions over all sequences $(v_t)_{t=1}^T$, where $v_t \in \left[0, \bar{V}\right]$ such that $\mathbb{E}_{v\sim F}\left[v_t\right] = \bar{v}_t$. This is a subset of N^*, which can generate every demand curve $1 - F_t(p)$, $1 \leq t \leq T$. Clearly, for each $P \in D^*$, a minimizer for Nature can then be found in N^*. Therefore, to conclude using Theorem 7, there only remains to show that condition (3) holds. This follows from Sion's minimax theorem and Theorem 13. To see this, note that any F is nonnegative nondecreasing and bounded by 1, hence is a nonnegative Borel function bounded by 1. Therefore F belongs to the closure (for the topology associated with the pointwise convergence) of the set of continuous functions on $[0, \bar{V}]$ bounded by 1. Finally, $f \to \int_0^{\bar{V}} f dP_t(p)$ is continuous by the dominated convergence theorem.

So, by Theorem 7, there exists a robust ϵ-optimal mechanism that consists in a sequence of random posted prices.

Proof of Corollary 8: This follows immediately from Proposition 26 and Proposition 5 in Carrasco et al. (2018).

C.6 Proof of Proposition 27

For the next constructions, we will rely heavily on the projects $A_y = (y - x, \delta_y)$ and $A_x = (0, \delta_x)$ for $y > x > 0$. Let r_y and r_x denote the corresponding indices. By construction $r_y = r_x = x$. Under any contract w, the indices are $r_y^w = w(y) - (y - x)$ and $r_x^w = w(x)$.

Order-preserving $\implies$ debt contract:

First, we show that if w is order-preserving, then it is non-decreasing. Suppose to the contrary that for some $y > x$, $w(y) < w(x)$. Then, w reverses the order of projects $(0, \delta_y)$ and $(0, \delta_x)$, so w is not order preserving.

Next, we show that if w is order-preserving, it is 1-Lipschitz. That is, for any $y > x$, $w(y) - w(x) \leq y - x$. Fix any $y > x$. Consider the contracts A_y and A_x. If $r_y^w \leq 0$, then $w(y) - (y - x) \leq 0$, meaning $w(y) - w(x) \leq y - x$ because $w(x) \geq 0$. If $r_y^w > 0$, then by the order preserving property, we must have $r_y^w = r_x^w$, giving us $w(y) - (y - x) = w(x)$, so we have linearity: $w(y) - w(x) = y - x$.

The same argument shows that for any x such that $w(x) > 0$, $y > x$ implies $w(y) - w(x) = y - x$, meaning that the contract has a slope of 1 above x. Fix any $x > 0$ with $w(x) > 0$ and $y > x$. Then $r_y = r_x = x > 0$, and $r_x^w > 0$. By the order-preserving property, $r_y^w = r_x^w$, implying $w(y) - w(x) = y - x$.

Finally, we show that any 1-Lipschitz and monotone contract with the previous property is a debt contract. Define $z = \inf\{x \mid w(x) > 0\}$. Since w is 1-Lipschitz, and therefore continuous, we have $\lim_{x \to z^-} w(x) = \lim_{x \to z^+} w(x) = 0$. Therefore, $w(x) = (x - z)^+$, so w is a z-debt contract.

IIA $\implies$ debt contract: The outline of the proof is largely the same as the previous section, but we work with the principal's payoff rather than the agent's. First, we show that if w satisfies IIA, it must be non-decreasing. Suppose to the contrary that $y > x$ and $w(y) < w(x)$. Then,

$$V_P(w \mid \{(0, \delta_y), (0, \delta_x)\}) = x - w(x) < y - w(y) = V_P(w \mid \{(0, \delta_y)\}).$$

Since the principal is made worse-off by the inclusion of the project $(0, \delta_x)$, this violates IIA.

Next, we show that if w satisfies IIA, it must be continuous. Suppose towards the contrary that w has a discontinuity at some $y > 0$. Since w is non-decreasing, this is an upwards jump. Then $x - w(x)$, the principal's payoff, has a downwards jump discontinuity at y. For small enough values of $\epsilon > 0$,

$$V_P(w \mid \{(0, \delta_{y-\epsilon}), (0, \delta_y)\}) = y - w(y) < (y - \epsilon) - w(y - \epsilon) = V_P(w \mid \{(0, \delta_{y-\epsilon})\}),$$

which violates IIA.

Finally, we show that for any x such that $w(x) > 0$, $y > x$ implies $w(y) - w(x) = y - x$. First, consider any $\epsilon < w(x)$, and the projects (ϵ, δ_x) and $(0, \delta_y)$. If w satisfies IIA, we require

$$V_P(w \mid \{(\epsilon, \delta_x), (0, \delta_y)\}) \geq V_P(w \mid \{(\epsilon, \delta_x)\}),$$

which implies

$$x - w(x) \leq y - w(y) \tag{C.1}$$

Alternatively, consider projects $A'_x = (0, \delta_x)$ and $A'_y = (w(y) - w(x) + \epsilon, \delta_y)$. Notice by construction that $r_x^w = w(x)$ and $r_y^w = w(x) - \epsilon$. If w satisfies IIA, we require

$$V_P(w \mid \{A'_x, A'_y\}) \geq V_P(w \mid \{A'_y\}),$$

which implies

$$x - w(x) \geq y - w(y). \tag{C.2}$$

Combining equations (C.1) and (C.2) gives the desired result that if $w(x) > 0$ and $y > x$, then $w(y) - w(x) = y - x$. The same continuity argument as in the order-preserving case concludes that w is a debt contract.

C.7 Extending the minimax theorem

Theorem 13 *Let X and Y be subsets of a metrizable space and $\pi : X \times Y \to \mathbb{R}$ be a function such that*

1. *for all $x \in X$, $\inf_{y \in Y} \pi(x, y) = \inf_{y \in \bar{Y}} \pi(x, y)$, and*
2. $\sup_{x \in X} \inf_{y \in Y} \pi(x, y) = \inf_{y \in Y} \sup_{x \in X} \pi(x, y)$.

Then

$$\sup_{x \in X} \inf_{y \in \bar{Y}} \pi(x, y) = \inf_{y \in \bar{Y}} \sup_{x \in X} \pi(x, y).$$

Proof. The statement follows from the following chain of inequalities:

$$
\begin{aligned}
\inf_{y \in \bar{Y}} \sup_{x \in X} \pi(x, y) &\geq \sup_{x \in X} \inf_{y \in \bar{Y}} \pi(x, y) \\
&= \sup_{x \in X} \inf_{y \in Y} \pi(x, y) \\
&= \inf_{y \in Y} \sup_{x \in X} \pi(x, y) \\
&\geq \inf_{y \in \bar{Y}} \sup_{x \in X} \pi(x, y).
\end{aligned}
$$

The first equality follows from condition 1., the second from condition 2., and the last inequality follows from the inclusion $Y \subseteq \bar{Y}$. ■