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Empirical Measures for Integrable Eigenfunctions Restricted to Invariant Curves

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ABSTRACT

Empirical Measures for Integrable Eigenfunctions Restricted to Invariant Curves

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We introduce empirical measures to study the L^2 norms of restrictions of quantum integrable eigenfunctions to the unique rotationally invariant geodesic H on a convex surface of revolution. The weak* limit of these measures describes the dependence of their size on H in terms of the angular momentum. The limit measures blow up $(1-c^2)^{-1/2}$ at the end points $c = \pm 1$, reflecting the fact that the Gaussian beam sequence is the largest on H.

We then use the quantized action operators constructed by Colin de Verdière on these surfaces to show that there is a unitary Fourier integral operator which conjugates them to the standard action operators on the round sphere up to finite rank error, showing that all of these surfaces are essentially equivalent in terms of quantum integrability of the Laplacian.

Afterwards we move on to study asymptotics of ladder sequences of spherical harmonics and show that they have Airy-type behavior in a shrinking neighborhood of these circles. This provides a more explicit calculation of the quantities appearing in classical expansions by Thorne and Olver for the Legendre functions in terms of the geometry on the sphere.

We also include expository notes which are intended to be a practical introduction to homogeneous Lagrangian distributions, Fourier integral operators, and the symbol calculus of composition. The primary focus is on examples and line-by-line calculation, including a description of the singularities of the Duistermaat-Guillemin wave trace using the symbol calculus.

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CHAPTER 1

Introduction and Organization

We begin by discussing the organization of this document. Chapters 3 and 4 are dedicated to the proofs of the main results, discussed below in section 1.1. Chapter 2 is a synthesis of personal notes that were written during the process of learning the symbol calculus of Fourier integral operators. The aim of these notes is to provide a practical introduction to the symbol calculus of Lagrangian distributions and Fourier integral operators. For this reason, we keep the quotation of abstract theory to a minimum, stating only what is needed to parse the examples and calculations that follow, and defer to the excellent sources [**21**],[**9**],[**23**],[**6**],[**18**],[**19**],[**7**] for the detailed general theory. The focus is instead on explaining the symbolic data of commonly encountered examples of Lagrangian distributions and Fourier integral operators, as well as an algorithmic presentation of the symbol calculus. The goal is to leave the reader with the ability to fill in the details in the proofs of many such calculations in the literature and execute their own. Chapter 2 begins with the bare minimum basic theory needed to understand the examples that follow in section 2.2. From there, we briefly recall the basics of Fourier integral operators before moving on to describe the 'recipe' of symbolic composition in section 2.4 in general terms. Finally, section 2.5 is a detailed, symbolic, line-by-line calculation of the celebrated trace of the wave group calculated by Duistermaat and Guillemin [**8**].

In chapter 3 we consider a convex surface of revolution (S^2, g). Letting ∂_θ be the vector field that generates the S^1 symmetry, Colin de Verdiére [**4**] has shown the existence of a first order pseudo-differential operator $\widehat{I}_2$ which commutes with the Laplacian Δ_g and $D_\theta = -i\partial_\theta$ such that the joint spectrum of $\widehat{I}_2$ and D_θ consists of a lattice of simple eigenvalues,

$$\mathrm{Spec}(\widehat{I}_2, D_\theta) = \{(\ell, m) \in \mathbb{Z}^2 \mid \ell \geq 0; |m| \leq \ell\}. \tag{1.0.1}$$

The operator $\widehat{I}_2$ is analogous to the degree operator A on the round sphere (S^2, g_{can}),

$$A = \sqrt{-\Delta_{g_{\mathrm{can}}} + \frac{1}{4}} - \frac{1}{2} \tag{1.0.2}$$

whose joint spectrum with D_θ is exactly (1.0.1). The similarity of $\widehat{I}_2$ and A suggests that, in terms of the spectral theory of the Laplacian, all convex surfaces should be 'equivalent' to the round sphere. We codify this by showing the existence of a unitary Fourier integral operator which commutes with D_θ and conjugates $\widehat{I}_2$ into A up to finite rank error. Next, we fix an orthonormal basis of joint eigenfunctions φ_m^ℓ satisfying $\widehat{I}_2\varphi_m^\ell = \ell\varphi_m^\ell$, $D_\theta\varphi_m^\ell = m\varphi_m^\ell$ and study how these joint eigenfunctions concentrate across a fixed ℓ-eigenspace of $\widehat{I}_2$ as $\ell \to \infty$ via the **empirical measures**

$$\mu_\ell = \frac{1}{M_\ell} \sum_{m=-\ell}^{\ell} ||\varphi_m^\ell||_{L^2(H)}^2 \delta_{\frac{m}{\ell}} \tag{1.0.3}$$

which are normalized to be probability measures on $[-1, 1]$. Here $H \subset S^2$ is the unique rotationally invariant geodesic on (S^2, g) and we study the weak limit of this sequence of measures, which is absolutely continuous with respect to Lebesgue measure on $[-1, 1]$

and whose shape describes the relative concentration of the φ_m^ℓ on H as m varies across the $\widehat{I}_2$ eigenspace.

In chapter 4 we study special sequences of the standard spherical harmonics,

$$Y_N^m(\phi,\theta) = \sqrt{\frac{2N+1}{4\pi}\frac{(N-m)!}{(N+m)!}} P_N^m(\cos\phi)e^{im\theta}, \tag{1.0.4}$$

on the round sphere (S^2, g_{can}). Let $0 \leq m_k \leq N_k$ be sequences of integers so that $m_k/(N_k+\frac{1}{2}) = c \in (0,1)$ for all $k \in \mathbb{N}$. The **ladder sequence** $Y_{N_k}^{m_k}$ is actually a semi-classical Lagrangian distribution with respect to the semi-classical parameter $h_k = (N_k + \frac{1}{2})^{-1}$ and its associated Lagrangian is the torus $T_c \subset T^*S^2$, consisting of all geodesics in S^*S^2 which make an angle ψ with the equator satisfying $\cos\psi = c$. There are two latitude circles on S^2 which are tangent to every geodesic making a fixed angle $\cos\psi = c$ with the equator. The projection of T_c down to the sphere has a fold singularity over these latitude circles which makes them caustics for the such a ladder sequence. We study the scaled asymptotics of ladder sequences in an $h_k^{2/3}$ neighborhood of these caustic latitude circles and derive full asymptotic expansions in terms of the Airy function. These correspond to classical asymptotic expansions for the Legendre functions proven by Thorne and later Olver [**33**],[**30**]. The advantage of this approach is that the quantities appearing in the expansion have clear meaning in terms of classical mechanics on the Lagrangian T_c, while the ODE methods used previously did not yield as explicit expressions.

1.1. Statement and discussion of the main results

We begin with the results of chapter 3 concerning convex surfaces of revolution. In order to state the results, we need to briefly describe the underlying geometry. Let (S^2, g) be a convex surface of revolution. Fix a meridian geodeisc γ_0 joining the poles of length

$L > 0$ and let θ be the polar angle measured relative to γ_0, $r > 0$ be the geodesic distance from a chosen pole. Then we have coordinates

$$S^2 \cong \{(r, \theta) \mid r \in (0, L), \theta \in [0, 2\pi)\}$$

in which the metric is of the form

$$g = dr^2 + a(r)d\theta^2$$

where $a(r)$ has a single critical point $a'(r_0) = 0$ with $a''(r_0) < 0$. Let $\widehat{I_2}$ be first order pseudo-differential operator of [**4**]. $\widehat{I_2}$ is self-adjoint and elliptic, commutes with $-\Delta_g$ and $D_\theta = -i\partial_\theta$, and has joint spectrum (1.0.1). In what follows, we fix an orthonormal basis φ_m^ℓ of joint eigenfunctions of $\widehat{I_2}$ and D_θ. The principal symbols of $\widehat{I_2}$ and $\widehat{I_1} = D_\theta$, I_2 and $I_1 = p_\theta$, are homogeneous, Poisson commuting smooth functions on $T^*S^2 \setminus 0$ and are called the action variables for the geodesic flow. Their Hamiltonian flows are 2π-periodic so their joint flow $\Phi_{\mathbf{t}}$ defines a homogeneous, Hamiltonian action of the torus T^2. The joint flow preserves level sets of both I_2 and p_θ and by homogeneity, all of the information is contained in the $I_2 = 1$ level set, which we denote by $\Sigma \subset T^*S^2 \setminus 0$. On Σ, $|I_1| \leq 1$ and for $c \in [-1, 1]$, we let $T_c = I_1^{-1}(c) \cap \Sigma$. For $c \neq \pm 1$, these level sets are diffeomorphic to T^2 and consist of a single orbit of the joint flow. The levels $T_{\pm 1}$ consist of I_2 unit covectors tangent to H with the sign reflecting the orientation relative to ∂_θ. The first result is a global conjugation to a normal form:

Theorem 1.1.1. *Let (S^2, g) be a convex surface of revolution and $A = \sqrt{-\Delta_{g_{can}} + \frac{1}{4}} - \frac{1}{2}$ be the degree operator on the round sphere. There exists a homogeneous unitary Fourier integral operator*

$$W : L^2(S^2, g_{can}) \to L^2(S^2, g)$$

such that $[W, D_\theta] = 0$ *and* $W^*\widehat{I_2}W = A + R$ *where* R *is a finite rank operator. Consequently, if* Y^ℓ_m *denotes the standard orthonormal basis of* $L^2(S^2, g_{can})$ *such that* $AY^\ell_m = \ell Y^\ell_m$, $D_\theta Y^\ell_m = mY^\ell_m$, *then for* ℓ *large enough, there are constants* c^ℓ_m *with* $|c^\ell_m| = 1$ *so that*

$$WY^\ell_m = c^\ell_m \varphi^\ell_m \tag{1.1.1}$$

In **[25]**, Lerman proves that there is only one homogeneous hamiltonian action of the torus T^2 on $T^*S^2 \setminus 0$ up to symplectomorphism. In particular, letting $p_2(x, \xi) = |\xi|_{g_{can}(x)}$ be the principal symbol of A, p_θ and p_2 generate such an action, so there is a homogeneous symplectomorphism χ on $T^*S^2 \setminus 0$ satisfying $\chi^* p_\theta = p_\theta$ and $\chi^* I_2 = p_2$. Theorem 1.1.1 is essentially an operator theoretic upgrade of this symplectic equivalence.

To state the result regarding the empirical measures (1.0.3), we need to set more notation. Let $d\mu_L$ denote Liouville measure on Σ and $d\mu_{c,L} = d\mu_L/dp_\theta$ denote Liouville measures on the regular tori T_c. The torus action $\Phi_{\mathbf{t}}$ commutes with the geodesic flow $G^t = \exp tH_{|\xi|_g}$ and we can write

$$|\xi|_g = K(I_1, I_2) \tag{1.1.2}$$

For a smooth function K on $\mathbb{R}^2 \setminus 0$, homogeneous of degree 1. We let $(\omega_1, \omega_2) = \nabla_I K(I_1, I_2)$ be the so-called **frequency vector** associated to this action. The ω_i are themselves functions of the action variables I_1, I_2. For a homogeneous pseudo-differential operator $B \in \Psi^0(S^2)$, we let $\sigma(B)$ denote its principal symbol and set $\widehat{\sigma(B)}(c) = \int_{T_c} \sigma(B)\, d\mu_{c,L}$. We also let $\omega(B) = \int_\Sigma \sigma(B) d\mu_L$ be the Liouville state on

B. Letting $T^*_H S^2 = \{(x,\xi) \in T^*S^2 \mid x \in H\}$ be the set of all covectors based on H, we observe that for $(x,\xi) \in T_H S^2 \cap T_c$,

$$p_\theta(x,\xi)^2 = |\xi|_g^2 a(r_0)^2 \cos^2\phi = K(c,1)^2 a(r_0)^2 \cos^2\phi \tag{1.1.3}$$

where ϕ is the angle between the covector ξ and H and r_0 is the distance from the north pole to H so that $H = \{r = r_0\}$. Let $\mathcal{L}(H)$ be the length of H. Then $a(r_0) = \mathcal{L}(H)/2\pi$.

Theorem 1.1.2. *Let (S^2, g) be a convex surface of revolution where $g = dr^2 + a(r)^2 d\theta^2$ in geodesic polar coordinates. Let $H \subset S^2$ be the equator, the unique rotationally invariant geodesic. Then in terms of action-angle variables,*

(a) For every $f \in C^0([-1,1])$,

$$\int_{-1}^{1} f(c)\, d\mu_\ell(c) = \frac{1}{M_\ell} \sum_{m=-\ell}^{\ell} ||\varphi_m^\ell||^2_{L^2(H)} f\left(\frac{m}{\ell}\right) \to \frac{1}{M}\int_{-1}^{1} f(c) \frac{\omega_2(c,1)}{\sqrt{1 - \frac{(2\pi)^2 c^2}{K(c,1)^2 \mathcal{L}(H)^2}}}\, dc$$

(b) For any $f \in C^0([-1,1])$,

$$\int_{-1}^{1} f(c)\, d\nu_\ell(c) = \frac{1}{N_\ell(B)} \sum_{m=-\ell}^{\ell} \langle B\varphi_m^\ell, \varphi_m^\ell\rangle_{L^2(S^2,g)} f\left(\frac{m}{\ell}\right) \to \frac{1}{\omega(B)}\int_{-1}^{1} f(c)\widehat{\sigma(B)}(c)\, dc$$

The constant appearing in (a) is

$$M = \int_{-1}^{1} \frac{\omega_2(c,1)}{\sqrt{1 - \frac{(2\pi)^2 c^2}{K(c,1)^2 \mathcal{L}(H)}}}\, dc$$

and normalizes the limit measure to have mass 1 on $[-1,1]$.

When (S^2, g_{can}) is the standard sphere, $\mathcal{L}(H) = 2\pi$, $K(c,1) = 1$ and $\omega_2(c,1) = 1$, hence

$$\frac{\omega_2(c,1)}{\sqrt{1 - \frac{(2\pi)^2 c^2}{K(c,1)^2 \mathcal{L}(H)^2}}} = \frac{1}{\sqrt{1-c^2}} \tag{1.1.4}$$

When $c = \pm 1$, T_c collapses to the set of I_2 unit covectors tangent to H. On $T_{\pm 1}$, $\omega_2 = \frac{\partial K}{\partial I_2} = \frac{\partial K}{\partial I_1} = a(r_0)^{-2} = \frac{(2\pi)^2}{\mathcal{L}^2(H)}$ and $\phi = 0$. The left hand side of (1.1.4) therefore blows up at $c = \pm 1$. Blow-up at the end points is to be expected since the end points of the interval correspond $c = m/\ell = \pm 1$. The eigenfunctions $\varphi^{\ell}_{\pm \ell}$ are Gaussian beam sequences which concentrate microlocally on $\Sigma \cap T^*H$ and are extremizing sequences for the universal L^2 restriction bounds proven by Burq, Gerard, and Tzvetkov [**2**]. The measures μ_ℓ considered here are closely related to the empirical measures associated to a polarized toric Kähler manifold $L \to M^n$ studied in [**39**],

$$\mu_k^z = \frac{1}{\Pi_{h^k}(z,z)} \sum_{\alpha \in kP \cap \mathbb{Z}^d} |s_\alpha(z)|^2_{h^k} \delta_{\frac{\alpha}{k}}$$

Here, $s_\alpha(z)$ are the holomorphic sections of L^k. These correspond to lattice points inside the k^{th} dialate of a certain Delzant polytope $P \subset \mathbb{R}^n$. This polytope is the image of the moment map $\mu : M \to P$ associated to the torus action on M. In our setting, M is analogous to the phase space energy surface $\Sigma = \{I_2 = 1\} \subset T^*S^2$ with the moment map $I_1 : \Sigma \to [-1,1]$. The joint eigenfunctions of $\widehat{I}_2$-eigenvalue ℓ correspond to the lattice points inside the ℓ^{th} dialate of $I_1(\Sigma) = [-1,1]$ and are analogous to the holomorphic sections s_α. In both cases the measures are dialated back to be supported on the image of the moment map and normalized to have mass 1. The submanifold H plays the role of the continuous parameter $z \in M$ in the Kähler setting. In [**39**] it is shown that as $k \to \infty$, a central limit theorem type rescaling of these measures tends to a Gaussian measure centered on $\mu(z)$ while in our case the measures μ_ℓ tend to an absolutely continuous limit which blows up at the end points with a $(1-c^2)^{-\frac{1}{2}}$ type singularity.

Chapter 4 is concerned with the round sphere (S^2, g_{can}). We use standard polar coordinates, so in the notation of chapter 3, $\phi = r$, $a(\phi) = \sin\phi$, $I_2 = |\xi|_{g_{can}}$. Therefore $\Sigma = S^*S^2$ in this case and $T_c = \{p_\theta = c\} \cap S^*S^2$. Suppose that $Y_{N_k}^{m_k}$ is a ladder sequence of spherical harmonics with $m_k/(N_k + \frac{1}{2}) = c \in (0,1)$. This sequence concentrates microlocally on the torus T_c and the boundary of the projection $\pi(T_c)$ consists of the two latitude circles $\gamma_c^{\pm}$ corresponding to the two solutions of $\sin\phi_{\pm} = c$. The projection has a fold singularity over each of these latitude circles which is responsible for the 'Airy bump' in the asymptotics near these circles. We prove the following scaled asymptotics for such ladder sequences:

Theorem 1.1.3. *For integers $0 \le m \le N$, let Y_N^m be the standard spherical harmonics (1.0.4) on (S^2, g_{can}) and let $x = (\phi, \theta)$ be geodesic polar coordinates from a pole. Suppose $0 \le m_k \le N_k$ are sequences of integers such that $m_k/(N_k + \frac{1}{2}) = c$ for all k. Then there exists an $\varepsilon > 0$ such that if $x = (\phi, \theta)$ with $c < \sin\phi < c + \varepsilon$, with $h_k = \left(N_k + \frac{1}{2}\right)^{-1}$,*

$$\begin{aligned} Y_{N_k}^{m_k}(x)\sqrt{dV_g(x)} \sim & Ai\left(-h_k^{-\frac{2}{3}}\rho(x)\right)\sum_{n=0}^{\infty} u_{0,n}(x) h_k^{-\frac{1}{6}+n} + \\ & Ai'\left(-h_k^{-\frac{2}{3}}\rho(x)\right)\sum_{n=0} u_{1,n}(x) h_k^{\frac{1}{6}+n} \end{aligned} \tag{1.1.5}$$

The argument of the Airy function and its derivative is

$$\rho(x) = \left(\frac{4}{3}\int_{\gamma_x} \alpha\right)^{\frac{2}{3}} \tag{1.1.6}$$

Here, γ_x is the geodesic arc joining the two pre-images $\pi^{-1}(x) \in T_c$ and α is the canonical 1-form on T^*S^2. The arc is oriented so as to make the integral positive. The $u_{i,j}$ are smooth half densities on S^2 and the leading order coefficient $u_{0,0}$ is

$$u_{0,0}(x) = \left(\frac{4\rho(x)}{\sin^2\phi - c^2}\right)^{\frac{1}{4}} e^{im_k\theta}\sqrt{dV_g} = (2\pi)\rho(x)^{\frac{1}{4}} e^{im_k\theta}\pi_*\sqrt{d\mu_{L,c}} \tag{1.1.7}$$

where $d\mu_{L,c}$ is the normalized joint flow invariant density on T_c and $\pi : T^*S \to S^2$ is the natural projection.

The ladder sequence is of size $\sim h^{-1/6}$ in an $h^{2/3}$ neighborhood of the caustic latitude circles which is characteristic for caustics caused by fold singularities. It is size $O(1)$ at points $x = (\phi, \theta)$ for which $\sin\phi > c$, and $O(h^\infty)$ in the case $\sin\phi < c$. It is interesting to note that we only have asymptotics in an $h^{2/3}$ strip around the caustic on the side contained in the projection $\pi(T_c)$. It should be possible to obtain two-sided Airy asymptotics, but it seems like it would require complexifying the Lagrangian T_c and extending the dynamics into the complex domain. The half density $\sqrt{d\mu_{L,c}}$ that shows up in the leading order amplitude $u_{0,0}$ is essentially the principal symbol of the ladder sequence $Y^{m_k}_{N_k}$ as a semi-classical Lagrangian distribution.

The asymptotics are obtained by constructing explicit quasi-modes for the Legendre operator using the well-known Maslov-WKB quantization procedure, (see section (4.1.1.2), [**5**],[**7**] for background) which approximate the Legendre functions P^m_N locally uniformly up to $O(h^\infty)$ error. The quasi-mode is expressible as an oscillatory integral with a degenerate critical point in the phase near the turning points, which correspond to the caustic latitude circles on the sphere. From this, the Airy expansion is obtained by putting the phase function in a cubic normal form. This idea was first due to Chester Friedman, and

Ursell [**3**], [**26**],[**27**] and further refinements can be found in [**22**] [**18**]. The main interest in this approach is that it connects the mysterious quantities appearing in the classical asymptotic expansions of Thorne and Olver [**33**],[**30**] for the Legendre functions to the relevant classical mechanics on T^*S^2.

CHAPTER 2

Lagrangian Distributions

2.1. Lagrangian distributions: basic definitions

Let X^n be an n dimensional smooth manifold, compact without boundary. The theory of Lagrangian distributions and Fourier integral operators requires us to work with **smooth half densities** and **half density distributions** on X. The spaces of these are denoted $C^\infty(\Omega^{\frac{1}{2}}, X)$ and $D'(\Omega^{\frac{1}{2}}, X)$. The reader unfamiliar with these notions should consult chapter 6 of [**19**]. X may or may not carry a Riemannian metric g, but when it does, the metric provides a canonical smooth half density, namely the square root of the metric volume, which in local coordinates is equal to

$$\sqrt{dV_g} = (\det\, g_{ij})^{\frac{1}{4}} |dx|^{\frac{1}{2}}$$

Homogeneous Lagrangian distributions on X are a special class of half density distributions. A Lagrangian submanifold of T^*X is called **homogeneous** if it is invariant under dilation in the fiber variable, that is, if for each real number $t > 0$

$$(x, \xi) \in \Lambda \iff (x, t\xi) \in \Lambda$$

Let $U \subset \mathbb{R}^n$ be an open set. A smooth function $a \in C^\infty(U \times \mathbb{R}^N \setminus 0)$ is a **classical amplitude of order k** if there exists a sequence of smooth functions $a_j(x, \theta) \in C^\infty(U \times \mathbb{R}^N \setminus 0)$ such that each a_j is a homogeneous function of θ of degree j for $|\theta| \geq 1$ and for each positive integer N,

$$a(x,\theta) - \sum_{j=0}^{N} a_{k-j}(x,\theta)$$

is homogeneous of degree $k - N - 1$ in θ. In this case we write

$$a(x,\theta) \sim \sum_{j=0}^{\infty} a_{k-j}(x,\theta).$$

A **non-degenerate homogeneous phase function** is a smooth function $\phi \in C^\infty(U \times \mathbb{R}^N \setminus 0)$ which is real valued, homogeneous of degree 1 in θ and such that $d\phi \neq 0$ on its domain. We also require the differentials $d\left(\partial_{\theta_j}\phi\right)$ to be linearly independent everywhere. If ϕ satisfies these assumptions then the **phase critical set**,

$$C_\phi = \{(x,\theta) \in U \times \mathbb{R}^N \setminus 0 \mid d_\theta \phi = 0\}$$

is a smooth, homogeneous submanifold of dimension n. Associated to ϕ is the map

$$i_\phi : C_\phi \to T^*U \qquad i_\phi(x,\theta) = (x, d_x\phi(x,\theta))$$

For any such phase function, the map i_ϕ is a Lagrangian immersion and its image is an immersed homogeneous Lagrangian submanifold of T^*U. We say that ϕ **parametrizes** the image of i_ϕ.

Definition 2.1.1. Let $\Lambda \subset T^*X \setminus 0$ be a homogeneous Lagrangian submanifold. We say that $u \in D'(\Omega^{\frac{1}{2}}, X)$ is a **Lagrangian distribution of order m with respect to $\boldsymbol{\Lambda}$** if for each $p \in X$, there are local coordinates around p in which u can be written in the form

$$u(x) = (2\pi)^{-(n+2N)/4} \int_{\mathbb{R}^N} a(x,\theta) e^{i\phi(x,\theta)}\, d\theta\, |dx|^{\frac{1}{2}} \tag{2.1.1}$$

Where $a(x, \theta)$ is a classical amplitude of order $k = m + (n - 2N)/4$ and ϕ is a non-degenerate phase function parametrizing the open subset of Λ lying over this coordinate patch. We write $I^m(X, \Lambda)$ for the set of all homogeneous Lagrangian distributions of order m on X.

2.1.1. The symbol of a Lagrangian distribution

If $u \in I^m(X, \Lambda)$, the leading order part of the amplitude $a_k(x, \theta)$ appearing in the representation (2.1.1) is not invariant and depends on the choice of phase function. However there is a notion of the 'leading order part' of u which does not depend its local representation. This is called the **symbol** of u and is written as $\sigma(u)$. The symbol of a Lagrangian distribution is a smooth, homogeneous half density on its associated Lagrangian Λ, tensored with a section of a flat, complex line bundle $\mathbb{L} \to \Lambda$ called the Maslov bundle. We now review how this is defined. Writing u locally in the form (2.1.1), we observe there is a canonical density on the critical set C_ϕ, namely

$$d_{C_\phi} = F^*\delta_0 \qquad\qquad F(x, \theta) = (\partial_{\theta_1}\phi(x, \theta), \ldots, \partial_{\theta_N}\phi(x, \theta))$$

Since we will discuss the pullback operation as a Fourier integral operator later on, we pause here to explain exactly what $F^*\delta_0$ means. Of course, for $u \in C^\infty(U \times \mathbb{R}^N \setminus 0)$, the pairing $\langle F^*\delta_0, u\rangle$ should mean 'integrate u over the fiber $F^{-1}(0)$', but this requires a density on $F^{-1}(0)$. Hence we are identifying $F^*\delta_0$ with this density. To calculate d_{C_ϕ}, notice that the non-degeneracy condition of ϕ implies that $C_\phi = F^{-1}(0)$ is a submanifold and for each $p \in C_\phi$, dF induces the exact sequence

$$0 \to T_pC_\phi \to T_p(U \times \mathbb{R}^N \setminus 0) \to T_0\mathbb{R}^N \to 0$$

Letting $(y_1, \ldots, y_N)$ be coordinates on the co-domain $\mathbb{R}^N$, the densities $|dx \otimes d\theta|$ and $|dy|$ on $U \times \mathbb{R}^N \setminus 0$ and $\mathbb{R}^N$ define a density on C_ϕ in the following way: choose any basis $v_1, \ldots, v_n$ of T_pC_ϕ. Complete this to a basis of $T_p(U \times \mathbb{R}^N \setminus 0)$ by adjoining $(w_1, \ldots, w_N)$. Then let

$$d_{C_\phi}(v_1, \ldots, v_n) = \frac{|dx \otimes d\theta|(v_1, \ldots, v_n, w_1, \ldots, w_N)}{|dy|(dF_p(w_1), \ldots, dF_p(w_N))}$$

If one chooses local coordinates $(\lambda_1, \ldots, \lambda_n)$ on C_ϕ and sets $v_i = \partial_{\lambda_i}$, w_i to be a basis of $T_p(U \times \mathbb{R}^N \setminus 0)/T_pC_\phi$ dual to the basis $d(\partial_{\theta_i}\phi)$ of $(T_pC_\phi)^\perp \subset T_p^*(U \times \backslash R^N \setminus 0)$, then this formula reduces to

$$d_{C_\phi} = \left| \frac{\partial(\lambda, d_\theta\phi)}{\partial(x, \theta)} \right|^{-1} |d\lambda| \tag{2.1.2}$$

where the prefactor is the reciprocal of the Jacobian determinant of the map

$$G : U \times \mathbb{R}^N \setminus 0 \to \mathbb{R}^{n+N} \qquad G(x, \theta) = (\lambda(x, \theta), d_\theta\phi(x, \theta))$$

Definition 2.1.2. Suppose that $u \in I^m(X, \Lambda)$ with the local representation (2.1.1). We define the half density part of the symbol, $\sigma(u)$ by

$$\sigma(u)(\lambda) = (i_\phi^{-1})^* \left(a_k(x, \theta)\sqrt{d_{C_\phi}} \right) \qquad \lambda \in i_\phi(C_\phi) \tag{2.1.3}$$

And this does not depend on the choice of local representation.

There are three separate notions of 'order' appearing so it useful to pause and relate them. They are

(1) The order of u as a Lagrangian distribution, i.e. the m so that $u \in I^m(X, \Lambda)$

(2) The order of the amplitude of u in any local representation (2.1.1), which is $m + (n - 2N)/4$.

(3) The order of $\sigma(u)$ as a homogeneous half density on Λ. This is $m + n/4$.

To see that (3) holds, notice that any choice of local coordinates λ_j on C_ϕ are necessarily homogeneous of degree 1, so the Jacobian matrix whose determinant appears in (2.1.2) written out in block form is

$$\frac{\partial(\lambda, d_\theta \phi)}{\partial(x, \theta)} = \begin{pmatrix} \lambda_x & \lambda_\theta \\ \phi_{x\theta} & \phi_{\theta\theta} \end{pmatrix}.$$

The upper diagonal block has entries which are homogeneous of degree 1, the lower degree -1. The off diagonal block entries are homogeneous of degree 0. One can check from the formula for the determinant of a block matrix that the determinant is homogeneous of degree $n - N$. Hence $\sqrt{d_{C_\phi}}$ is homogeneous of degree $(N - n)/2 + n/2 = N/2$. This makes the half density symbol homogeneous of degree $m + n/4$ on Λ.

2.2. Examples of Lagrangian distributions

In this section we enumerate commonly encountered Lagrangian distributions and calculate their order and symbol.

2.2.1. Lagrangian distributions in one dimension

Suppose that $\Lambda \subset T^*\mathbb{R} \setminus 0$ is a connected, homogeneous Lagrangian submanifold. Say $(t, \tau) \in \Lambda$, with $\tau > 0$. Then Λ contains the entire half line $\{(t, \tau) \mid \tau > 0\}$ by homogeneity. On the other hand, it cannot contain any point in the lower half line over t or in the fiber over any other point by connectedness. Hence $\Lambda = \Lambda_t^+ = \{(t, \tau) \mid \tau > 0\}$. This means that the only (second countable) homogeneous Lagrangian submanifolds of $T^*\mathbb{R}$ are countable disjoint union of positive and negative half lines. Thus the basic example to understand are Lagrangian distributions associated to Λ_0^+. Any $u \in I^m(\mathbb{R}, \Lambda_0^+)$ can be written as

$$u(t) = (2\pi)^{-\frac{3}{4}} \int_0^\infty a(t,\tau) e^{it\tau}\, d\tau \tag{2.2.1}$$

Where $a(t,\tau) \sim a_k(t)\tau^k + a_{k-1}(t)\tau^{k-1} + \cdots$ and $k = m - \frac{1}{4}$. Since $\phi(t,\tau) = t\tau$, $C_\phi = \{(0,\tau) \mid \tau > 0\}$ and $d_{C_\phi} = |d\tau|$. Therefore,

$$\sigma(u)(\tau) = a_0(0)\tau^{m+\frac{1}{4}} |d\tau|^{\frac{1}{2}} \tag{2.2.2}$$

Indeed, by proposition 1.2.5 **[21]**, replacing $a_k(t)$ with another function having the same value at $t = 0$ only changes u by an element of $I^{m-1}(\mathbb{R}, \Lambda_0^+)$. Therefore the leading order behavior of u is controlled only by $a_k(0)$ and iterating this shows u is equivalent, modulo $I^{-\infty}(X, \Lambda_0^+)$, to a Lagrangian distribution whose amplitude has constant coefficients,

$$a(\tau) \sim a_k\tau^k + a_{k-1}\tau^{k-1} + \cdots .$$

This means that the basic objects to understand are the distributions

$$v_\lambda(t) = (2\pi)^{-\frac{1}{2}} \int_0^\infty \tau^{\lambda-1} e^{it\tau}\, d\tau. \tag{2.2.3}$$

We notice that $v_\lambda(t)$ is nothing more than the inverse Fourier transform of the power distribution $\tau_+^{\lambda-1}$ defined, for $\operatorname{Re}\tau > 0$ by

$$(2.2.4) \qquad \langle \tau_+^\lambda, \psi \rangle = \int_0^\infty \tau^{\lambda-1} \psi(\tau)\, d\tau.$$

It turns out that $\tau_+^{\lambda-1}$ can be extended to a meromorphic family of distributions with poles at the non-positive integers. See **[22]**,**[15]**, or **[11]** for an extensive treatment of these distributions. The inverse Fourier transform (2.2.3) has the explicit formula

$$(2.2.5) \qquad v_\lambda(t) = \frac{e^{\frac{i\lambda\pi}{2}} \Gamma(\lambda)}{\sqrt{2\pi}} (t + i0)^{-\lambda}.$$

Here, the distribution $(t+i0)^{-\lambda}$ is the entire analytic continuation of the distribution defined as the boundary limit of the principal branch of $z^{-\lambda}$ on upper half plane when $\operatorname{Re}\lambda > -1$,

$$\langle (t+i0)^{-\lambda}, \psi \rangle = \lim_{\varepsilon \to 0} \int_{-\infty}^\infty (t + i\varepsilon)^{-\lambda} \psi(t)\, dt$$

To understand (2.2.5), first suppose that $\operatorname{Re}\lambda > 0$. Let A be a complex number with positive real part. By changing variables in the formula for the gamma function, one has that

$$\int_0^\infty \frac{\tau^{\lambda-1}}{\Gamma(\lambda)} e^{-A\tau}\, d\tau = A^{-\lambda}.$$

Setting $A = (\varepsilon - it)$, one gets, for each $\varepsilon > 0$,

$$\int_0^\infty \frac{\tau^{\lambda-1}}{\Gamma(\lambda)} e^{it\tau} e^{-\varepsilon\tau}\, d\tau = (\varepsilon - it)^{-\lambda} = e^{i\frac{\pi}{2}\lambda} (t + i\varepsilon)^{-\lambda}.$$

Now we just note that as $\varepsilon \to 0$, the left hand side converges to the integral

$$\int_0^\infty \frac{\tau^{\lambda-1}}{\Gamma(\lambda)} e^{it\tau}\, d\tau$$

which has meaning as a distribution in t; it is the inverse Fourier transform of the regularized power distriubtion $\tau_+^{\lambda-1}/\Gamma(\lambda)$. The right hand side converges to $e^{i\frac{\pi}{2}\lambda}(t+i0)^{-\lambda}$, so these are equal as distributions. Since both sides are analytic in λ, (the left being the Fourier transform of an analytic family of tempered distributions), the (2.2.5) holds for all $\lambda \in \mathbb{C}$. To summarize, we have shown the following:

Proposition 2.2.1. *Suppose that $u \in I^{m-\frac{3}{4}}(\mathbb{R}, \Lambda_0^+)$. Then u is equivalent, modulo $C^\infty(\mathbb{R})$, to the asymptotic sum*

$$u(t) \sim a_m(t+i0)^{-m} + a_{m-1}(t+i0)^{-(m-1)} + \cdots \tag{2.2.6}$$

If b_m is the coefficient of $\tau^{m+\frac{1}{4}}|d\tau|^{\frac{1}{2}}$ in $\sigma(u)$, then

$$a_m = b_m \frac{e^{i\frac{m\pi}{2}}\Gamma(m)}{(2\pi)^{\frac{3}{4}}}$$

This formula will appear crucially when we consider the Duistermaat-Guillemin wave trace in section 2.5.

2.2.2. Conormal distributions

Let $S \subset X^n$ be an embedded submanifold of codimenison d. The conormal bundle of S the sub-bundle of T^*X whose fiber at each point is the annihilator of the tangent space of S,

$$N^*S = \{(x,\xi) \in T^*X \mid x \in S,\, \xi \in T_x^*X,\, \xi|_{T_xS} = 0\} \tag{2.2.7}$$

N^*S is a rank d vector bundle over S. For any S, N^*S is a homogeneous Lagrangian submanifold of T^*X. We say that u is a **conormal distribution relative to S** if $u \in I^*(X, N^*S)$. To describe them more explicitly, let p be a point in S and select local coordinates $x = (x', x'')$ such that $S = \{x'' = 0\}$. In these coordinates the linear function $\phi(x,\theta) = x'' \cdot \theta$ (where $v \cdot w$ is the Euclidean inner product) is a non-degenerate phase function parametrizing N^*S near p. Therefore, we can write u locally as

$$u(x) = (2\pi)^{-\frac{n+2d}{4}} \int_{\mathbb{R}^d} a(x,\theta) e^{ix''\cdot\theta}\, d\theta |dx|^{\frac{1}{2}} \tag{2.2.8}$$

The critical set of the phase is $C_\phi = \{(x', 0, \theta)\}$ so we may take (x', θ) as coordinates on C_ϕ and an easy calculation shows that $d_{C_\phi} = |dx' \otimes d\theta|$. The Lagrangian immersion associated to ϕ is

$$i_\phi : (x', 0, \theta) \mapsto (x', 0, 0, \theta),$$

so if a_0 is the leading order term of the amplitude of u then the half density part of the symbol of u is

$$\sigma(u)(x', 0, 0, \theta) = a_0(x', 0, \theta) |dx' \wedge d\theta|^{\frac{1}{2}}. \tag{2.2.9}$$

Although this is an explicit formula, the half density $|dx' \otimes d\theta|$ depended on the choice of local coordinates at p. It is desirable to have a canonical half density on N^*S which can be used to view the symbol of u as a scalar. We now explain how a choice of half densities μ_X and μ_X on X and S determine a half density Ξ_S on N^*S. This gives a useful geometric interpretation to the symbol of both examples of conormal distributions which follow. For $p \in S$, let $N_p \subset T_p^*X$ denote the subspace of covectors which annihilate T_pS, i.e. the fiber of N^*S over p. Suppose we have fixed a point $\zeta \in N^*S$ with $\pi(\zeta) = p$. Taking the derivative of the projection $\pi : N^*S \to S$ at ζ gives the exact sequence

$$0 \to T_\zeta N_p \to T_\zeta N^*S \to T_pS \to 0 \tag{2.2.10}$$

Using this, half densities on the fiber N_p and S determine one on N^*S. We already have a half density on S, so we need a canonical half density on the fiber N_p. We have the second exact sequence coming from the restriction map of a covector to TS,

$$0 \to N_p \to T_p^*X \to T_p^*S \to 0. \tag{2.2.11}$$

Letting Ω_X and Ω_S be the half densities on T^*X and T^*S induced by the symplectic volume form, the ratios Ω_X/μ_X and Ω_S/μ_S determine half densities on the fibers T_p^*X, T_p^*S. The exact sequence then determines a half density ν on N_p. The first exact sequence together with ν and μ_S then determine the half density on N^*S. To calculate this concretely, we again work in local coordinates such that $S = \{x'' = 0\}$ and write $p = (x', x'')$. Let $\mu_X = f_X(x', x'')|dx' \otimes dx''|^{\frac{1}{2}}$ and $\nu = f_S(x')|dx'|^{\frac{1}{2}}$. The dual coordinates also split $\xi = (\xi', \xi'')$. The half densities Ω_X/μ_X and Ω_S/μ_S are $f_X^{-1}(x', 0)|d\xi' \otimes d\xi''|^{\frac{1}{2}}$ and $f_S^{-1}(x')|d\xi'|^{\frac{1}{2}}$. On N_p, these determine the half density

$$\frac{f_S(x')}{f_X(x',0)}|d\xi''|^{\frac{1}{2}}$$

Using the first exact sequence we finally arrive at

$$\Xi_S = \frac{f_S^2(x')}{f(x',0)}|dx' \wedge d\xi''|^{\frac{1}{2}}$$

Where the coordinate half density $|dx' \wedge d\xi''|^{\frac{1}{2}}$ is the same as $\sqrt{d_{C_\phi}}$ in the local calculation. The next three examples are all conormal distributions.

2.2.3. The δ_S density along a submanifold

Suppose that $S \subset X^n$ is an embedded submanifold of codimension d. If $f \in C^\infty(X)$, the pairing

$$\langle f, \delta_S \rangle \tag{2.2.12}$$

Should of course mean 'restrict f to S and integrate'. However, we want to let δ_S act on smooth half densities, and we need a density on S to integrate against. Let $i : S \hookrightarrow X$ be the inclusion map. If we fix half densities μ_X and μ_S on X and S as before, we can simply extend the restriction map to half densities by the rule

$$i^*(f\,\mu_X) = f|_S\,\mu_S \tag{2.2.13}$$

and then define the action of δ_S on half densities by

$$\langle \delta_S, f\,\mu_X \rangle = \int_S f|_S\,\mu_S^2 \tag{2.2.14}$$

To analyze this further, we look at it locally. Suppose that we choose local coordinates $x = (x', x'')$ in an open set U such that $S \cap U = \{x'' = 0\}$. As before, let $\mu_X = f_X(x', x'')|dx' \wedge dx''|^{\frac{1}{2}}$ and $\mu_S = f_S(x')|dx'|^{\frac{1}{2}}$. Then by Fourier inversion,

$$\delta_S = (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} \frac{f_S^2(x')}{f_X(x', x'')} e^{-ix''\cdot\theta}\, d\theta |dx|^{\frac{1}{2}}. \tag{2.2.15}$$

The number of phase variables is d and the amplitude is order zero, so the order of δ_S as a Lagrangian distribution is $-\frac{n}{4} + \frac{d}{2}$. The phase critical set is

$$C_\phi = \{(x', 0, \theta)\} \subset U \times \mathbb{R}^d$$

The image of i_ϕ is the set $\{(x', 0, 0, -\theta)\} \subset T^*M$ which is N^*S. To compute d_{C_ϕ}, use the coordinates (x', θ) on C_ϕ. Then we have

$$d_{C_\phi} = \left| \frac{\partial(x', \theta, -x'')}{\partial(x, \theta)} \right|^{-1} |dx' \wedge d\theta| = |dx' \wedge d\theta| \tag{2.2.16}$$

In summary,

Proposition 2.2.2. *Suppose S is a codimension d submanifold of X. Let δ_S be the δ distribution on S relative to the half densities μ_X on X and ν_S on S. Then $\delta_S \in I^{-\frac{n}{4}+\frac{d}{2}}(X; N^*S)$. In coordinates $x = (x', x'')$ such that $S = \{x'' = 0\}$, the half density symbol of δ_S is*

$$\sigma(\delta_S) = \frac{f_S^2(x')}{f_X(x',0)}|dx' \wedge d\xi''|^{\frac{1}{2}} = \Xi_S \tag{2.2.17}$$

where $\mu_X = f_X(x', x'')|dx' \wedge dx''|^{\frac{1}{2}}$ *and* $\mu_S = f_S(x')|dx'|^{\frac{1}{2}}$. *The symbol is exactly the canonical half density on* N^*S *determined by* μ_X *and* μ_S.

2.2.4. The kernel of pullback along a smooth map

Let $F : Y^m \to X^n$ be a smooth map. The pullback map,

$$F^* : C^\infty(X) \to C^\infty(Y) \qquad F^*g = g(F(y)),$$

acts on smooth functions but we can extend to it act on half densities by choosing half densities μ_X and μ_Y on X and Y. We then define

$$F^* : C^\infty(X, \Omega^{\frac{1}{2}}) \to C^\infty(Y, \Omega^{\frac{1}{2}}) \qquad F^*(g\,\mu_X) = F^*g\,\mu_Y. \tag{2.2.18}$$

In local coordinates y_i in a neighborhood U of $p \in Y$ and x_i in a neighborhood V of $F(p) \in X$, write $\mu_X = f_X(x)|dx|^{\frac{1}{2}}$ and $\mu_Y = f_Y(y)|dy|^{\frac{1}{2}}$. Then

$$K_{F^*}(y,x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \frac{f_Y(y)}{f_X(F(y))} e^{i\theta\cdot(F(y)-x)} d\theta |dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}} \tag{2.2.19}$$

is the kernel of the pullback map (2.2.18) relative to these coordinates. The phase $\phi(x,y,\theta) = (F(y)-x)\cdot\theta$ is non-degenerate with $C_\phi = \{(y, F(y), \theta)\} \subset U \times V \times \mathbb{R}^n$ and the parametrization

$$i_\phi : (F(y), y, \theta) \mapsto (y, dF_y^T\theta, F(y), -\theta)$$

has image which is equal to $N^*\Gamma_F$, the conormal bundle to the graph $\Gamma_F \subset Y \times X$. If we take (y, θ) as coordinates on C_ϕ then one can check that $d_{C_\phi} = |dy \wedge d\theta|^{\frac{1}{2}}$ and thus, in the coordinates (y, θ) on $N^*\Gamma_f$, the half density part of the symbol is equal to

$$\sigma(K_{F^*}) = \frac{f_Y(y)}{f_X(F(y))}|dy \wedge d\theta|^{\frac{1}{2}} \tag{2.2.20}$$

In order to give a more satisfying interpretation of the symbol, notice that our choice of half densities $\mu^{\frac{1}{2}}$ on X and $\nu^{\frac{1}{2}}$ on Y gives us a half density on $Y \times X$ (their product, $\nu^{\frac{1}{2}} \otimes \mu^{\frac{1}{2}}$) and one on Γ_f (identity Γ_f with Y and take $\nu^{\frac{1}{2}}$). According to section 3.2, these then determine a half density, $\zeta(\nu^{\frac{1}{2}} \otimes \mu^{\frac{1}{2}}, \nu^{\frac{1}{2}})$ on $N^*\Gamma_f$. We claim that ζ written in (y, θ) coordinates on $N^*\Gamma_f$ is exactly (2.2.20). Lastly, note that the order is $0 - (n+m-2n)/4 = (n-m)/4$.

Proposition 2.2.3. *Let $F : Y^m \to X^n$ be smooth map and μ_X,μ_Y be smooth half densities on X and Y. The pullback map $F^*(g\, d\mu_X)(y) = g(F(y))\mu_Y(y)$ has distribution kernel $K_{F^*} \in I^{\frac{n-m}{4}}(Y \times X, N^*\Gamma_F)$ where Γ_F is the graph of F. The half density symbol is equal to*

$$\sigma(K_{F^*}) = \Xi_{\Gamma_F} \tag{2.2.21}$$

Where Ξ_{Γ_F} is the canonical half density on $N^*\Gamma_F$ induced by the density μ_Y on $\Gamma_F \subset Y \times X$ and $\mu_Y \otimes \mu_X$ on $Y \times X$ as described in section 2.2.2

2.2.5. The kernel of pushforward along a submersion

Suppose that $F : X^{n+k} \to Y^k$ is a smooth submersion. As usual we fix half densities μ_X and μ_Y on X and Y. In the previous section, we saw that this gives us a pullback map on half densities

$$F^* : C^\infty(Y, \Omega^{\frac{1}{2}}) \to C^\infty(X, \Omega^{\frac{1}{2}}).$$

We now define pushforward as the formal L^2 adjoint of this map. That is, for half densities α, β on X and Y we define the pushforward map

$$F_* : C^\infty(X, \Omega^{\frac{1}{2}}) \to C^\infty(Y, \Omega^{\frac{1}{2}}) \qquad \langle F_*\alpha, \beta\rangle_{L^2(Y)} = \langle \alpha, F^*\beta\rangle_{L^2(X)} \tag{2.2.22}$$

Write $\alpha = a\,\mu_X$ and $\beta = b\,\mu_Y$ with a, b smooth functions and define F_*a by $F_*\alpha = (F_*a)\,\mu_Y$. Then we can rewrite the definition of F_* as

$$\int_Y (F_*a)(y)b(y)\,\mu_Y^2 = \int_X a(x)b(F(x))\,\mu_X^2 \tag{2.2.23}$$

At each point $p \in X$, let $q = F(p)$. The derivative dF_p induces the exact sequence

$$0 \to T_pF^{-1}(q) \to T_pX \to T_qY \to 0.$$

As we have seen several times now, the half densities μ_X and μ_Y determine the 'quotient' half density on the fibers $F^{-1}(q)$, μ_X/μ_Y. Using the same exact sequence, we may write $\alpha = \frac{\alpha}{\mu_Y} \otimes \mu_Y$, where the quotient is another half density on the fibers. Define

$$(F_*\alpha)(y) = \left(\int_{F^{-1}(y)} \frac{\alpha}{\mu_Y} \otimes \frac{\mu_X}{\mu_Y} \right) \mu_Y \tag{2.2.24}$$

To compare this to the original definition of pushforward, we observe that

$$\langle F_*\alpha, \beta \rangle_{L^2(Y)} = \int_Y \left(\int_{F^{-1}(y)} a(x) \frac{\mu_X^2}{\mu_Y^2} \right) b(y) \mu_Y^2$$

$$\langle \alpha, F^*\beta \rangle = \int_X a(x) b(F(x)) \mu_X^2$$

are equal by Fubini's theorem, so formula (2.2.24) is correct. To make this more concrete, we can write F_* in local coordinates chosen so that F has the form

$$F(x', x'') = x''$$

Then, with $\mu_X = f_X(x', x'')|dx' \wedge dx''|^{\frac{1}{2}}$, $\mu_Y = f_Y(y)|dy|^{\frac{1}{2}}$,

$$\int_{F^{-1}(y)} \frac{\alpha}{\mu_Y} \otimes \frac{\mu_X}{\mu_Y} = \int a(x', y) \frac{f_X^2(x', y)}{f_Y^2(y)} |dx'|$$

so the half density kernel of F_* is

$$K_{F^*}(y, x) = \int_{\mathbb{R}^k} e^{i\theta\cdot(y - x'')} \frac{f_X(x', x'')}{f_Y(x'')} |d\theta||dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}}. \tag{2.2.25}$$

The phase $\phi(x, y, \theta) = (y - x'') \cdot \theta$ is non-degenerate and the critical set is $C_\phi = \{(x'', (x', x''), \theta)\}$. The parametrizing map is

$$i_\phi : (x'', (x', x''), \theta) \mapsto (x'', \theta, (x', x''), (0, -\theta)) \subset T^*(Y \times X)$$

and if we view θ as the dual coordinate to x'', then the image of i_ϕ can be identified with

$$(N^*\Gamma_F)^T = \{F(x), \theta, x, -dF_x^T\theta) \mid \theta \in T^*_{F(x)}Y\}$$

Where we write the superscript 'T' to mean 'reverse the X and Y components' i.e. for a subset $C \subset T^*X \times T^*Y$,

$$C^T = \{(y, \eta, x, \xi) \mid (x, \xi, y, \eta) \in C\}.$$

We also see that the symbol of F_* is the same canonical half density on $N^*\Gamma_F$ induced by μ_X and μ_Y as for the pullback map. The order is of course $-(n+k+k-2k)/4 = -n/4$, which is equal to $(\dim Y - \dim X)/4$, the same as for pullback.

Proposition 2.2.4. *Let $F : X^{n+k} \to Y^k$ be smooth submersion and μ_X,μ_Y be smooth half densities on $X$and Y. The pushforward map on half densities defined by*

$$\langle F_*\alpha, \beta\rangle_{L^2(Y)} = \langle \alpha, F^*\beta\rangle_{L^2(X)}$$

has distribution kernel $K_{F_*} \in I^{\frac{-n}{4}}(Y \times X, (N^*\Gamma_F)^T)$ where Γ_F is the graph of F and the transpose 'T' means flip the T^*X and T^*Y components. The half density symbol is equal to

$$\sigma(K_{F^*}) = \Xi_{\Gamma_F} \tag{2.2.26}$$

Where Ξ_{Γ_F} is the canonical half density on $N^*\Gamma_F$ induced by the density μ_Y on $\Gamma_F \subset Y \times X$ and $\mu_Y \otimes \mu_X$ on $Y \times X$, the same as for the pullback map.

2.2.6. The half wave kernel on a Riemannian manifold

Let (X^n, g) be a closed Riemannian manifold, Δ be the non-negative Laplace operator, and $D_t = -i\partial_t$. The smooth half density $|dt|^{\frac{1}{2}} \otimes |dV_g|^{\frac{1}{2}}$ determines an isomorphism between smooth half densities and smooth functions with which we lift the **half wave operator**, $D_t + \sqrt{\Delta}$ to act on $C^\infty(\Omega^{\frac{1}{2}}, \mathbb{R} \times X)$ and we consider the initial value problem

$$\begin{aligned} \left(D_t + \sqrt{\Delta}\right) u(t,x) &= 0 \\ u(0,x) &= f(x). \end{aligned} \tag{2.2.27}$$

Let $U(t,x,y)$ be the kernel of the unitary operator $\exp -it\sqrt{\Delta}$, the propagator of the initial value problem. Since our primary goal is to understand Lagrangian distributions symbolically, we choose not to include a detailed construction of a parametrix (approximate kernel) for the propagator. For these details see one of the many excellent expositions [**32**][**1**],[**37**],[**24**]. A consequence of these local parametrix constructions is that the propagator kernel is a Lagrangian distribution,

$$U(t,x,y) \in I^{-\frac{1}{4}}(\mathbb{R} \times X \times X, C') \tag{2.2.28}$$

where C' is the space-time graph of the geodesic flow inside $T^*\mathbb{R} \setminus 0 \times T^*X \setminus 0 \times T^*X \setminus 0$

$$C' = \{(t,\tau,x,\xi,y,-\eta) \mid \tau + |\xi|_{g(x)} = 0,\, G^{-t}(x,\xi) = (y,\eta)\}. \tag{2.2.29}$$

There is a Lagrangian immersion parametrizing C',

$$\iota : \mathbb{R} \times T^*X \setminus 0 \to C' \qquad \iota(t,x,\xi) = (t, -|\xi|_{g(x)}, x, \xi, G^{-t}(x,\xi)) \tag{2.2.30}$$

and the domain carries a natural half density $|dt|^{\frac{1}{2}} \otimes |\Omega|^{\frac{1}{2}}$, where $|\Omega_X|^{\frac{1}{2}}$ is the symplectic half density on T^*X. We describe how to calculate the half density symbol $\sigma(U)$ of $U(t,x,y)$ in a rather indirect way. Let $P = D_t + \sqrt{\Delta}_x$ and p denote its principal symbol. Since $PU(t,x,y) = 0$ and $\sigma_{\mathrm{sub}}(P) = 0$, the first transport equation, theorem 5.3.1, **[9]** implies that

$$\frac{1}{i}\mathcal{L}_{H_p}\sigma(U) = 0 \tag{2.2.31}$$

If we write $\iota^*\sigma(U) = \sigma(t,x,y)|dt|^{\frac{1}{2}} \otimes |\Omega_X|^{\frac{1}{2}}$ then we are reduced to calculating the scalar σ. The integral curves of vector field H_p pull back to curves of the form $(t,x,\xi) \to (t+s, G^s(x,\xi))$. The half density $|dt|^{\frac{1}{2}} \otimes |\Omega_X|$ is constant along these curves because G^s preserves the symplectic form on T^*X, so the transport equation implies that $\sigma(t,x,\xi)$ must be as well. Furthermore, we know that at $t = 0$, $\sigma(U) = \sigma(\mathrm{Id}) = |\Omega_X|^{\frac{1}{2}}$, thus $\sigma(0,x,y) = 1$. If we write an arbitrary point $(t,x,\xi) \in \mathbb{R} \times T^*X \setminus 0$ as

$$(t,x,\xi) = \exp(tH_p)(0, G^{-t}(x,\xi))$$

then the fact that σ is constant along the integral curves of H_p implies that $\sigma(t,x,y) = 1$. Thus

$$\iota^*(\sigma(U)) = |dt|^{\frac{1}{2}} \otimes |\Omega|^{\frac{1}{2}} \tag{2.2.32}$$

2.3. Fourier integral operators

Recall that a linear operator $A : C^\infty(X, \Omega^{\frac{1}{2}}) \to D'(Y, \Omega^{\frac{1}{2}})$ is equivalent to a distribution in $D'(Y \times X, \Omega^{\frac{1}{2}})$ by the Schwartz kernel theorem. As we saw in the case of pullback and pushforward, it may be that the kernel of an operator is a Lagrangian distribution. In this case we call the operator A an **Fourier integral operator**.

Definition 2.3.1. A linear operator A is a Fourier integral operator if its kernel K_A is eual to a Lagrangian distribution; $K_A \in I^m(Y \times X, \Lambda)$ modulo C^∞ kernels for some homogeneous Lagrangian $\Lambda \subset T^*(Y \times X) \setminus 0$. If this is the case, the set

$$C = \Lambda' = \{(y, \eta, x, -\xi) \mid (y, \eta, x, \xi) \in \Lambda\}$$

is a Lagrangian submanifold of $T^*Y \times T^*X$ with respect to the symplectic form $\omega_Y \oplus -\omega_X$. We call C a canonical relation and we will write $A \in I^m(Y \times X; C)$ to mean $K_A \in I^m(Y \times X; C')$ modulo C^∞ kernels.

Our main goal is to study the composition of Fourier integral operators which isn't possible unless they map smooth half densities to smooth half densities. This is not always the case, but is true when their canonical relations satisfy a mild condition.

Proposition 2.3.2. *Suppose that $A \in I^m(Y \times X, C)$ and $C \subset T^*Y \setminus 0 \times T^*X \setminus 0$. Then*

$$A : C^\infty(\Omega^{\frac{1}{2}}, X) \to C^\infty(\Omega^{\frac{1}{2}}, Y).$$

And A extends uniquely to a continuous mapping on half density distributions,

$$A : D'(\Omega^{\frac{1}{2}}, X) \to D'(\Omega^{\frac{1}{2}}, Y).$$

Note that in general the canonical relation of a Fourier integral operator A cannot contain a point which lies in the zero section of both factors, but it can contain points of the form $(y, 0, x, \xi)$ or $(y, \eta, x, 0)$. It is these cases we need to rule out in order for A to preserve smoothness. To prove this, observe that theorem 2.2.2 [**6**] implies that that $\mathrm{WF}'(A) \subset C$ and hence $\mathrm{WF}'(A)_Y = \varnothing$. Then by the standard wavefront set calculus,

$$\mathrm{WF}(Af) \subset \mathrm{WF}'(A) \circ \mathrm{WF}(f) \cup \mathrm{WF}'(A)_Y \tag{2.3.1}$$

so if f is smooth this set is empty, and Af is smooth. Note that since $A^T \in I^m(Y \times X, C^T)$, the same argument implies that A^T maps smooth half densities to smooth half densities and thus A is extendable as a continuous map on half density distributions

$$A : D'(X, \Omega^{\frac{1}{2}}) \to D'(Y, \Omega^{\frac{1}{2}}).$$

We state a preliminary version of the composition theorem for Fourier integral operators which we refine in section 2.4

Theorem 2.3.3. *Suppose $A_1 \in I^{m_1}(Y \times X; C_1)$ and $A_2 \in I^{m_2}(Z \times Y; C_2)$. Suppose that C_i contain no elements of the zero section in either factor and that $C_2 \times C_1$ intersects $T^*Z \times \Delta_{T^*Y} \times T^*X$ cleanly, where $\Delta_{T^*Y} \subset T^*Y \times T^*Y$ is the diagonal. Then the set theoretic composition*

$$C_2 \circ C_1 = \{(z, \omega, x, \xi) \mid \exists (y, \eta) \in T^*Y; (z, \omega, y, \eta) \in C_2 \text{ and} (y, \eta, x, \xi) \in C_1\} \subset T^*Z \backslash 0 \times T^*X \backslash 0$$

is an immersed canonical relation and

$$A_2 \circ A_1 \in I^{m_1+m_2}(Z \times X, C_2 \circ C_1).$$

Under the clean intersection assumption, there is a bilinear composition map on half densities,

$$\circ : C^\infty(C_2; \Omega^{\frac{1}{2}}) \otimes C^\infty(C_1; \Omega^{\frac{1}{2}}) \to C^\infty(C_2 \circ C_1; \Omega^{\frac{1}{2}})$$

and

$$\sigma(A_2 \circ A_1) = \sigma(A_2) \circ \sigma(A_1).$$

2.3.1. Fourier integral operators associated to a canonical graph

Let $\chi : T^*X \setminus 0 \to T^*Y \setminus 0$ be a homogeneous symplectomorphism (canonical transformation). The graph of χ,

$$\Gamma_\chi = \{(y, \eta, x, \xi) \mid (y, \eta) = \chi(x, \xi)\} \subset T^*Y \setminus 0 \times T^*X \setminus 0$$

is a homogeneous canonical relation from T^*X to T^*Y. Suppose that $A \in I^m(Y \times X; \Gamma_\chi)$ is a Fourier integral operator associated to this canonical relation. A local oscillatory integral representation for the kernel K_A of A can be obtained by choosing a local generating function for χ. That is, a function $\psi(x, \theta)$ defined on $U \times \mathbb{R}^n$ for an open subset $U \subset X$ which is homogeneous of degree one in θ and satisfies

$$\chi(x, d_x\psi(x, \theta)) = (d_\theta\psi(x, \theta), \theta). \tag{2.3.2}$$

We claim that $\phi(x,y,\theta) = \psi(x,\theta) - y\cdot\theta$ is a non-degenerate phase function which locally parametrizes the canonical relation Γ_χ. We observe $C_\phi = \{(x,y,\theta) \mid y = d_\theta\psi(x,\theta)\}$, $d_x\phi = d_x\psi$, and $d_y\phi = -\theta$ Hence image of the immersion i_ϕ is the set of points

$$i_\phi(C_\phi) = \{(d_\theta\psi(x,\theta), -\theta, x, d_x\psi(x,\theta)\} \subset T^*(Y\times X) \tag{2.3.3}$$

Which is equal to Γ'_χ when we identity $\theta\in\mathbb{R}^n$ with the covector $\theta\cdot dy$ lying over the point $y\in Y$. We also note that C_ϕ is a graph over the (x,θ) coordinates. It follows from this observation that

$$d_{C_\phi} = \left|\frac{\partial(x,\theta,d_\theta\psi - y)}{\partial(x,y,\theta)}\right|^{-1} |dx\wedge d\theta|. \tag{2.3.4}$$

The prefactor is $|\det J|^{-1}$ where

$$J = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & I \\ \psi''_{x\theta} & -I & \psi''_{\theta\theta} \end{pmatrix}, \tag{2.3.5}$$

which is equal to 1 by expanding along the top row, so $d_{C_\phi} = |dx\wedge d\theta|$. Now since Γ_χ is a canonical graph, its right projection $\pi_R : \Gamma_\chi \to T^*X\setminus 0$ is a diffeomorphism. This means we have a canonical half density on Γ_χ given by the pullback of the symplectic half density on T^*X, $|\Omega_X|^{\frac{1}{2}} = |dx\wedge d\xi|^{\frac{1}{2}}$. Since $\xi = d_x\psi$ under the immersion i_ϕ, we have that

$$\frac{\partial(x,\xi)}{\partial(x,\theta)} = \begin{pmatrix} \mathrm{Id} & 0 \\ \psi''_{xx} & \psi''_{x\theta} \end{pmatrix}, \tag{2.3.6}$$

which means

$$|dx \wedge d\theta| = |\psi''_{x\theta}|^{-1} |dx \wedge d\xi|. \tag{2.3.7}$$

Relative to the pulled back symplectic half density $\Omega_X^{\frac{1}{2}} = |dx \wedge d\xi|^{\frac{1}{2}}$, we can view the principal symbol as the scalar $\sigma(u)\,|dx \wedge d\xi|^{-\frac{1}{2}}$. We have just shown that the half density part of the symbol is

$$i_\varphi^* \left(\sigma(u)\right) = a_0(x,\theta) |\psi''_{x\theta}|^{-\frac{1}{2}}. \tag{2.3.8}$$

where as usual, a_0 is the leading order part of the amplitude in a local expression for K_A.

Proposition 2.3.4. *Suppose that $A \in I^m(Y \times X, C)$ where C is the graph of the homogeneous canonical transformation $\chi : T^*X \to T^*Y$. Suppose that $\psi(x,\theta)$ is a local generating function for χ in the sense of* (2.3.2). *Then the kernel of A can be written locally as*

$$(2\pi)^{-n} \int_{\mathbb{R}^n} a(x,y,\theta) e^{i(\psi(x,\theta) - y\cdot\theta)}\, d\theta.$$

For a classical amplitude of order $m + \frac{2n-2n}{4} = m$*. Let* $\pi_R : C \to T^*X$ *be the projection onto the right factor and* $|dx \wedge d\xi|^{\frac{1}{2}} = \pi_R^* |\Omega_X|^{\frac{1}{2}}$ *be the pullback of the symplectic half density on* T^*X*. Then the half density part of the symbol of* A *is equal to*

$$\sigma(A)(x,\xi) = a_m(x, d_\theta\psi(x,\theta), \theta)|\psi''_{x\theta}(x,\theta)|^{-\frac{1}{2}}|dx \wedge d\xi|^{\frac{1}{2}} \tag{2.3.9}$$

where $(x,\theta) \in C_\phi$ *is such that* $(x, d_x\psi(x,\theta)) = (x,\xi) \in T^*X$*..*

As an important example of this class of operators, suppose that $X = Y$ and χ is the identity map on T^*X. It is easy to verify that $\psi(x,\theta) = x\cdot\theta$ is a local generating function for the identity. Thus, if $A \in I^m(X \times X; \Gamma_{\mathrm{Id}})$ we can represent the kernel of A locally as

$$K_A(x,y) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-y)\cdot\theta} a(x,y,\theta)\, d\theta |dx|^{\frac{1}{2}}|dy|^{\frac{1}{2}} \tag{2.3.10}$$

where $a(x,y,\theta)$ is a classical amplitude of order m. On the critical set, $y = x$ and the image of the point (x,x,θ) under i_ϕ is the covector $(x,\theta,x,\theta) \in T^*X \setminus 0 \times T^*X \setminus 0$. Since $\psi_{x\theta} = \mathrm{Id}$, the symbol of this operator is just the top order part of the amplitude on the diagonal times the symplectic half density

$$\sigma(A) = a_0(x,x,\theta)|dx \wedge d\theta|^{\frac{1}{2}}.$$

2.4. Symbolic composition

A homogeneous canonical relation is called **weighted** if it comes equipped with a smooth, homogeneous half density. Suppose that $(C_1, \sigma_1) \subset T^*X \setminus 0 \times T^*Y \setminus 0$ and $(C_2, \sigma_2) \subset T^*Z \setminus 0 \times T^*X \setminus 0$ are weighted homogeneous canonical relations from T^*Y to T^*X and

T^*X to T^*Z. In this section we describe how to compose these to get a third weighted homogeneous canonical relation, $(C_2 \circ C_1, \sigma_2 \circ \sigma_1)$ from T^*Y to T^*Z,

$$C_2 \circ C_1 = \{(z,\zeta,y,\eta) \mid \exists (x,\xi) \in T^*Y\backslash 0, (z,\zeta,x,\xi) \in C_2, (x,\xi,y,\eta) \in C_1\} \subset T^*Z\backslash 0 \times T^*Y\backslash 0$$

As mentioned in section 2.3, this is only possible when C_1 and C_2 intersect cleanly, to be discussed later on in this section. First we describe the linear algebraic version of composition which is always possible.

2.4.1. Composition of linear weighted canonical relations: abstract description

Let V, W, U be symplectic vector spaces of dimensions $2m$,$2d$, and $2n$ respectively. Suppose that $C_1 \subset W^- \times V$ and $C_2 \subset U^- \times W$ are linear canonical relations (Lagrangian subspaces) from V to W and from W to U. We define the set theoretic composition

$$C_2 \circ C_1 = \{(u,v) \in U^- \times V \mid \exists w \in W, (u,w) \in C_2, (w,v) \in C_1\}, \tag{2.4.1}$$

and also the set

$$F = (C_2 \times C_1) \cap (U \times \Delta_W \times V) = \{(u,w,w,v) \mid (u,w) \in C_2, (w,v) \in C_1\}. \tag{2.4.2}$$

We also define two maps central to the theory of composition to follow,

$$\tau : C_2 \times C_1 \to W \qquad \tau : (u,w,w',v) \mapsto w' - w \tag{2.4.3}$$

$$\alpha : F \to C_2 \circ C_1 \qquad \alpha : (u, w, w, v) \mapsto (u, v) \tag{2.4.4}$$

Associated to each of these maps is an exact sequence which we will make heavy use of in what follows. They are,

$$0 \to F \to C_1 \times C_2 \to W \to \text{coker } \tau \to 0 \tag{2.4.5}$$

$$0 \to \ker \alpha \to F \to C_2 \circ C_1 \to 0. \tag{2.4.6}$$

The first thing to understand is the so-called canonical pairing between $\ker \alpha$ and $\operatorname{coker} \tau$.

Lemma 2.4.1. *Identifying* $\ker \alpha$ *with a subspace of* W *in the obvious way, we have*

$$\ker \alpha = (\operatorname{im} \tau)^{\perp}. \tag{2.4.7}$$

If $w \in \ker \alpha$*, then the association*

$$w \mapsto \ell_w(\cdot) = \omega_W(w, \cdot)$$

defines an isomorphism between $\ker \alpha$ *and* $(\operatorname{coker} \tau)^*$*. In the terminology of Guillemin, '*$\ker \alpha$ *and* $\operatorname{coker} \tau$ *are canonically paired by the symplectic form.'*

Proof. We first prove the inclusion $\operatorname{im}\tau \subset (\ker\alpha)^\perp$. Fix some $w' - w \in \operatorname{im}\tau$, with $(u, w, w', v) \in C_2 \times C_1$. Observe that $w_0 \in \ker\alpha$ means that $(0, w_0) \in C_2$ and $(w_0, 0) \in C_1$. For any such w_0,

$$\omega_W(w' - w, w_0) = \omega_W(w', w_0) - \omega_W(w, w_0). \tag{2.4.8}$$

Since $(u, w) \in C_2$, $(0, w_0) \in C_2$, and $C_2 \subset U^- \times W$ is Lagrangian,

$$\omega_W(w, w_0) - \omega_U(u, 0) = \omega_W(w, w_0) = 0$$

By the same argument, we find that $\omega_W(w', w_0) = 0$. Hence (2.4.8) is equal to zero, which proves the first inclusion. Next, we have to prove that $(\operatorname{im}\tau)^\perp \subset \ker\alpha$. To this end, suppose that $w_0 \in W$ satisfies

$$\omega_W(w_0, w' - w) = 0 \tag{2.4.9}$$

Whenever there exists u and v such that $(u, w, w', v) \in C_2 \times C_1$. In particular, using the fact that $(0, 0) \in C_i$, we may assume that

$$\omega_W(w_0, w) = 0$$

for all w in the right projection of C_2 and all w in the left projection of C_1. We have to show that $w_0 \in \ker\alpha$, or equivalently, that $(0, w_0) \in C_2$ and $(w_0, 0) \in C_1$. To this end, suppose that $(u'', w'') \in C_2$ is arbitrary. Then

$$\omega_{U^- \times W}((u'', w''), (0, w_0)) = \omega_W(w'', w_0) = 0$$

Since C_2 is Lagrangian, $(0, w_0) \in C_2$. The same argument shows that $(w_0, 0) \in C_1$, i.e. $w_0 \in \ker \alpha$. For the last assertion, suppose that $w \in \ker \alpha$. Then $\ell_w = \omega_W(w, \cdot)$ is a linear functional on W that vanishes on the symplectic complement of $\ker \alpha$ which is $\operatorname{im} \tau$. Hence ℓ_w descends to a well-defined linear functional on $\operatorname{coker} \tau$. This association is an isomorphism since the dimensions of $\ker \alpha$ and $\operatorname{coker} \tau$ are the same.

□

The first consequence of this lemma is the fact that $C_2 \circ C_1$ is a canonical relation.

Proposition 2.4.2. *$C_2 \circ C_1$ is a Lagrangian subspace of $U^- \times V$.*

Proof. It is easy to observe that $C_2 \circ C_1$ is isotropic; if $(u_1, v_1), (u_2, v_2) \in C_2 \circ C_1$, choose $w_1, w_2 \in W$ so that $(u_j, w_j, w_j, v_j) \in C_2 \times C_1$. Then

$$\omega_{U^- \times V}((u_1, v_1), (u_2, v_2)) = \omega_V(v_1, v_2) - \omega_W(w_1, w_2) + \omega_W(w_1, w_2) - \omega_U(u_1, u_2) = 0$$

since both terms on the right hand side are zero using the fact that C_i are isotropic. Using the exact sequences (2.4.5),(2.4.6) together with the lemma we can count dimensions to verify that $C_2 \circ C_1$ is Lagrangian.

□

Next, suppose we have half densities $\sigma_i \in |C_i|^{\frac{1}{2}}$ on each canonical relation. The next proposition is almost the recipe for composition. It is the crucial isomorphism to understand concretely.

Proposition 2.4.3. *There is an isomorphism $T : |C_2|^{\frac{1}{2}} \otimes |C_1|^{\frac{1}{2}} \to |C_2 \circ C_1|^{\frac{1}{2}} \otimes |\ker \alpha|$*

Proof. The entire point is to understand this isomorphism explicitly. In order to do this we begin by fixing a non-zero half density μ on $C_2 \circ C_1$. Then the content of this proposition is that σ_2, σ_1 determine a unique density ν on $\ker \alpha$ by the rule $T(\sigma_2 \otimes \sigma_1) = \mu \otimes \nu$. To describe the density ν, we will fix another non-zero half density η on F. Using the exact sequence (2.4.6), μ and η determine the quotient half density μ/η on $\ker \alpha$. Next, using the other exact sequence (2.4.5), the product half density $\sigma_2 \otimes \sigma_1$ on $C_2 \times C_1$ together with μ determine the quotient half density $\frac{\sigma_2 \otimes \sigma_1}{\eta}$ on $\operatorname{im} \tau$. So far, schematically, we have

$$\sigma_2 \otimes \sigma_1 = \eta \otimes \frac{\sigma_2 \otimes \sigma_1}{\eta} = \frac{\eta}{\mu} \otimes \mu \otimes \frac{\sigma_2 \otimes \sigma_1}{\eta} \tag{2.4.10}$$

Now the short exact sequence

$$0 \to \operatorname{im} \tau \to W \to \operatorname{coker} \tau \to 0 \tag{2.4.11}$$

determines a half density ξ on $\operatorname{coker} \tau$, the ratio of the symplectic half density on W and the $\sigma_2 \otimes \sigma_1/\eta$ from above, so that

$$|\Omega_W|^{\frac{1}{2}} = \frac{\sigma_2 \otimes \sigma_1}{\eta} \otimes \xi$$

Now ξ^{-1} is a $-\frac{1}{2}$ density on $\operatorname{coker} \tau$, and the isomorphism $(\operatorname{coker} \tau)^* \cong \ker \alpha$ means that ξ^{-1} uniquely determines a half density on $\ker \alpha$, which we call ξ'. In the end we get

$$\frac{\sigma_2 \otimes \sigma_1}{\eta} \cong \xi^{-1} \cong \xi'$$

and putting this back into (2.4.10),

$$\sigma_2 \otimes \sigma_1 \cong \mu \otimes \frac{\eta}{\mu} \otimes \xi' \tag{2.4.12}$$

The last two factors are both half densities on $\ker \alpha$, and their product is the density ν.

$\square$

It may seem as though this construction depended on the choice of η, but roughly speaking ξ' goes like η^{-1} which cancels out the other η factor appearing. The argument of lemma 2, page 27 of [**17**] shows this independence rigorously. However, to practically compute these objects it is useful to make a choice of η. It may happen that $\ker \alpha = 0$ and in this case the isomorphism T becomes much simpler. The next proposition explains this case.

Proposition 2.4.4. *In the terminology of this section, suppose that* $\ker \alpha = 0$. *Then* $|\ker \alpha| \cong \mathbb{R}$ *and there is a unique half density* $\mu \in |C_2 \circ C_1|^{\frac{1}{2}}$ *such that* $T(\sigma_2 \otimes \sigma_1) = \mu \otimes 1$. *The half density* μ *is the quotient of* $\sigma_2 \otimes \sigma_1$ *by the symplectic half density* $|\Omega_W|^{\frac{1}{2}}$ *according to the exact sequence*

$$0 \to F \to C_1 \times C_2 \to W \to 0. \tag{2.4.13}$$

Proof. The assumption $\ker \alpha = 0$ means that $F \cong C_2 \circ C_1$ and τ is surjective. The exact sequence (2.4.5) reduces to (2.4.13). Let μ be the half density on $F \cong C_2 \circ C_1$ determined by (2.4.13) and $|\Omega_W|^{\frac{1}{2}}$. The exact sequence (2.4.6) with μ determines the half density 1 on $\ker \alpha$. On the other hand, the half density that μ determines on $\operatorname{im} \tau = W$ is by definition $|\Omega_W|^{\frac{1}{2}}$. This means the half density on $\operatorname{coker} \tau$ is 1, and so is the half density on $\ker \alpha$ we get by duality. Thus in the end we get $\sigma_2 \circ \sigma_1 \cong 1 \otimes \mu$ as stated.

□

2.4.2. Examples of compositions of weighted linear canonical relations

Before describing how to pass from the case of linear canonical relations to cleanly intersecting canonical relations on cotangent bundles, we work out three examples of linear composition. The first are included in the special case $\ker\alpha = 0$ of the above proposition.

Proposition 2.4.5 (Composition of linear canonical graphs)**.** *Let $A_1 : V \to W$ and $A_2 : W \to Z$ be canonical transformations and let*

$$C_1 = \{(A_1 v, v) \mid v \in V\} \subset W^- \times V \qquad C_2 = \{(A_2 w, w) \mid w \in W\} \subset U^- \times W$$

be the graph canonical relations of each A_i. Let σ_i be the canonical graph half densities coming from the symplectic form on V, W. That is, σ_i assigns the value 1 to the basis $(Ae_j, e_j), (Af_j, f_j)$ where (e_j, f_j) is any symplectic basis of $\operatorname{dom} A_i$. Then

$$C_2 \circ C_1 = \{(A_2 A_1 v, A_1 v, A_1 v, v) \mid v \in V\}$$

is the graph canonical relation of $A_2 \circ A_1$ and $T(\sigma_2 \otimes \sigma_1)$ is identified with the canonical graph half density.

Proof. It is easy to check that the set theoretic composition is the graph of $A_2 \circ A_1$ and clear that $\ker\alpha = 0$. We need to verify the assertion about the composite half density. Choose a symplectic basis (e_j, f_j) of V. This gives us the basis $(A_2 A_1 e_j, A_1 e_j, A_1 e_j, e_j), (A_2 A_1 f_j, A_1 f_j, A_1 f_j, f_j)$ of F. We complete this to a basis of $C_2 \times C_1$ by adding $(0, 0, A_1 e_j, e_j)$ and $(0, 0, A_1 f_j, f_j)$. Call this basis $\mathcal{B}$. Now the composite half density $\mu \in |C_2 \circ C_1|^{\frac{1}{2}} \cong |F|^{\frac{1}{2}}$ is, according to proposition 2.4.4,

$$\mu((A_2A_1e_j, e_j), (A_2A_1f_j, f_j)) = \frac{\sigma_1 \otimes \sigma_2(\mathcal{B})}{|\Omega_W|^{\frac{1}{2}}(A_1e_j, A_1f_j)} \tag{2.4.14}$$

Now since A_1 is a canonical transformation, the denominator is equal to 1. The numerator is equal to $|\det A|$ where A is the matrix taking the product basis

$$\mathcal{B}' = \{(A_2A_1e_j, A_1e_j, 0, 0), (A_2A_1f_j, A_1f_j, 0, 0), (0, 0, A_1e_j, e_j), (0, 0, Af_j, f_j)\}$$

into the basis $\mathcal{B}$. The matrix A is a block matrix of the form

$$A = \begin{pmatrix} I & * \\ 0 & I \end{pmatrix}$$

Hence $|\det A| = 1$. This proves μ is the canonical graph half density on the composite.

□

To preface the next example, suppose that V is a symplectic vector space. A Lagrangian subspace $L \subset V$ can be viewed as a canonical relation from V to the zero vector space. Given two Lagrangian subspaces L_1 and L_2 of V, we let $L_1^T = \{(\ell, 0) \mid \ell \in L_1\} \subset V^- \times \mathbf{0}$. Then $L_2 \circ L_1^T$ is a canonical relation from the zero vector space to itself. Half densities on each L_i determine a half density on the zero canonical relation, otherwise known as a number. We call this number the **canonical pairing of σ_1 and σ_2** and denote it by (σ_2, σ_1)

Proposition 2.4.6. *Let L_1 and L_2 be two transverse Lagrangian subspaces of a symplectic vector space V equipped with half densities σ_1 and σ_2. In the composition,* $\ker \alpha = 0$ *and $L_2 \circ L_1^T = \mathbf{0}$, the zero canonical relation. The composite half density is identified with*

$$\sigma_2 \otimes \sigma_1 \cong \frac{\sigma_2 \otimes \sigma_1}{|\Omega_V|^{\frac{1}{2}}} \otimes 1 \tag{2.4.15}$$

The pairing (σ_2, σ_1) *is the number*

$$(\sigma_2, \sigma_1) = \frac{\sigma_2 \otimes \sigma_1}{|\Omega_V|^{\frac{1}{2}}} \tag{2.4.16}$$

Proof. The fact that $L_1 \cap L_2 = \{0\}$ means that $\ker \alpha = 0$. By Proposition 2.4.4 the composite half density is determined by the symplectic half density on V and the exact sequence

$$0 \to \mathbf{0} \times L_1 \times L_2 \times \mathbf{0} \to V \to 0$$

associated to τ.

$\square$

Here is one important special case of the previous proposition. Suppose W is a symplectic vector space and $P : W \to W$ a canonical transformation. Let $V = W^- \times W$, $L_1 = \Delta \subset W^- \times W$ and $L_2 = \{(Pw, w) \mid w \in W\}$ be the graph of P. Equip each L_i with the the canonical graph half density σ_i described in proposition 2.4.5. Then

$$L_1 \cap L_2 = \{(v, v) \mid v \in \ker I - P\}$$

so L_i are transverse if and only if $I - P$ is invertible.

Proposition 2.4.7. *The canonical pairing* (2.4.16) *in the case just described is equal to* $|\det I - P|^{-\frac{1}{2}}$

Proof. From the formula, the pairing (σ_2, σ_1) is equal to

$$\frac{\sigma_1 \otimes \sigma_2((e_i, e_i), (f_i, f_i), (Pe_i, e_i), (Pf_i, f_i))}{|\Omega_{W^- \times W}|^{\frac{1}{2}}((e_i, e_i), (f_i, f_i), (Pe_i, e_i), (Pf_i, f_i))} \tag{2.4.17}$$

The numerator is equal to one, and the denominator is equal to $|\det A|^{\frac{1}{2}}$ where A is the change of basis matrix from the symplectic basis $(-e_i, 0), (-f_i, 0), (0, e_i), (0, f_i)$ to the one we are evaluating on. We just have to verify that $|\det A| = |\det(I - P)|$. The matrix of A in $2n \times 2n$ block form is

$$A = \begin{pmatrix} -I & P \\ -I & I \end{pmatrix} \tag{2.4.18}$$

Subtracting the top $2n$ rows from the bottom makes the bottom left block zero and we then see that $|\det A| = |\det I - P|$.

□

The final example is the generalization of the last example in which we allow $I - P$ to have a kernel. We again suppose that $P : W \to W$ is a symplectic map and claim that it is always the case that

$$(\ker I - P)^{\perp} = \operatorname{im} I - P \tag{2.4.19}$$

To see this, notice that for any $w_1, w_2 \in W$,

$$\omega_W((I - P)w_1, w_2) = \omega_W(w_1, (I - P^{-1})w_2).$$

Therefore w_2 is symplectic orthogonal to the image of $I-P$ if and only if $(I-P^{-1})w_2 = 0$, or $(I-P)w_2 = 0$. This equality also implies that $I-P$ preserves the subspace $V = \operatorname{im} I - P$. Now we suppose further that

$$\ker I - P \cap \operatorname{im} I - P = \varnothing. \tag{2.4.20}$$

In this case, the restriction of $I-P$ to V,

$$I - P^{\#} : V \to V \tag{2.4.21}$$

has no kernel and the subspaces $\ker I-P$ and V inherit a symplectic structure by the restriction of ω_W.

Proposition 2.4.8. *Suppose that $P : W \to W$ is a symplectic map satisfying* (2.4.20). *Let $L_1 = \{(w,w) \mid w \in W\}$ and $L_2 = \{(Pw,w) \mid w \in W\}$ be the graphs of the identity map and P on W equipped with the symplectic graph half densities σ_i. Then $L_2 \circ L_1^T$ is the zero canonical relation,*

$$\ker \alpha = \{(w,w) \mid w \in \ker I - P\},$$

and the composite half density can be identified with

$$\sigma_2 \otimes \sigma_1 \cong |\det I - P^{\#}|^{-\frac{1}{2}} |\Omega| \otimes 1 \tag{2.4.22}$$

where $|\Omega|$ is the symplectic density on $\ker I - P \cong \ker \alpha$

Proof. Choose a symplectic basis (e_i, f_i), $i = 1, \ldots, k$ of $\ker I - P$ and complete this to a symplectic basis of W by adding $(e'_{k+1}, \ldots, e'_n, f'_{k+1}, \ldots, f'_n)$. The map τ in this case has image equal to $L_2 + L_1 \subset W \times W$. The exact sequence

$$0 \to F \to L_2 \times L_1 \to L_2 + L_1 \to 0$$

gives us the quotient half density

$$\nu = \frac{\sigma_2 \otimes \sigma_1}{|\Omega|^{\frac{1}{2}}}$$

where $|\Omega|^{\frac{1}{2}}$ is the symplectic half density on $F \cong \ker I - P$. To describe ν concretely, start with the basis $(e_i, e_i, e_i, e_i), (f_i, f_i, f_i, f_i)$ of $F = \{(w, w) \mid w \in L_2 \cap L_1\}$. We complete this to a basis $\mathcal{B}$ of $L_2 \times L_1$ by adding the vectors $(Pe'_i, e'_i, 0, 0)$,$(Pf'_i, f'_i, 0, 0)$,$(0, 0, e_i, e_i)$, $(0, 0, f_i, f_i)$,$(0, 0, e'_i, e'_i)$, and $(0, 0, f'_i, f'_i)$. If we choose bases of L_2,L_1 by taking the graphs of the basis $(e_1, \ldots, e_k, e'_{k+1}, \ldots, e'_n, f_1, \ldots, f_k, f'_{k+1}, \ldots, f'_n)$ on W, then the basis $\mathcal{B}$ of $L_2 \times L_1$ differs from the product of these bases by a matrix of determinant one. The image of the basis $\mathcal{B}$ under τ is the basis of $L_2 + L_1$ consisting of the vectors

$$(e_i, e_i), (f_i, f_i), (e'_i, e'_i), (f'_i, f'_i), (-Pe'_i, -e'_i), (-Pf'_i, -f'_i) \tag{2.4.23}$$

By definition, ν assigns the value 1 to (2.4.23). Now consider the exact sequence

$$0 \to L_2 + L_1 \to W^- \times W \to W^- \times W/(L_2 + L_1) \to 0$$

we now get a half density ξ on the quotient $W^- \times W/(L_2 + L_1)$,

$$\xi = \frac{|\Omega_{W^- \times W}|^{\frac{1}{2}}}{\nu}.$$

To describe it, start with the basis (2.4.23) of $L_1 + L_2$, we add the vectors $(e_i, 0), (f_i, 0)$ in order to complete this to a basis of $W \times W^-$. The change of basis matrix, A, from the symplectic product basis, $(-e_i, 0)$,$(-f_i, 0)$,$(-e_i', 0)$,$(-f_i', 0)$,$(0, e_i)$,$(0, f_i)$,$(0, e_i')$,$(0, f_i')$ of $W^- \times W$ to the basis of $W^- \times W$ just constructed has determinant equal to $|\det I - P^{\#}|$. To see this, we let A be this matrix and decompose A into blocks,

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$$

and note that each block can be further written as a block matrix reflecting the decomposition $W \cong \ker I - P \oplus \operatorname{im} I - P$. If we subtract the last $2n - 2k$ rows of the top half of A from the bottom $2n - 2k$ rows of A, the D block becomes 0 and the E block becomes

$$E' = \begin{pmatrix} I & 0 \\ 0 & I - P^{\#} \end{pmatrix}$$

And since

$$B = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

We get $|\det A| = |\det B \det E'| = |\det I - P^{\#}|$, as claimed. Now this means that ξ assigns the value $|\det I - P^{\#}|^{\frac{1}{2}}$ to the basis $(e_i, 0), (f_i, 0)$ in the quotient, $W^- \times W/(L_2 + L_1)$. One can check that the bases $(e_i, 0)$,$(f_i, 0)$ of $W^- \times W/(L_2 + L_1)$ and $(e_i, e_i), (f_i, f_i)$ of $\ker \alpha \cong \ker I - P$ are dual with respect to the canonical pairing in lemma 2.4.1, at least up to a change of basis of determinant one. This means that ξ^{-1} induces the density on $\ker I - P$ which assigns the value $|\det I - P^{\#}|^{-\frac{1}{2}}$ to the basis (e_i, e_i),(f_i, f_i). Therefore we finally end up with the density

$$|\Omega|^{\frac{1}{2}} \otimes |\det I - P^{\#}|^{-\frac{1}{2}} |\Omega|^{\frac{1}{2}}$$

on $\ker \alpha$, as claimed. □

2.4.3. Composition of cleanly intersecting weighted homogeneous canonical relations

Let X,Y, and Z be smooth manifolds, compact without boundary, of dimensions n, m, and d respectively. Suppose that $C_1 \subset T^*X \setminus 0 \times T^*Y \setminus 0$ and $C_2 \subset T^*Z \setminus \times T^*X \setminus 0$ are homogeneous canonical relations from T^*Y to T^*X and T^*X to T^*Z. In this section we explain how the clean intersection assumption guarantees that the composition $C_2 \circ C_1 \subset T^*Z \setminus 0 \times T^*Y \setminus 0$ is an immersed, homogeneous canonical relation.

Definition 2.4.9. We say that C_2 and C_1 **intersect cleanly** the if the following is a clean fiber product:

(2.4.24)
$$\begin{tikzcd} C_1 \arrow[d, "\pi_L"] & F \arrow[l] \arrow[d] \\ T^*X \setminus 0 & C_2 \arrow[l, "\pi_R"'] \end{tikzcd}$$

This means that $F = (C_2 \times C_1) \cap (T^*Z \times \Delta_{T^*X} \times T^*Y)$ is a submanifold of $T^*Z \times T^*X \times T^*X \times T^*Y$ and if $(p, q) \in F$, then

(2.4.25)
$$T_{(p,q)}F = \{(u, w, w, v) \mid (u, w) \in T_pC_2, (w, v) \in T_qC_1\}$$

Proposition 2.4.10. *Suppose that C_2 and C_1 are intersect cleanly. Then the map π_α,*

$$\pi_\alpha : F \to C_2 \circ C_1 \qquad \pi_\alpha(p,q) = (\pi_L(p), \pi_R(q)) \tag{2.4.26}$$

is a surjective, proper submersion whose image is an immersed, homogeneous canonical relation in $T^*Z \times T^*Y$. We define the ***excess****, e, to be the dimension of the fibers of π_α,*

$$e = \dim F - \frac{1}{2}\dim(T^*Z \times T^*Y) \tag{2.4.27}$$

Proof. We begin by showing that π_α is a submersion. The derivative of π_α is the map

$$d(\pi_\alpha)_{(p,q)} : T_{(p,q)}F \to T_{\pi_\alpha(p,q)}(T^*Z \times T^*Y) \qquad d(\pi_\alpha)_{(p,q)} : (u,w,w,v) \mapsto (u,v)$$

For $(p,q) \in F$, $p = (\zeta,\xi) \in C_2$, $q = (\xi,\eta) \in C_1$, let $\tau_{(p,q)}$ be the map on tangent spaces,

$$\tau_{(p,q)} : T_pC_2 \times T_qC_1 \to T_\xi T^*X \qquad \tau_{(p,q)}(u,w,w',v) = w' - w$$

Then we can consider the linearized exact sequences,

$$0 \to \ker d(\pi_\alpha)_{(p,q)} \to T_{(p,q)}F \to T_pC_2 \circ T_qC_1 \to 0 \tag{2.4.28}$$

$$0 \to T_{(p,q)}F \to T_pC_2 \times T_qC_1 \to \operatorname{im}\tau_{(p,q)} \to \operatorname{coker}\tau_{(p.q)} \to 0 \tag{2.4.29}$$

Because F is a submanifold, the dimension of $T_{(p,q)}F$ is constant and therefore the dimension of $\operatorname{im}\tau_{(p,q)}$ is constant, equal to $\dim C_2 + \dim C_1 - \dim F$. We know from the linear case that $\ker d(\pi_\alpha)_{(p,q)}$ is identified with the symplectic complement of $\operatorname{im}\tau_{(p,q)}$ and so has constant dimension in $T_{(p,q)}F$. This shows that π_α is a constant rank, homogeneous degree one map. The second condition implies that π_α is proper, and these together ensure that the image of π_α is an immersed submanifold of $T^*Z \setminus 0 \times T^*Y \setminus 0$. It is clearly homogeneous, and it is a canonical relation because the linear composition $T_pC_2 \circ T_qC_1$, which is the tangent space to $C_2 \circ C_1$ at $\pi_\alpha(p,q)$, is a canonical relation from $T_\eta T^*Y$ to $T_\zeta T^*Z$.

$\square$

Now suppose that these canonical relations are weighted with smooth, homogeneous half densities $\sigma_i \in C^\infty(\Omega^{\frac{1}{2}}, C_i)$. For each point $(p,q) \in F$, the exact sequences (2.4.28), (2.4.29) determine an object an object $\sigma_2 \,\square\, \sigma_1$ which is a half density on $T_{(\zeta,\eta)}C_2 \circ C_1$ tensored with a density on $\ker d(\pi_\alpha)_{(p,q)}$, which is identified with the tangent space of the fiber $\pi_\alpha^{-1}(\zeta,\eta)$ at the point (p,q). The composite half density on $C_2 \circ C_1$ is then defined, for $(\zeta,\eta) \in C_2 \circ C_1$, by

$$\sigma_2 \circ \sigma_1(\zeta,\eta) = \int_{\pi_\alpha^{-1}(\zeta,\eta)} \sigma_2 \,\square\, \sigma_1. \tag{2.4.30}$$

In the discussion of the composition of linear canonical relations, we considered several cases in which the linear map α had no kernel. The geometric situation that corresponds to is the following special case of clean intersection. We say that C_2 and C_1 **intersect transversely** if they intersect cleanly and the co-dimension of F inside $T^*Z \times T^*X \times T^*X \times T^*Y$ is the sum of the co-dimensions of $C_2 \times C_1$ and $T^*Z \times \Delta_{T^*X} \times T^*Y$. This assumption ensures that the excess of $\pi_\alpha : F \to C_2 \circ C_1$ is zero, i.e. that π_α is a proper, local diffeomorphism onto its image. The following is an important case in which we always have transverse intersection.

Proposition 2.4.11. *Let $C_2 \subset T^*Z \setminus 0 \times T^*X \setminus 0$ be a homogeneous canonical relation from T^*X to T^*Z and $C_1 = \Gamma_\chi \subset T^*X \setminus 0 \times T^*Y \setminus 0$ be the graph canonical relation of the homogeneous canonical transformation*

$$\chi : T^*Y \to T^*X$$

Then C_2 and C_1 intersect transversely.

Proof. Fix a point $p = (\zeta, \chi(\eta), \chi(\eta), \eta) \in F$. We need to show that the tangent spaces

$$T_{(\zeta,\chi(\eta),\chi(\eta),\eta)}(C_2 \times C_1) = \{(u, w, d\chi v, v) \mid (u, w) \in T_{(\zeta,\chi(\eta)}C_2, (d\chi v, v) \in T_{(\chi(\eta),\eta)}C_1\}$$

and

$$T_{(\zeta,\chi(\eta),\chi(\eta),\eta)}(T^*Z \times \Delta_{T^*X} \times T^*Y) = \{(a, b, b, c) \mid a \in T_\zeta T^*Z, b \in T_{\chi(\eta)}T^*X, c \in T_\eta T^*Y\}$$

together generate the tangent space of the four fold product at p. Suppose we pick an arbitrary tangent vector $(\alpha, \beta, \delta, \gamma)$ of the four-fold product at p. Choose any $(u, w) \in T_{(\zeta,\chi(\eta))}C_2$, and pick $a \in T_\zeta T^*Z$ and $b \in T_{\chi(\eta)}T^*X$ so that $(u, w) + (a, b) = (\alpha, \beta)$. Now pick $v \in T_\eta T^*Y$ so that $d\chi v + b = \delta$ (which is possible because $d\chi$ is surjective) and finally $c \in T_\eta T^*Y$ so that $c + v = \gamma$. Then

$$(u, w, d\chi v, v) + (a, b, b, c) = (\alpha, \beta, \delta, \gamma)$$

□

As an example of transverse symbol composition, we consider the composition of two canonical graphs.

Proposition 2.4.12. *Suppose that $\chi_1 : T^*Y \to T^*X$ and $\chi_2 : T^*X \to T^*Z$ are homogeneous canonical transformations and let C_i be their corresponding graph canonical relations. Let π_L be the projection map onto the left factor and define the half densities*

$$|dy \wedge d\eta|^{\frac{1}{2}} = \pi_L^*|\Omega_{T^*Y}|^{\frac{1}{2}} \tag{2.4.31}$$

$$|dx \wedge d\xi|^{\frac{1}{2}} = \pi_L^*|\Omega_{T^*X}|^{\frac{1}{2}} \tag{2.4.32}$$

on C_1 and C_2. Then let $\sigma_2(\chi_2(x,\xi),(x,\xi)) = a(x,\xi)|dx \wedge d\xi|^{\frac{1}{2}}$ and $\sigma_1(\chi_1(y,\eta),(y,\eta)) = b(y,\eta)|dy \wedge d\eta|^{\frac{1}{2}}$. where a and b are smooth and homogeneous in ξ and η. Then

$$C_2 \circ C_1 = \{(\chi_2\chi_1(y,\eta),(y,\eta)\} \subset T^*Z \setminus \times T^*Y \setminus 0 \tag{2.4.33}$$

$$\sigma_2 \circ \sigma_1(\chi_2\chi_1(y,\eta),(y,\eta)) = a(\chi(y,\eta)b(y,\eta)|dy \wedge d\eta|^{\frac{1}{2}}. \tag{2.4.34}$$

Proof. By proposition 2.4.11, the composition is transverse. We fix a point $(y,\eta) = q \in T^*Y$. The fiber of π_α over $(\chi_2\chi_1(q),q)$ is the single point $(\chi_2\chi_1(q),\chi_1(q),q)$. Since C_2 intersects C_1 transversely, the map $\tau = \tau_{(\chi_2\chi_1(q),\chi_1(q),\chi_1(q),q)}$ is

$$\tau : (d\chi_2\beta, \beta, d\chi_1\alpha, \alpha) \mapsto d\chi_1\alpha - \beta \in T_{\chi(q)}T^*X \tag{2.4.35}$$

Let (e_i, f_i) be a symplectic basis for T_qT^*Y. Then we have the basis $(d\chi_2 d\chi_1 e_i, d\chi_1 e_i, d\chi_1 e_i, e_i)$, $(d\chi_2 d\chi_1 f_i, d\chi_1 f_i, d\chi_1 f_i, f_i)$ of $T_{\chi_2\chi_1(q),q)}F$. If we add the linearly independent vectors $(0, 0, d\chi_1 e_i, e_i)$, $(0, 0, d\chi_1 f_i, f_i)$ we get a basis of $T_{(\chi_2\chi_1(q),\chi_1(q))}C_2 \times T_{(\chi_1(q),q)}C_1$ which differs from the product of the symplectic bases on each factor by a A with $|\det A| = 1$. The image of this basis under τ is the symplectic basis $d\chi_1 e_i, d\chi_1 f_i$ of $T_{\chi_1(q)}T^*X$. By proposition 2.4.4, the composite half density assigns the value $a(\chi_1(q))b(q)$ to this basis of $T_{(\chi_2\chi_1(q),q)}F$, which proves the stated formula.

□

2.5. The Duistermaat-Guillemin wave trace

Let Δ be the positive Laplace operator on (X^n, g). The goal of this section is to explain the leading order asymptotics of the trace of the wave group on X found in [**8**]. Let (φ_j, λ_j^2) be an orthonormal basis of $L^2(X, dV_g)$ of eigenfunctions of the Laplacian,

$$\Delta\varphi_j = \lambda_j^2\varphi_j.$$

We first begin by fixing notation for the remainder of this section. Let $U(t)$ be the **wave group** of X, the operator

$$U(t) : C^\infty(\Omega^{\frac{1}{2}}, X) \to C^\infty(\Omega^{\frac{1}{2}}, \mathbb{R} \times X) \qquad U(t) : f(x)|dV_g|^{\frac{1}{2}} \mapsto u(t, x)\,|dt|^{\frac{1}{2}} \otimes |dV_g(x)|^{\frac{1}{2}}$$

where $u(t, x)$ is the solution of the initial value problem,

$$\begin{aligned} &\left(\frac{1}{i}\frac{\partial}{\partial t} - \sqrt{\Delta}\right) u(t,x) = 0 \\ &u(0,x) = f(x). \end{aligned} \tag{2.5.1}$$

We will write $p(x,\xi) = |\xi|_{g(x)} = \sqrt{g^{ij}(x)\xi_i\xi_j}$ for the principal symbol of the operator $\sqrt{\Delta}$. We also let $G^t = \exp tH_p$ be the homogeneous geodesic flow, where H_p is the Hamiltonian vector field of p on T^*X. We are interested in the trace of $U(t)$ as a distribution half density on $\mathbb{R}$,

$$e(t) = \operatorname{Tr} U(t) = \sum_j e^{it\lambda_j} |dt|^{\frac{1}{2}}. \tag{2.5.2}$$

If we write $U(t,x,y) = \sum_{j=0}^{\infty} e^{it\lambda_j}\varphi_j(x)\overline{\varphi_j(y)}\,|dV_g(x)|^{\frac{1}{2}} \otimes |dV_g(y)|^{\frac{1}{2}} \otimes |dt|^{\frac{1}{2}}$, then formally we have

$$\operatorname{Tr} U(t) = \int_X U(t,x,x) = \sum_j e^{it\lambda_j}\left(\int_X |\varphi_j(x)|^2 |dV_g(x)|\right)|dt|^{\frac{1}{2}} = e(t). \tag{2.5.3}$$

In section 2.2.6, we saw that $U(t,x,y)$ is a Lagrangian kernel of order $-1/4$ associated to the Lagrangian submanifold

$$\Lambda = \{(t, |\xi|_{g(x)}, x, \xi, y, \eta) \mid (y, -\eta) = G^t(x,\xi)\} \subset T^*(\mathbb{R} \times X \times X). \tag{2.5.4}$$

For the remainder of the section, we will make frequent use of the embedding

$$(2.5.5) \qquad i:\mathbb{R}\times T^*X \hookrightarrow T^*(\mathbb{R}\times X\times X) \qquad i(t,x,\xi)\mapsto (t,p(x,\xi),x,\xi,G^t(x,\xi)).$$

whose image is Λ. The strategy for understand $e(t)$ is to the view 'restrict to the diagonal and integrate' operation appearing in (2.5.3) is a Fourier integral operator which can be composed with $U(t,x,y)$ if the geodesic flow satisfies a nice geometric assumption. In this case the composition theory will allow us to prove that $e(t)$ is a Lagrangian distribution on $\mathbb{R}$.

2.5.1. The trace as a Fourier integral operator

Let $\Delta:\mathbb{R}\times X\to\mathbb{R}\times X\times X$ be the inclusion of the spacial diagonal, $\Delta:(t,x)\mapsto(t,x,x)$ and $\pi:\mathbb{R}\times X\to\mathbb{R}$ be the projection onto the first factor. Pulling back along Δ is restriction to the diagonal and pushing forward along π is integration in the spacial variables. We extend this to half densities by setting

$$\pi_*\Delta^* f(t,x,y)\,|dt|^{\frac{1}{2}}\otimes|dV_g(x)|^{\frac{1}{2}}\otimes|dV_g(y)|^{\frac{1}{2}} = \left(\int_X f(t,x,x)\,|dV_g(x)|\right)|dt|^{\frac{1}{2}}$$

Letting $K(t,s,p,q)$ be the half density distribution kernel of $\pi_*\Delta^*$, we may write

$$K(t,s,p,q)=\widetilde{K}(t,s,p,q)\,|ds|^{\frac{1}{2}}\otimes|dV_g(p)|^{\frac{1}{2}}\otimes|dV_g(q)|^{\frac{1}{2}}\otimes|dt|^{\frac{1}{2}}$$

It is easy to check that the scalar kernel $\widetilde{K}$ is equal to

$$\widetilde{K}(t,s,p,q)=\delta_0(s-t)\otimes\delta_p(q).$$

If we view $\widetilde{K}$ as the kernel of an operator $A:C^\infty(\mathbb{R}\times M)\to C^\infty(\mathbb{R}\times M)$ by setting $Af(t,p) = \int K(t,s,p,q)f(s,q)$ then it is clear that A is the identity operator. To summarize, if $\Phi : \mathbb{R}\times\mathbb{R}\times M\times M \to \mathbb{R}\times M\times\mathbb{R}\times M$ is the map $\Phi(t,s,p,q) = (t,p,s,q)$. Then we have $\Phi^* K_{Id} = K$. Recall that

$$\mathrm{Id} \in I^0(\mathbb{R}\times X\times\mathbb{R}\times X; \Gamma_{Id})$$

where $\Gamma_{\mathrm{Id}} = (t,\tau,x,\xi,t,\tau,x,\xi)$ is the graph of the identity map on $T^*(\mathbb{R}\times M)$. The symbol of the identity map is equal to the pullback of the canonical symplectic volume $|dt\wedge d\tau\wedge dx\wedge d\xi|^{\frac{1}{2}}$ on $T^*(\mathbb{R}\times M)$. To summarize,

Proposition 2.5.1. *The operator $\pi_*\Delta^*$ is a Fourier integral operator in the class $I^0(\mathbb{R}\times\mathbb{R}\times X\times X; C)$ where*

$$C = \{(t,\tau,t,\tau,x,\xi,x,\xi)\} \subset T^*\mathbb{R}\setminus 0 \times T^*(\mathbb{R}\times X\times X)\setminus 0$$

and whose principal symbol is $i^*\sigma = |dt\wedge d\tau\wedge dx\wedge d\xi|^{\frac{1}{2}}$ where $i : T^*(\mathbb{R}\times M) \to C$ is the obvious parametrization.

2.5.2. The clean fixed point set assumption

Our goal is to compose the operator $\pi_*\Delta^*$ with the Lagrangian kernel $U(t,x,y)$. To do this, we need to ensure that the canonical relation C intersects Λ cleanly. This section describes an assumption on the geodesic flow that guarantees clean intersection.

Definition 2.5.2. Let $Z_t = \{(x,\xi)\in T^*X \mid G^t(x,\xi) = (x,\xi)\}$. We say that G^t has a **clean fixed point set** if for each $t\in\mathbb{R}$, Z_t is a submanifold and the tangent space at a point $p\in Z_t$ is equal to

$$T_pZ_t = \{v\in T_pT^*M \mid dG^tv = v\} \subset T_pT^*X. \tag{2.5.6}$$

Notice that this is just the statement that the graph of the geodesic flow intersects the graph of the identity cleanly in $T^*X \times T^*X$. We will see that closed geodesics control the singularities of the trace of $U(t)$. An important geometric consequence of the clean fixed point set is that the length spectrum of X is a countable set of isolated points.

Proposition 2.5.3. *If G^t has clean fixed points, then the* ***length spectrum of*** $\mathbf{X}$*,*

$$\mathrm{LSP}(X) = \{T \in \mathbb{R} \mid \exists (x, \xi) \in T^*X; G^T(x, \xi) = (x, \xi)\}$$

is a closed, discrete set.

We need the following lemma in the proof that the clean fixed point assumption implies the clean intersection of C with Λ.

Lemma 2.5.4. *Let R be the vector field which generates the scaling action $\lambda \cdot (x, \xi) \mapsto (x, \lambda\xi)$ on the fibers of T^*X. In local symplectic coordinates, $R = \xi\partial_\xi$ and at each point $(x, \xi) \in S^*M$,*

$$\omega(H_p, R) = 1 \tag{2.5.7}$$

Proof. The formula $R = \xi_i \partial_\xi$ just follows from differentiating the curve $(x, t\xi)$ at $t = 1$ in local coordinates. Now

$$\omega(H_p, R) = dp(R) = \frac{\partial p}{\partial \xi_k}\xi_k,$$

while

$$\frac{\partial p^2}{\partial \xi_k} = 2p\frac{\partial p}{\partial \xi_k} = 2g^{ik}\xi_i.$$

Since $p = 1$ on S^*M, $\omega(H_p, R) = g^{ik}\xi_i\xi_k = 1$.

$\square$

Proposition 2.5.5. *If G^t has clean fixed point sets, then the canonical relation C of $\pi_*\Delta^*$ intersects Λ cleanly.*

Proof. We need to check two things, that

$$F = (C \times \Lambda) \cap T^*\mathbb{R} \times \Delta_{T^*(\mathbb{R}\times M\times M)}$$

is a submanifold and that its tangent space is equal to the intersection of the tangent spaces of each factor. Since

$$C = \{(t, \tau, t, \tau, x, \xi, x, \xi)\} \subset T^*R \setminus 0 \times T^*(\mathbb{R} \times X \times X) \setminus 0,$$

we can identify a point in F by its projection onto Λ. In this sense we have

$$F \cong \{(t, \tau, x, \xi, x, \xi) \in \Lambda\} = \{(t, p(x, \xi), x, \xi, x, \xi) \mid G^t(x, \xi) = (x, \xi)\} \subset T^*\mathbb{R}\times T^*X\times T^*X.$$

There is a natural embedding

(2.5.8)

$$i : \bigcup_{T\in LSP(X)} \{T\} \times Z_T \to T^*\mathbb{R} \times T^*X \times T^*X \qquad i : (T, x, \xi) \mapsto (T, p(x, \xi), x, \xi, x, \xi)$$

whose image is F. Since the domain is diffeomorphic to a disjoint union of all the fixed point sets, the clean fixed point set assumption guarantees that F is a manifold. To verify the condition on spaces, we fix a point $i(T, x, \xi) = (p, q) \in F$. Here, $p =$

$(T, p(x,\xi), T, p(x,\xi), x, \xi, x, \xi)$ and $q = (T, p(x,\xi), x, \xi, x, \xi)$. If we identify a tangent vector in $T(T^*\mathbb{R})$ by a pair of real numbers,

$$(a,b) \cong a\partial_t + b\partial_\tau,$$

then according to the embedding i, the tangent space of F is

$$T_{(p,q)}F \cong \{(0, dp(v), v, v) \mid v \in T_{(x,\xi)}Z_T\}, \tag{2.5.9}$$

where this is really shorthand for

$$(0, dp(v), v, v) \cong (0, dp(v), 0, dp(v), v, v, 0, dp(v), v, v) \in T_pC \times T_q\Lambda. \tag{2.5.10}$$

We need to verify that (2.5.9) is equal to $(T_pC \times T_q\Lambda) \cap T_{(p,q)}(T^*\mathbb{R} \times \Delta_{T^*(\mathbb{R}\times X\times X)})$. The intersection is equal to the set of tangent vectors of the form $(\alpha, \beta, \beta) \in T_pC \times T_q\Lambda$ where $\alpha \in T(T^*\mathbb{R})$ and $\beta \in T(T^*(\mathbb{R} \times X \times X))$. Write $\beta = (a, b, v, w)$ where $(a,b) \in T(T^*\mathbb{R})$ and $v, w \in T(T^*X)$. Using the parametrization (2.5.5) we can write

$$T_q\Lambda = \{(s, dp(v), v, sH_p + dG^t(v)) \mid s \in \mathbb{R}, v \in T(T^*X)\}$$

On the other hand, if $(\alpha, \beta) \in T_pC$, then

$$(\alpha, \beta) = (a, b, a, b, v', v') \qquad v' \in T(T^*X)$$

This means the intersection can be identified with the set of points

$$\{(s, dp(v), v, v) \mid v = sH_p + dG^T v\} \subset T(T^*\mathbb{R} \times T^*X \times T^*X)$$

If $v = sH_p + dG^T v$, then by the lemma (2.5.4),

$$s = \omega(sH_p, R) = \omega((I - dG^T)v, R) = 0 \tag{2.5.11}$$

where the last equality is because R is fixed by dG^T by homogeneity. Hence $s = 0$ and $v = dG^T v$. By the clean fixed point assumption, this is equal to the expression (2.5.9) for the tangent space of F.

□

2.5.3. Lagrangian structure of the wave trace

The clean intersection (2.5.5) guarantees that $e(t) = \pi_* \Delta^* U(t)$ is a Lagrangian distribution on $\mathbb{R}$. From our description of the submanifold F of the previous section, if $(T, \tau) \in C \circ \Lambda$, then there exists $(x, \xi) \in T^* X$ so that $G^T(x, \xi) = (x, \xi)$ and $\tau = p(x, \xi)$. By homogeneity, of G^T, $(t, \tau) \in C \circ \Lambda$ implies that $(t, \tau) \in C \circ \Lambda$ for all $\tau > 0$. Therefore, in the notation of section 2.2.1,

$$C \circ \Lambda = \bigcup_{T \in \mathrm{LSP}(X)} \Lambda_T^+ \tag{2.5.12}$$

Fix some $T \in \mathrm{LSP}(X)$ and split the cross section of the fixed point set Z_T into connected components,

$$Z_T \cap S^* X = \cup_{j=1}^k Z_{T,j}^1 \tag{2.5.13}$$

If Z_T has dimension $e+1$, then each component is a clean fixed point set of dimension i. Since the fiber $\pi_\alpha^{-1}(T,1)$ is identified with $Z_T \cap S^*X$, the excess of the composition at $t = T$ is equal to $e/2$, therefore

$$\mathrm{Ord}(\, e(t)|_{t=T}) = \frac{e}{2} - \frac{1}{4} + \frac{3}{4} - \frac{3}{4} = \frac{e+1}{2} - \frac{3}{4}$$

From section 2.2.1, in a neighborhood of $t = T$, modulo smooth functions of t, $e(t)$ has the expansion

(2.5.14) $$e(t) \sim a_0(t - T + i0)^{-\frac{e+1}{2}} + a_1(t - T + i0)^{-\frac{e-1}{2}} + \cdots$$

where the leading coefficient is equal to

(2.5.15) $$a_0 = \exp\frac{i(e+1)\pi}{4}\Gamma\left(\frac{e+1}{2}\right) b_0$$

where the symbol of $e(t)$ at $(T,1)$ is equal to

(2.5.16) $$\sigma(T,1) = b_0 \tau^{\frac{e}{2}} |d\tau|^{\frac{1}{2}}$$

The coefficient b_0 will be a sum of terms each of which is contributed by one of the components $Z^1_{T,j}$. This differs from the result of theorem 4.5 in [**8**] only by the omission of the Maslov factors which they write as $i^{-\sigma_j}$. These turn out to be the common Morse index of the geodesics belonging to component j of the fixed point set $Z_T \cap S^*X$.

Proposition 2.5.6. *The wave trace $e(t) = \pi_* \Delta^* U(t)$ can be written as a sum*

$$e(t) = \sum_{T \in \mathrm{LSP}(X)} u_T(t) \mod C^\infty(\mathbb{R}) \tag{2.5.17}$$

where each $u_T \in I^{\frac{e+1}{2}-\frac{3}{4}}(\mathbb{R}, \Lambda_T^+)$. *Each* u_T *has the asymptotic expansion,*

$$u_T(t) \sim a_{0,T}(t - T + i0)^{-\frac{e+1}{2}} + a_{1,T}(t - T + i0)^{-\frac{e-1}{2}} + \cdots \tag{2.5.18}$$

and the leading coefficient is equal to

$$a_{0,T} = \sum_{j=1}^{k} \exp i\pi \left(\frac{m_j}{2} + \frac{e+1}{4} \right) \Gamma \left(\frac{e+1}{2} \right) b_{0,j,T}. \tag{2.5.19}$$

The principal symbol of u_T *is*

$$\sigma(u_T)(T, 1) = \left(\sum_j \exp \frac{i\pi m_j}{2} b_{0,j,T} \right) \tau^{\frac{e}{2}} |d\tau|^{\frac{1}{2}} \tag{2.5.20}$$

The next section contains the calculation of the symbol of u_T which we split into two cases, $T = 0$ and $T \neq 0$.

2.5.4. The symbol of $\mathbf{u_0}$

The fixed point set Z_0 is all of T^*X, so the excess is $e = 2n - 1$ and the order of u_0 is $n - 3/4$. Our goal is to prove

Proposition 2.5.7. *Let* μ_L *be the Leray volume form on* S^*X, *the positive* $2n+1$ *form on* S^*X *determined by*

$$\mu_L \wedge dp = \omega^n \tag{2.5.21}$$

where ω^n is the symplectic volume form on T^*X. Then

$$\sigma(u_0) = \mathrm{Vol}(S^*X)\tau^{n-\frac{1}{2}}|d\tau|^{\frac{1}{2}} \tag{2.5.22}$$

Since the symbol is homogeneous it is determined by its value at $\tau = 1$. We just need to verify that this is equal to $\mathrm{Vol}(S^*M)$. Put $F_0 = \pi_\alpha^{-1}(\Lambda_0^+)$ which is equal to the image of (2.5.8) restricted to $\{0\} \times T^*X$. Writing $(p, q) = i(0, x, \xi)$, (2.5.9) tells us that

$$T_{(p,q)}F_0 = \{(0, dp(v), v, v) \mid v \in T_{(x,\xi)}T^*X\}$$

We let $|\Omega|^{\frac{1}{2}}$ be the half density on F_0 which is equal to one on the pushforward by (2.5.8) of any symplectic basis of $T_{(x,\xi)}(T^*X)$. Now we choose a basis of $T_{(x,\xi)}T^*X$ of the form

$$\{e_1, \ldots, e_{n-1}, H_p, f_1, \ldots, f_{n-1}, R\} \tag{2.5.23}$$

where $\{e_1, \ldots, e_{n-1}, H_p, f_1, \ldots, f_{n-1}\}$ is a basis for $T(S^*X)$, R is the radial vector field, and the entire collection forms a symplectic basis for $T(T^*X)$. To work out the symbol composition, begin with the exact sequence

$$0 \to \ker d(\pi_\alpha)_{(p,q)} \to T_{(p,q)}F_0 \to T_{(0,1)}\Lambda_0^+ \to 0.$$

In terms of the identification (2.5.9) of $T_{(p,q)}F_0$, $d(\pi_\alpha)_{(p,q)}$ acts by

$$d(\pi_\alpha)_{(p,q)} : (0, dp(v)\partial_\tau, v, v) \mapsto (0, dp(v))$$

hence $\ker d(\pi_\alpha)_{(p,q)} = T_{(x,\xi)}S^*X$. The pushforward by the embedding (2.5.8) of the linearly independent set of vectors $(e_1, \ldots, e_{n-1}, H_p, f_1, \ldots, f_{n-1}) \subset T_{(x,\xi)}T^*X$ is a basis of $T_{(x,\xi)}S^*X \cong \ker d(\pi_\alpha)_{(p,q)}$ and we complete it to a symplectic basis of $T(p,q)F_0$ by adding the pushforward of the radial vector field, R. Then we may write

$$|\Omega|^{\frac{1}{2}} = |\mu_L|^{\frac{1}{2}} \otimes |d\tau|^{\frac{1}{2}} \tag{2.5.24}$$

Because μ_L equals to one on the basis $(e_1, \ldots, e_{n-1}, H_p, f_1, \ldots, f_{n-1})$ of $T_{(x,\xi)}S^*X$ and $dp(R) = \omega(H_p, R) = 1$ by lemma 2.5.4. Now consider the exact sequence

$$0 \to T_{(p,q)}F_0 \to T_pC \times T_q\Lambda \to \operatorname{im}\tau_{(p,q)} \to 0$$

We have natural bases on T_pC and $T_q\Lambda$ from the which come from pushing forward the product of our basis (2.5.23) of $T_{(x,\xi)}T^*X$ with the standard basis of $T\mathbb{R}$ and the standard symplectic basis of $T(T^*\mathbb{R})$ along the parametrizing maps

$$i : \mathbb{R} \times T^*X \to \Lambda$$

$$i' : T^*\mathbb{R} \times T^* \times X \to C$$

The product half density $\sigma_2 \otimes \sigma_1$ on $T_pC \times T_q\Lambda$ is equal to one on this pushforward basis.

Lemma 2.5.8. *The basis* (2.5.23) *of* $T_{(p,q)}F_0$ *can be completed to a basis of* $T_pC \times T_q\Lambda$ *which differs from the pushforward product basis above by a matrix* A *with* $|\det A| = 1$.

We can do this by adding the $2n+3$ vectors $\mathbf{0} \times (0, dp(v_i), v_i, v_i)$, $\mathbf{0} \times (1, 0, 0, H_p)$, $(1, 0, 1, 0, 0, 0) \times \mathbf{0}$, and $(0, 1, 0, 1, 0, 0) \times \mathbf{0}$, where v_i ranges over the basis (2.5.23) This means that we can write

$$\sigma_1 \otimes \sigma_2 = |\Omega|^{\frac{1}{2}} \otimes \nu \tag{2.5.25}$$

where ν is the half density which assigns 1 to the basis $(0, dp(v_i), v_i, v_i)$,$(1, 0, 0, H_p)$, $(0, -1, 0, 0)$,$(-1, 0, 0, 0)$ of $\operatorname{im} \tau_{(p,q)} \subset T(T^*\mathbb{R} \times T^*X \times T^*X)$. All that's left is to figure out which half density on $\ker d(\pi_\alpha)_{(p,q)} \cong T(S^*X)$ ν corresponds to. For this we use the last exact sequence

$$0 \to \operatorname{im} \tau_{(p,q)} \to T(T^*\mathbb{R} \times T^*X \times T^*X) \to \operatorname{coker} \tau_{(p,q)} \to 0.$$

We take the basis which ν assigns the value 1 to and complete it to a basis of the middle factor by adding the $2n-1$ vectors $(0, 0, 0, v_i)$ where v_i ranges over $e_1, \dots, e_{n-1}, f_1, \dots, f_{n-1}, R$. This differs by a matrix of $|\det| = 1$ from a symplectic basis on $T(T^*\mathbb{R} \times T^*X \times T^*X)$. Hence we get the half density on $\operatorname{coker} \tau$ which assigns the value 1 to the basis consisting of the residues of $(0, 0, 0, v_i)$. Its reciprocal is a negative half density which assigns the same value to this basis. The final step is to note that this basis of $\operatorname{coker} \tau$ is the dual basis to the 1-forms

$$\omega_{T(T^*\mathbb{R} \times T^*X \times T^*X)}((0, dp(v_i), v_i, v_i), \cdot)$$

where v_i ranges over the basis $e_1, \dots, e_{n-1}, H_p, f_1, \dots, f_{n-1}$ of $\ker d(\pi_\alpha)_{(p,q)} \cong T_{(x,\xi)}S^*X$. Thus, in the end, ν is equivalent to the half density on $\ker d\alpha \cong T_{(x,\xi)}S^*X$ which equals one on $e_1, \ldots, e_{n-1}, H_p, f_1, \ldots, f_{n-1}$, which we have already observed is $|\mu_L|^{\frac{1}{2}}$. We have just proved that

$$\sigma_1 \otimes \sigma_2 \cong |\mu_L| \otimes |d\tau|^{\frac{1}{2}}$$

The half density symbol of the trace at $\tau = 1$ is then just the integral of μ_L over $\pi_\alpha^{-1}(0,1) \cong S^*M$ times $|d\tau|^{\frac{1}{2}}$, which is what was stated.

2.5.5. The symbol of $\mathbf{u_T}$ for $\mathbf{T \neq 0}$

To make the calculations more explicit, we make the simplifying assumption that the fixed point set $Z_T \cap S^*X$ consists of one closed geodesic γ. Fix a base point $(x_0, \xi_0) \in \gamma$ and let $T^{\#}$ be the minimal period of γ, the smallest positive $T > 0$ so that $G^T(x_0, \xi_0) = (x_0, \xi_0)$.

Definition 2.5.9. The **Poincaré map**, P_γ, is the derivative of the geodesic flow at our chosen base point of γ,

$$P_\gamma = d(G^T)_{(x_0,\xi_0)} : T_{(x_0,\xi_0)} S^*X \to T_{(x_0,\xi_0)} S^*X \tag{2.5.26}$$

The clean fixed point condition means that H_p is the only tangent vector in $T_{(x_0,\xi_0)}S^*X$ fixed by P_γ.

Lemma 2.5.10. *For each point $(x, \xi) \in \gamma$, the map*

$$I - d(G^T)_{(x,\xi)} : T_{(x,\xi)} S^*M/H_p \to T_{(x,\xi)} S^*M/H_p \tag{2.5.27}$$

is an isomorphism.

Proof. To avoid cluttered notation, we drop the base point subscripts. To say that this map has no kernel is to say that

$$\operatorname{im} I - dG^T \cap \ker I - dG^T = \{0\}.$$

To verify the above equation, suppose that $v \in T(S^*X)$ satisfies $(I - dG^T)v = \alpha H_p$. Then

$$\alpha = \omega(\alpha H_p, R) = \omega((I - dG^T)v, R) = \omega(v, (I - (dG^T)^{-1})R) = 0$$

so $(I - dG^T)v = 0$. But then by cleanliness, v is a multiple of H_p.

$\square$

Since the Poincaré maps at different base points are conjugate by the linearized geodesic flow, the determinant of $I - P_\gamma$ on $T(S^*X)/H_p$ is independent of the chosen base point.

Proposition 2.5.11. *Assume that $T \neq 0$ and $Z_T \cap S^*X$ consists of a single closed geodesic. Then the half density symbol of u_T is*

$$\sigma(u_T) = \frac{T^{\#}}{|\det I - P_\gamma|^{\frac{1}{2}}} \tau^{\frac{1}{2}} |d\tau|^{\frac{1}{2}} \tag{2.5.28}$$

Proof. The order of u_T is $1/2 - 1/4 = 1/4$, so we just need to calculate $\sigma(u_T)$ at $(T, 1)$ as in the previous section. Let $F_T = \pi_\alpha^{-1}(\Lambda_T^+)$. We fix a point $(x, \xi) \in \pi_\alpha^{-1}(T, 1) \cong \gamma \subset S^*X$ and let $(p, q) = i(0, x, \xi) \in F_T$ where i is the embedding (2.5.8). The basis H_p,R of $T_{(x,\xi)} Z_T$ pushes forward under i to the basis

$$(0, 0, H_p, H_p), (0, 1, R, R) \in T_{(p,q)} F_T \tag{2.5.29}$$

where we again use the identification (2.5.10). Let $|\Omega|^{\frac{1}{2}}$ be the half density on $T_{(p,q)}F_T$ which is equal to 1 on this basis and consider the exact sequence

$$0 \to \ker d(\pi_\alpha)_{(p,q)} \to T_{(x,\xi)}F_T \to T_{(T,1)}\Lambda_T^+ \to 0.$$

Now $d(\pi_\alpha)_{(p,q)}(0,1,R,R) = dp(R)\partial_\tau = \partial_\tau$ by lemma (2.5.4). The quotient half density,

$$|dt|^{\frac{1}{2}} = \frac{|\Omega|^{\frac{1}{2}}}{|d\tau|^{\frac{1}{2}}}$$

on $\ker d(\pi_\alpha)_{(p,q)} \cong T_{(x,\xi)}\gamma$ equals 1 on H_p. Therefore the exact sequence splits the symplectic half density on F_T as

$$|\Omega|^{\frac{1}{2}} = |dt|^{\frac{1}{2}} \otimes |d\tau|^{\frac{1}{2}} \tag{2.5.30}$$

Next, to use the exact sequence

$$0 \to T_{(p,q)}F \to T_pC \times T_q\Lambda \to \operatorname{im}\tau_{(p,q)} \to 0,$$

again choose vectors $e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1} \in T_{(x,\xi)}(S^*X)$ so that, together with H_p and R, they form a symplectic basis of $T_{(x,\xi)}T^*X$. We need to complete the basis (2.5.29) of $T_{(p,q)}F_T$ into a basis of $T_pC \times T_q\Lambda$. We then add the $4n+1$ vectors

$$\mathbf{0} \times (0, dp(v_i), v_i, P_\gamma v_i), \mathbf{0} \times (1, 0, 0, H_p)$$

$$(0,0,0,0,w_i,w_i) \times \mathbf{0}, (1,0,1,0,0,0) \times \mathbf{0}, (0,1,0,1,0,0) \times \mathbf{0}$$

where $v_i \in \{e_1, \ldots, e_{n-1}, H_p, f_1, \ldots, f_{n-1}, R\}$ and $w_i \in \{e_1, \ldots, e_{n-1}, f_1, \ldots, f_{n-1}\}$. It is easy to see that this basis differs from the product pushforward basis on $T_pC \times T_q\Lambda$ by a matrix of determinant ± 1, so $\sigma_1 \otimes \sigma_2 = 1$ the completed basis. Therefore,

$$\sigma_1 \otimes \sigma_2 = |\Omega|^{\frac{1}{2}} \otimes \nu \tag{2.5.31}$$

where ν is a half density on $\operatorname{im} \tau$ which equals 1 on the basis

$$(0, dp(v_i), v_i, P_\gamma v_i), (0, 0, -w_i, -w_i), (-1, 0, 0, 0), (0, -1, 0, 0), (1, 0, 0, H_p)$$

of $\operatorname{im} \tau_{(p,q)}$. Here v_i and w_i range over the same sets as before. Finally we use the exact sequence

$$0 \to \operatorname{im} \tau_{(p,q)} \to T(T^*\mathbb{R} \times T^*X \times T^*X) \to \operatorname{coker} \tau_{(p,q)} \to 0.$$

We complete this to a basis of the middle factor by adding the single vector $(0, 0, 0, R)$.

Lemma 2.5.12. *Let $|\Omega|^{\frac{1}{2}}$ be the symplectic half density on $T_q(T^*\mathbb{R} \times T^*X \times T^*X)$ and let $\mathcal{B}$ be the basis*

$$\begin{aligned} \mathcal{B} = \{&(0, dp(v_i), v_i, P_\gamma v_i), (0, 0, -w_i, -w_i) \\ &(-1, 0, 0, 0), (0, -1, 0, 0), (1, 0, 0, H_p), (0, 0, 0, R)\}. \end{aligned} \tag{2.5.32}$$

Then

$$|\Omega|^{\frac{1}{2}}(\mathcal{B}) = |\det I - P^{\#}|^{\frac{1}{2}} \tag{2.5.33}$$

Proof. By applying a matrix with determinant ± 1, we can assume that $\mathcal{B}$ is the basis

$$
\begin{aligned}
&(0,0,R,0),(0,0,0,R),(0,0,0,H_p),(0,0,H_p,0)\\
&(1,0,0,0),(0,1,0,0),(0,0,0,e_i,e_i),(0,0,f_i,f_i) \qquad (2.5.34)\\
&(0,0,e_i,P_\gamma e_i),(0,0,f_i,P_\gamma f_i)
\end{aligned}
$$

The $2n-2 \times 2n-2$ block matrix

$$M = \begin{pmatrix} I & I \\ I & P \end{pmatrix}$$

takes the product basis $(e_i,0),(f_i,0),(0,e_i),(0,f_i)$ into (e_i,e_i),(f_i,f_i),(e_i,Pe_i),(f_i,Pf_i). Therefore the matrix

$$A = \begin{pmatrix} I_{6\times 6} & 0 \\ 0 & M \end{pmatrix}$$

Takes obvious symplectic basis of $T_q(T^*\mathbb{R} \times T^*X \times T^*X)$ into (2.5.34), where the upper left block preserves the subspace spanned by $(0,0,R,0)$,$(0,0,0,R)$, $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,H_p,0)$, and $(0,0,0,H_p)$. But the determinant of A is $\det M$, which is equal to $\det I - P$ by subtracting the last $2n-2$ rows from the first.

$\square$

Taking the reciprocal of the half density on $\operatorname{coker} \tau_{(p,q)}$ that is the quotient of the symplectic half density on the middle factor by ν, we get the $-1/2$ density on $\operatorname{coker} \tau$ which assigns the value $|\det I - P^{\#}|^{-\frac{1}{2}}$ to the residue class of $(0,0,0,R)$. This is symplectic dual to the basis $(0,0,H_p,H_p)$ of $\ker d(\pi_\alpha)_{(p,q)} \cong T_{(x,\xi)}\gamma$, so finally identify the half density ν on $\operatorname{im}\tau_{(p,q)}$ with $|\det I - P|^{-\frac{1}{2}}|dt|^{\frac{1}{2}}$. Therefore

$$\sigma_1 \otimes \sigma_2 = |dt|^{\frac{1}{2}} \otimes |d\tau|^{\frac{1}{2}} \otimes |\det I - P^{\#}|^{-\frac{1}{2}}|dt|^{\frac{1}{2}} = |\det I - P^{\#}|^{-\frac{1}{2}}|dt| \otimes |d\tau|^{\frac{1}{2}}$$

The principal symbol at $\tau = 1$ is the integral of this density over the fiber $\alpha^{-1}(0,1) = \gamma \subset S^*M$ times $|d\tau|^{\frac{1}{2}}$, which gives the stated result.

$\square$

The same calculation works when the fixed point set consists of finitely many distinct closed geodesics. The calculation of the symbol $\sigma(u_T)$ at $(T,1)$ is the same except we get a sum of terms each corresponding to each geodesic. As we have mentioned, because we are ignoring Maslov factors, we would miss the important feature that the terms in this sum are weighted by unit modulus complex numbers, and therefore the individual contributions to the trace may cancel. This is one of the main obstacles to extracting geometric information from the trace.

CHAPTER 3

Convex Surfaces of Revolution

3.1. Introduction

In this section we study the behavior of quantum integrable eigenfunctions on a convex surface of revolution (S^2, g). We begin by reviewing the integrability of the geodesic flow, the moment map, and the quantum toric integrability of the Laplacian on such surfaces. Colin de Verdiére [**4**] has shown that there exists a first order pseudo-differential operator $\widehat{I}_2$ which commutes with Δ and the generator of the S^1 symmetry, $D_\theta = -i\partial_\theta$, such that the joint spectrum of $\widehat{I}_2$ and D_θ consists of a lattice of simple eigenvalues,

$$\operatorname{Spec}(\widehat{I}_2, D_\theta) = \{(\ell, m) \in \mathbb{Z}^2 \mid \ell \geq 0; |m| \leq \ell\}. \tag{3.1.1}$$

The operator $\widehat{I}_2$ is analogous to the degree operator A on the round sphere (S^2, g_{can}),

$$A = \sqrt{-\Delta_{g_{\text{can}}} + \frac{1}{4}} - \frac{1}{2}. \tag{3.1.2}$$

In section 3.2 we prove theorem (1.1.1) which states that there is a unitary, homogeneous Fourier integral operator W which conjugates the pair $(\widehat{I}_2, D_\theta)$ to the standard pair (A, D_θ) up to finite rank error. The basis for the argument is the fact that up to homogeneous symplectomorphism, there is only one homogeneous Hamiltonian action of the torus T^2 on $T^*S^2 \setminus 0$, proved by Lerman [**25**]. In particular if $I_2 = \sigma(\widehat{I}_2)$ and $p_\theta = \sigma(D_\theta)$ are the principal symbols of the action operators, then both of the pairs (I_2, p_θ) and $(|\xi|_{g_{\text{can}}}, p_\theta)$ generate such an action. Therefore there is a homogeneous symplectomorphism pulling back the symbol I_2 to $|\xi|_{g_{\text{can}}}$ and which fixes p_θ. We then quantize this symplectic map into a unitary Fourier integral operator and adapt the averaging argument first due to Weinstein **[36]** and later refined by Guillemin **[16]** to make sure the conjugation commutes with D_θ. From there we move on to studying the concentration of the joint eigenfunctions φ_m^ℓ of $(\widehat{I_2}, D_\theta)$ on the unique rotationally invariant closed geodesic H in section 3.3. We do this by calculating the weak limit of the empirical measures

$$\mu_\ell = \frac{1}{M_\ell} \sum_{m=-\ell}^{\ell} ||\varphi_m^\ell||^2_{L^2(H)} \delta_{\frac{m}{\ell}} \tag{3.1.3}$$

as well as a phase space version of these measures,

$$\nu_\ell(B) = \frac{1}{N_\ell(B)} \sum_{m=-\ell}^{\ell} \langle B\varphi_m^\ell, \varphi_m^\ell \rangle_{L^2(S^2)} \delta_{\frac{m}{\ell}}, \tag{3.1.4}$$

where $B \in \Psi^0$ is a homogeneous pseudo-differential operator of order zero. The idea is to compute the weak limits of both sequences of measures by expressing their un-normalized versions as the trace of a semi-classical Fourier integral operator. The leading order term is then computed using the symbol calculus.

3.1.1. Background

Let (S^2, g) be a surface of revolution. We denote the two fixed points of the S^1 action by N and S. Fix a meridian geodesic γ_0 which joins N to S and let (r, θ) denote geodesic polar coordinates from N, i.e. so that the curve $r \mapsto (r, 0)$ is the arc length parametrized geodesic γ_0. In these coordinates the metric takes the form

$$g = dr^2 + a(r)^2 d\theta^2$$

for some smooth function $a : [0, L] \to \mathbb{R}$ such that $a^{2k}(0) = a^{2k}(L) = 0$ and $a'(0) = 1$, $a'(L) = 1$. Here L is the distance between the poles. A convex surface of revolution is one such that $a(r)$ has exactly one non-degenerate critical point which is a maximum, $a''(r_0) < 0$. The latitude circle $H = \{(r = r_0)\}$ is the unique rotationally invariant geodesic.

Recall that we say the Laplacian $-\Delta_g$ of a Riemannian manifold (M^n, g) is quantum completely integrable if there exists n first order homogeneous pseudo-differential operators $P_1, \ldots, P_n \in \Psi^1(M)$ satisfying:

- $[P_i, P_j = 0]$
- $\sqrt{-\Delta_g} = K(P_1, \ldots, P_n)$ for some polyhomogeneous function $K \in C^\infty(\mathbb{R}^n \setminus 0)$
- If $p_j = \sigma(P_j)$ are the principal symbols, the regular values of the associated moment map $\mathcal{P} = (p_1, \ldots, p_n) : T^*M \setminus 0 \to \mathbb{R}^n \setminus 0$ form an open, dense subset of T^*M.

For background on quantum integrable Laplacians, see **[12]**,**[34]**, chapter 11 of **[37]**. If (S^2, g) any surface of revolution, and $D_\theta = \frac{1}{i}\partial_\theta$ is the self-adjoint differential operator associated to the generator of the S^1 action, it is clear by writing Δ_g in polar coordinates that $[\Delta_g, D_\theta] = 0$. Hence every surface of revolution is quantum completely integrable by taking $P_1 = \sqrt{-\Delta_g}$ and $P_2 = D_\theta$. The third condition is satisfied, for instance, if $a(r)$ is assumed to be Morse. In the special case of a convex surface of revolution, Colin de Verdière in **[4]** has shown that the Laplacian is quantum toric completely integrable. This means that there exists $\widehat{I}_1, \widehat{I}_2$ first order, homogeneous, commuting pseudo-differential operators satisfying the above conditions of quantum complete integrability, but with the additional property that

$$\exp 2\pi i \widehat{I}_j = \mathrm{Id} \tag{3.1.5}$$

In particular, one can take $\widehat{I}_1 = D_\theta$ and $\widehat{I}_2$ to be self-adjoint and elliptic. Note that condition (3.1.5) implies that the joint spectrum of $\widehat{I}_1, \widehat{I}_2$ is a subset of $\mathbb{Z}^2$. In fact it is shown in [**4**] that it consists of all simple eigenvalues and

$$\mathrm{Spec}(\widehat{I}_1, \widehat{I}_2) = \{(m, \ell) \in \mathbb{Z}^2 \mid |m| \leq \ell; \ell > 0\}. \tag{3.1.6}$$

We fix a particular orthonormal basis of joint eigenfunctions $\{\varphi_m^\ell\}$ satisfying $\widehat{I}_2 \varphi_m^\ell = \ell \varphi_m^\ell$ and $D_\theta \varphi_m^\ell = m \varphi_m^\ell$.

3.1.2. The moment map and classical toric integrability

Let $I_j = \sigma(\widehat{I}_j)$ be the principal symbols. The associated moment map $\mathcal{P} = (I_1, I_2)$: $T^*S^2 \setminus 0 \to \mathbb{R}^2 \setminus 0$ has image equal to the closed conic wedge

$$\mathcal{B} = \{(x, y) \mid |x| \leq y; y > 0\}.$$

The set of critical points, Z, of $\mathcal{P}$ consists of covectors lying tangent to the equator. If (ρ, η) are the dual coordinates to (r, θ) on the fibers of T^*S^2,

$$Z = \{(r_0, \theta, 0, \eta) \mid \eta \neq 0\} = T^*H \setminus 0$$

$\mathcal{P}$ maps Z to the boundary $\partial\mathcal{B}$, so the interior of $\mathcal{B}$ consists entirely of regular values. Consider a regular level set of the form $T_c = \mathcal{P}^{-1}(1,c)$, for $c \in (-1,1)$. By homogeneity, all other regular levels are dialates of these. For each c, T_c is connected and diffeomorphic to a torus $T^2 \cong \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$. The singular levels correspond to $c = \pm 1$ and are equal to the set of covectors $T_{\pm 1} = \{(r_0, \theta, 0, \pm 1)\}$. One consequence of quantum toric integrability is of course classical toric integrability. That is, letting H_{I_j} denote the hamilton vector fields of I_j, equation (3.1.5) implies that both H_{I_j} generate 2π-periodic flows. Since $\{I_1, I_2\} = 0$, we let for $\mathbf{t} = (t_1, t_2) \in T^2$,

$$\Phi_{\mathbf{t}} : T^2 \times T^*S^2 \setminus 0 \to T^*S^2 \setminus 0 \tag{3.1.7}$$

$$\Phi_{\mathbf{t}}(x,\xi) = \exp t_1 H_{I_1} \circ \exp t_2 H_{I_2}(x,\xi)$$

The joint flow $\Phi_{\mathbf{t}}$ thus defines a homogeneous, Hamiltonian action of T^2 on $T^*S^2 \setminus 0$ which commutes with the geodesic flow $G^t = \exp tH_{|\xi|_g}$. It preserves the level sets of the moment map and each torus T_c consists of a single orbit of the joint flow.

3.1.3. The standard torus action on T^*S^2

In **[25]**, Lerman shows that up to symplectic equivalence, there is only one homogeneous Hamiltonian action of T^2 on $T^*S^2 \setminus 0$. The simplest example of a convex surface of revolution is the standard sphere (S^2, g_{can}). For the standard sphere we can take $\widehat{I}_2 = A = \sqrt{-\Delta_{g_{can}} + \frac{1}{4}} - \frac{1}{2}$, the so-called degree operator. We refer to the associated torus action on T^*S^2 is generated by $p_2(x,\xi) = |\xi|_{g_{can}}$ and p_θ the standard torus action on T^*S^2. If $I_1 = p_\theta$ and I_2 are the action variables associated to a convex surface of revolution, there is a homogeneous symplectomorphism

$$\chi : T^*S^2 \setminus 0 \to T^*S^2 \setminus 0$$

such that $\chi^* p_\theta = p_\theta$ and $\chi^* I_2 = |\xi|_{g_{can}}$. Theorem (1.1.1) is the statement that the symplectic equivalence of the torus action on a convex surface of revolution to that of the standard action can be quantized. That is, the generators of the standard unitary torus action D_θ and A on the round sphere are unitarily conjugate via a homogeneous Fourier integral operator to the quantized action operators $\widehat{I_j}$ on any convex surface of revolution.

3.1.4. The quantum torus action

In this section we briefly review the fact that the commuting operators $\widehat{I_1} = D_\theta$ and $\widehat{I_2}$ on a convex surface of revolution (S^2, g) together generate an action of T^2 on $L^2(S^2, dV_g)$ by unitary Fourier integral operators. (See for instance p. 245 of [**37**]). For $\mathbf{t} = (t_1, t_2) \in T^2$ we set

$$U(\mathbf{t}) = \exp i[t_1 D_\theta + t_2 \widehat{I_2}] \tag{3.1.8}$$

Proposition 3.1.1. *The operator $U(t_1, t_2)$ is a homogeneous Fourier integral operator belonging to the class $I^{-\frac{1}{2}}(T^2 \times S^2 \times S^2; C_U)$. Its canonical relation is given by the space-time graph of the joint flow*

$$C_U = \{(t_1, p_\theta(x,\xi), t_2, I_2(x,\xi), y, \eta, x, \xi) \mid (y,\eta) = \Phi_{(t_1,t_2)}(x,\xi)\,;\ (x,\xi) \in T^*S^2 \setminus 0\}$$

The half density part of the symbol $\sigma(U)$ pulls back along the parametrizing map

$$\iota : (t_1, t_2, x, \xi) \mapsto (t_1, p_\theta(x, \xi), t_2, I_2(x, \xi), \Phi_{(t_1,t_2)}(x, \xi), x, \xi)$$

to the half density $|dt_1 \wedge dt_2|^{\frac{1}{2}} \otimes |dx \wedge d\xi|^{\frac{1}{2}}$ *on* $T^2 \times T^*S^2$.

Proof. Since $\exp it_1 D_\theta$ just acts by pulling back a function along the flow of the vector field ∂_θ, one can check in coordinates that this is a Fourier integral operator in the class $I^{-\frac{1}{4}}(S^1 \times S^2, S^2; C)$ where

$$C = \{t_1, p_\theta(x, \xi), y, \eta, x, \xi) \mid (y, \eta) = \exp t_1 H_{p_\theta}(x, \xi); (x, \xi) \in T^*S^2 \setminus 0\}$$

The half density symbol pulls back along the parametrizing map

$$\iota : (t_1, x, \xi) \mapsto (t_1, p_\theta(x, \xi), \exp t_1 H_{p_\theta}(x, \xi), x, \xi)$$

to $|dt_1|^{\frac{1}{2}} \otimes |dx \wedge d\xi|^{\frac{1}{2}}$. Now I_2 is a first order, self-adjoint, elliptic pseudo-differential operator with integer spectrum, so by **[8]** we have that $\exp it_2 \widehat{I}_2 \in I^{-\frac{1}{4}}(S^1 \times S^2 \times S^2; C')$ where

$$C' = \{t_2, I_2(x, \xi), y, \eta, x, \xi) \mid (y, \eta) = \exp t_2 H_{I_2}(x, \xi); (x, \xi) \in T^*S^2 \setminus 0\}$$

Now the composition of C with C' is transverse since they are essentially canonical graphs. By standard transverse composition of FIOs the orders add and we get the description of $U(\mathbf{t})$ stated in the proposition.

$\square$

3.2. Conjugation to the global normal form

This section contains the proof of

Theorem 3.2.1. *Let (S^2, g) be a convex surface of revolution and $A = \sqrt{-\Delta_{g_{can}} + \frac{1}{4}} - \frac{1}{2}$ be the degree operator on the round sphere. There exists a homogeneous unitary Fourier integral operator*

$$W : L^2(S^2, g_{can}) \to L^2(S^2, g)$$

such that $[W, D_\theta] = 0$ and $W^* \widehat{I_2} W = A + R$ where R is a finite rank operator. Consequently, if Y_m^ℓ denotes the standard orthonormal basis of $L^2(S^2, g_{can})$ such that $AY_m^\ell = \ell Y_m^\ell$, $D_\theta Y_m^\ell = mY_m^\ell$, then for ℓ large enough, there are constants c_m^ℓ with $|c_m^\ell| = 1$ so that

$$WY_m^\ell = c_m^\ell \varphi_m^\ell \tag{3.2.1}$$

The outline of the argument goes as follows. First, using the canonical transformation $\chi : T^*S^2 \setminus 0 \to T^*S^2 \setminus 0$ of section 3.1.3 which satisfies $\chi^* I_2 = |\xi|_{g_{can}}$, $\chi^* p_\theta = p_\theta$, we can find a unitary Fourier integral operator W_0 so that $[W_0, D_\theta] = 0$ and

$$W_0 \hat{I}_2 W_0^* = A + R_{-1} \tag{3.2.2}$$

where R_{-1} is a pseudo-differential operator of order -1. We then use the averaging argument of Guillemin (See [**16**]) to show that there exists a unitary pseudo-differential operator F of order zero such that

$$F(A + R_{-1})F^* = A + R_{-1}^{\#} \tag{3.2.3}$$

where $[A, R_{-1}^{\#}] = 0$ and $[F, D_\theta] = 0$. This is contained in propositions 3.2.2, 3.2.3, and 3.2.4. Then $W = FW_0$ is a unitary Fourier integral operator which commutes with D_θ

and conjugates $\hat{I}_2$ to $A + R^{\#}_{-1}$, where $R^{\#}_{-1}$ is an order -1 pseudo-differential operator commuting with A. Using the fact that $A + R^{\#}_{-1}$ and A have the same spectrum, we easily see that $R^{\#}_{-1}$ is a finite rank operator.

Proposition 3.2.2. *There exists a unitary Fourier integral operator W_0 such that $W_0 \hat{I}_2 W_0^* = A + R_{-1}$ where $R_{-1} \in \Psi^{-1}$ is self-adjoint and $[W_0, D_\theta] = 0$. In this case we also have $[R_{-1}, D_\theta] = 0$*

Proof. Let U_0 be any unitary Fourier integral operator whose canonical relation is the graph of χ. Then by Egorov's theorem,

$$U_0 \hat{I}_2 U_0^* = A + R \tag{3.2.4}$$

Where $R \in \Psi^0$. Both the left hand side and A are self-adjoint, so R is as well. The subprincipal symbols of both the left hand side and A vanish which implies that $\sigma(R) = 0$ so $R \in \Psi^{-1}$. We write R_{-1} from now on to emphasize this. The only thing left to do is to show that we can modify U_0 in order to make it commute with D_θ. We let $V(t) = \exp itD_\theta$ and set

$$W_0' = \frac{1}{2\pi} \int_0^{2\pi} V(t) U_0 V(-t)\, dt \tag{3.2.5}$$

W_0' is a Fourier integral operator with the same canonical relation as U_0, although it may not be unitary. To fix this, replace W_0' with $W_0 = [W_0'(W_0')^*]^{-\frac{1}{2}} W_0'$. Then $W_0 W_0^* = I$ and W is still a Fourier integral operator associated to the same canonical graph since $W_0'(W_0')^*$ is pseudo-differential. W_0' commutes with D_θ so W_0 does as well. Note that if one replaces U_0 by W_0, (3.2.4) is still valid since both operators are associated to the graph of χ. Since $\widehat{I}_2$ and A commute with D_θ, we automatically have that R_{-1} does as well.

□

The following two propositions constitute a slight modifcation of what Guillemin refers to as the averaging lemma, found in [**16**]. The goal of the modification is to make sure the conjugations commute with D_θ.

Proposition 3.2.3. *Let R_{-1} be as in proposition 3.2.2. Then there exists a unitary pseudo-differential operator $F \in \Psi^0$, a self-adjoint operator $R^{\#}_{-1} \in \Psi^{-1}$ which commutes with A and a smoothing operator $R_{-\infty}$ such that $F(A + R_{-1})F^* = A + R^{\#}_{-1} + R_{-\infty}$ and* $[F, D_\theta] = 0$

Proof. We let $U(t) = \exp(itA)$ be the unitary group generated by A and for a pseudo-differential operator B, define as before, its average with respect to $U(t)$ by

$$B_{av} = \frac{1}{2\pi} \int_0^{2\pi} U(t)BU(-t)\, dt \tag{3.2.6}$$

Then B_{av} commutes with A and is self-adjoint if B is. We recall the statement of lemma 2.1 in [**16**]: If R is any self-adjoint operator of order $-k$, $k \in \mathbb{N}$, there exists a skew-adjoint pseudo-differential operator S of order $-k$ so that $[A, S] = R - R_{av} + \Psi^{-k-1}$. This statement is equivalent to the vanishing of the principal symbol of $[A, S] - (R - R_{av})$ which is a first order transport equation for $\sigma(S)$. This can be solved for $\sigma(S)$ explicitly on S^*S^2 , which can be extended as a degree $-k$ homogeneous function to $T^*S^2 \setminus 0$. Since it is imaginary, we can choose S to be skew-adjoint. Given such an S, set $\bar{S} = (2\pi)^{-1} \int_0^{2\pi} V(t)SV(-t)\, dt$. Then $\bar{S}$ is still skew-adjoint and commutes with D_θ. If we further suppose that R commutes with D_θ then

$$[A, \bar{S}] = \frac{1}{2\pi} \int_0^{2\pi} V(t)[A, S]V(-t)\, dt \tag{3.2.7}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} V(t)(R - R_{av})V(-t)\, dt + \Psi^{-k-1} \tag{3.2.8}$$

$$= R - R_{av} + \Psi^{-k-1} \tag{3.2.9}$$

Hence we may assume from the outset that $[S, D_\theta] = 0$. This fact allows us to build the operator F in stages. If R_{-1} is the operator in proposition 3.2.2, then using the above procedure we can choose $S_{-1} \in \Psi^{-1}$ skew-adjoint such that

$$[A, S_{-1}] = R_{-1} - (R_{-1})_{av} + R_{-2} \tag{3.2.10}$$

where $R_{-2} \in \Psi^{-2}$ and so that $[S_{-1}, D_\theta] = 0$. Then setting $F_1 = \exp S_{-1}$, a direct calculation shows that

$$F_1(A + R_{-1})F_1^* = A + (R_{-1})_{av} + R_{-2} \tag{3.2.11}$$

By construction, F_1 is unitary and commutes with D_θ. We can now choose S_{-2} skew-adjoint commuting with D_θ such that

$$[A, S_{-2}] = R_{-2} - (R_{-2})_{av} + R_{-3} \tag{3.2.12}$$

Then, with $F_2 = \exp S_{-2} \exp S_{-1}$ we have

$$F_2(A+R_{-1}) = A + (R_{-1})_{av} + (R_{-2})_{av} + R_{-3} \tag{3.2.13}$$

Continuing in this way, we get a sequence of unitary operators

$$F_k = \exp S_{-k} \cdots \exp S_{-1}$$

so that F_k commutes with D_θ and

$$F_k(A+R_{-1})F_k^* = A + (R_{-1})_{av} + \cdots + (R_{-k})_{av} + R_{-k-1} \tag{3.2.14}$$

We also note that $F_{k+1} - F_k \in \Psi^{-k-1}$. Let $F' \sim \sum_{k=1}^{\infty}(F_{k+1} - F_k)$, $R \sim \sum_{k=1}^{\infty}(R_{-k})_{av}$, and $R^{\#}_{-1} = R_{av}$. Then we know that $R^{\#}_{-1} - R \in \Psi^{-\infty}$ and if we put $F = F' + F_1$ we have $F - F_k \in \Psi^{-k}$. It is then easy to check that

$$F(A+R_{-1})F^* - (A + R^{\#}_{-1}) \in \Psi^{-\infty} \tag{3.2.15}$$

Furthermore, since all of the F_k commute with D_θ, we can choose F so that it does as well. As in the proof of proposition 3.2.2, F may not be unitary. This is fixed in the same way, by replacing F with $(FF^*)^{-\frac{1}{2}}F$. More explicitly, let $G = FF^* - I$. Note that $F = F_k + \Psi^{-k}$ which implies that G is a smoothing operator. By the functional calculus, we can find a self-adjoint operator K so that $(I+K)^2 = (I+G)^{-1}$ and if we replace F by $(I+K)F$, then F is unitary, $[F, D_\theta] = 0$, and we still have $F - F_k \in \Psi^{-k}$ since K is a smoothing operator.

□

Proposition 3.2.4. *Suppose that* $R^{\#}_{-1}$ *and* $R_{-\infty} \in \Psi^{-\infty}$ *are as in proposition 3.2.3 and that* $\mathrm{Spec}(A + R^{\#}_{-1} + R_{-\infty}) = \mathrm{Spec}(A) = \mathbb{N}$. *Then there exists a unitary operator* L *and* $R^{\#} \in \Psi^{-1}$*, self-adjoint, such that* $[R^{\#}, A] = 0$ *and*

$$L(I + R + R_{-\infty})L^{*} = I + R^{\#} \tag{3.2.16}$$

Furthermore, $L - I$ *is a smoothing operator and* $[L, D_{\theta}] = 0$

Proof. Let V_k denote the k^{th} eigenspace of A and V'_k the k^{th} eigenspace of $A + R^{\#}_{-1} + R_{-\infty}$. Also let π_k and π'_k denote orthogonal projection onto these subspaces. Finally let $P_k = \pi'_k$ restricted to V'_k. First we show that there is a $C > 0$ so that for all $N \geq 0$ and k sufficiently large

$$||(A + R^{\#}_{-1})^{N}(P_k - \pi'_k)||_{L^2} \leq C||(A + R^{\#}_{-1})^{N} R_{-\infty}\pi'_k||_{L^2} \tag{3.2.17}$$

To do this, we note that the spectrum of $A + R^{\#}_{-1}$ consists of bands of the form $\lambda^{j}_{k} = k + \mu^{j}_{k}$ where $|\mu^{j}_{k}| = O(k^{-1})$. Hence for k sufficiently large, the entire band is contained in a ball of radius $\frac{1}{4}$ around k. Let γ_k be a circle of radius $\frac{1}{2}$ centered at $k \in \mathbb{N}$. Then for k sufficiently large,

$$\pi_k = \frac{1}{2\pi i}\int_{\gamma_k} (\lambda - (A + R^{\#}_{-1}))^{-1}\, d\lambda \tag{3.2.18}$$

and

$$\pi'_k = \frac{1}{2\pi i}\int_{\gamma_k} (\lambda - (A + R^{\#}_{-1} + R_{-\infty}))^{-1}\, d\lambda \tag{3.2.19}$$

Hence

(3.2.20)

$$(A+R^{\#}_{-1})^N(\pi_k\pi'_k - \pi'_k) = \frac{1}{2\pi i}\int_{\gamma_k} (\lambda - (A+R^{\#}_{-1}))^{-1}(A+R^{\#}_{-1})^N R_{-\infty}\pi'_k(\lambda - (A+R^{\#}_{-1}+R_{-\infty}))^{-1}\, d\lambda$$

For $\lambda \in \gamma_k$, the distance between λ and the spectrum of both $A+R^{\#}_{-1}$ and $A+R^{\#}_{-1}+R_{-\infty}$ is bounded below by $\frac{1}{4}$. Hence the norms of both resolvents are bounded by 4, which implies the norm of the left hand side is bounded by $2||(A + R^{\#}_{-1})^N R_{-\infty}\pi'_k||_{L^2}$. Now suppose that we choose $k \geq k_0$ so that the above estimate holds. Then, repeating the argument on p. 255 of [**16**] we build a sequence of unitary operators $L_k : V'_k \to V_k$. Since L_k is a function of P_k and A commutes with D_θ, each L_k does as well. Define the unitary operator L by declaring $L = L_k$ on V'_k for $k \geq k_0$ sufficiently large so that the above estimate holds. To define L on $\bigoplus_{1\leq k\leq k_0} V'_k$, let U_k denote the eigenspace of $\widehat{I_2}$ of eigenvalue k. and let φ^k_m be an orthonormal basis of U_k consisting of joint eigenfunctions of D_θ. Then $W\varphi^k_m$ is a basis of V'_k which are also joint eigenfunctions of D_θ. Define L by taking $W\varphi^k_m$ to the corresponding standard spherical harmonic of joint eigenvalue (k, m). L clearly commutes with D_θ as well as A. Also, by construction $L(A+R^{\#}_{-1}+R_{-\infty})L^* = A+L(R^{\#}_{-1}+R_{-\infty})L^*$ preserves each V_k eigenspace, so commutes with A. This implies that $L(R^{\#}_{-1}+R_{-\infty})L^* = R^{\#}$ commutes with A. Finally the estimate above is used to prove that $L - I$ is a smoothing operator in the same way as in [**16**].

□

Proposition 3.2.5. *Suppose that Spec*$(A+R^{\#}_{-1}) =$ *Spec*$(A) = \mathbb{N}$ *where* $R^{\#}_{-1} \in \Psi^{-1}$ *is self-adjoint and commutes with* A*. Then* $R^{\#}_{-1}$ *is a finite rank operator.*

Proof. Since $R^{\#}$ commutes with A, we can choose an orthonormal basis of V_k, e^k_j satisfying $R^{\#}e^k_j = \mu^k_j e_j$. Since $R^{\#} \in \Psi^{-1}$, we have $|\mu^k_j| = O(k^{-1})$. The fact that $\mathrm{Spec}(A+R^{\#}) = \mathbb{N}$ implies that for k large, $R^{\#}|_{V_k} = 0$ which shows that $R^{\#}$ is finite rank.

□

3.3. Concentration of quantum integrable eigenfunctions on the equator

This section contains the proof of:

Theorem 3.3.1. *Let* (S^2, g) *be a convex surface of revolution where* $g = dr^2 + a(r)^2 d\theta^2$ *in geodesic polar coordinates. Let* $H \subset S^2$ *be the equator, the unique rotationally invariant geodesic. Then in terms of action angle variables we have,*

(a) For every $f \in C^0([-1,1])$,

$$\int_{-1}^{1} f(c)\, d\mu_\ell(c) = \frac{1}{M_\ell}\sum_{m=-\ell}^{\ell} ||\varphi^\ell_m||^2_{L^2(H)} f\left(\frac{m}{\ell}\right) \to \frac{1}{M}\int_{-1}^{1} f(c)\frac{\omega_2(c,1)}{\sqrt{1-\frac{(2\pi)^2c^2}{K(c,1)^2\mathcal{L}(H)^2}}}\, dc$$

(b) For any $f \in C^0([-1,1])$*, any any pseudo-differential operator* $B \in \Psi^0$,

$$\int_{-1}^{1} f(c)\, d\nu_\ell(c) = \frac{1}{N_\ell(B)}\sum_{m=-\ell}^{\ell} \langle B\varphi^\ell_m, \varphi^\ell_m\rangle_{L^2(S^2,g)} f\left(\frac{m}{\ell}\right) \to \frac{1}{\omega(B)}\int_{-1}^{1} f(c)\widehat{\sigma(B)}(c)\, dc$$

The constant appearing in (a) is

$$M = \int_{-1}^{1} \frac{\omega_2(c,1)}{\sqrt{1 - \frac{(2\pi)^2 c^2}{K(c,1)^2 \mathcal{L}(H)}}}\, dc$$

and normalizes the limit measure to have mass 1 on $[-1,1]$.

Let $\Pi_\ell : L^2(S^2, dV_g) \to L^2(S^2, dV_g)$ denote the orthogonal projection onto the $\widehat{I}_2 = \ell$ eigenspace. Suppose that $A : C^\infty(S^2) \to C^\infty(S^2)$ is an operator which commutes with D_θ. Then the kernel of the operator $f\left(\frac{D_\theta}{\ell}\right) A\Pi_\ell$ is equal to

$$\sum_{m=-\ell}^{\ell} A\varphi_m^\ell(x)\overline{\varphi_m^\ell(y)} f\left(\frac{m}{\ell}\right), \tag{3.3.1}$$

and therefore

$$\mathrm{Trace}\, f\left(\frac{D_\theta}{\ell}\right) A\Pi_\ell = \sum_{m=-\ell}^{\ell} \langle A\varphi_m^\ell, \varphi_m^\ell\rangle f\left(\frac{m}{\ell}\right). \tag{3.3.2}$$

When A is a pseudo-differential operator, this forumula returns the unnormalized measures (3.1.4) tested against f. To use this formula for the measures (3.1.3), we express the L^2 norms on $H \subset S^2$ as a global matrix element as follows: let $\gamma_H : C^\infty(S^2) \to C^\infty(H)$ denote restriction to H and γ_H^* denote the L^2 adjoint of γ_H with respect to the Riemannian volume measure dV_g. Thus, for $g \in C^\infty(H)$, $f \in C^\infty(S^2)$ we have

$$\langle \gamma_H^* g, f\rangle_{L^2(S^2, dV_g)} = \int_H g f|_H \, dS$$

where dS is the induced surface measure. From this it follows that

$$||\varphi_m^\ell||^2_{L^2(H,dS)} = \langle \gamma_H^* \gamma_H \varphi_m^\ell, \varphi_m^\ell\rangle$$

One problem with this setup is that (3.3.2) requires the operator A to commute with D_θ, and this will not be true for every pseudo $B \in \Psi^0(S^2)$ nor for the operator $\gamma_H^* \gamma_H$. We deal with this by averaging against the torus action generated by D_θ and $\widehat{I}_2$. For $\mathbf{t} = (t_1, t_2) \in T^2$, let

$$U(\mathbf{t}) = \exp i[t_1 D_\theta + t_2 \widehat{I}_2]. \tag{3.3.3}$$

In section 3.1.4 we review that this is a torus action on $L^2(S^2, dV_g)$ by unitary Fourier integral operators. For any operator $A : C^\infty(S^2) \to C^\infty(S^2)$ we set

$$\bar{A} = (2\pi)^{-2} \int_{T^2} U(\mathbf{t})^* A U(\mathbf{t}) \, d\mathbf{t} \tag{3.3.4}$$

The average $\bar{A}$ commutes with both D_θ and $\widehat{I}_2$ since

$$[D_\theta, \bar{A}] = (2\pi)^{-2} \int_{T^2} -\partial_{t_1} [U(\mathbf{t})^* A U(\mathbf{t})] d\mathbf{t} = 0 \tag{3.3.5}$$

And similarly for $\widehat{I}_2$. We also note that

$$\langle A\varphi_m^\ell, \varphi_m^\ell \rangle_{L^2(S^2)} = \langle \bar{A}\varphi_m^\ell, \varphi_m^\ell \rangle_{L^2(S^2)}$$

This means replacing A with $\bar{A}$ in the trace will not change the right hand side of (3.3.2). When $A \in \Psi^0(S^2)$, Egorov's theorem tells us that $\bar{A} \in \Psi^0(S^2)$ as well, and

$$\sigma(\bar{A}) = (2\pi)^{-2} \int_{T^2} \Phi_{\mathbf{t}}^* \sigma(A) \, d\mathbf{t}$$

where $\Phi_{\mathbf{t}}$ is the joint flow generated by $I_1 = p_\theta$ and I_2. In section 3.3.1, we analyze the averaged restriction operator

$$\bar{V} = (2\pi)^{-2} \int_{T^2} U^*(\mathbf{t})(\gamma_H^* \gamma_H) U(\mathbf{t})\, d\mathbf{t} \tag{3.3.6}$$

And show that, after applying microlocal cutoffs to $\gamma_H^* \gamma_H$, it splits into the sum of a pseudo-differential operator and a Fourier integral operator. The canonical relation of the non-pseudo-differential part of $\bar{V}$ is related to the notion of the 'mirror reflection map' on co-vectors based on H. Both summands can be made to commute with $U(\mathbf{t})$. The details of the analysis of averaged, cutoff restriction operator are contained in section 3.3.1. The strategy of using the operator $\bar{V}$ to study restricted L^2 norms (and more generally restricted ΨDO matrix elements) has been used by Toth and Zelditch [**35**] and we closely follow their analysis here. As mentioned, for this analysis to work we need to microlocally cut off $\gamma_H^* \gamma_H$ away from both N^*H and T^*H. Literally speaking we fix $\varepsilon > 0$ and instead work with the operator

$$(\gamma_H^* \gamma_H)_{\geq \varepsilon} = (1 - \widehat{\chi}_{\varepsilon/2})(\gamma_H^* \gamma_H)(1 - \widehat{\chi}_\varepsilon) \tag{3.3.7}$$

Where $(I - \widehat{\chi}_\varepsilon)$ is a homogeneous pseudo-differential operator with wave front set outside conic neighborhoods of both N^*H and T^*H. The cutoff away from the normal directions is technical and related to the choice to use the homogeneous calculus, while the cutoff away from the tangential directions is necessary since otherwise the canonical relation of $\bar{V}$ would be singular.

3.3.1. The averaged restriction operator

Let $T^*_H S^2 = \{(x, \xi) \in T^*S^2 \mid x \in H\}$ denote the set of covectors with footprint on H. Since γ_H is just pullback along the inclusion map, it is a Fourier integral operator associated with the pullback canonical relation,

$$C = \{(x, \xi|_{TH}, x, \xi) \mid (x, \xi) \in T^*_H S^2 \setminus 0\} \subset T^*H \times T^*S^2.$$

The left factor contains elements of the zero section whenever $\xi \in N^*H$, so it is not a homogeneous Fourier integral operator in the sense of [**23**]. Because of this defect, the wave front set of $\gamma_H^* \gamma_H$ is

$$WF'(\gamma_H^* \gamma_H) = C_H \cup N^*H \times 0_{T^*M} \cup 0_{T^*M} \times N^*H, \tag{3.3.8}$$

where $C_H \subset T^*M \setminus 0 \times T^*M \setminus 0$ is the homogeneous canonical relation

$$C_H = \{(x, \xi, x, \xi') \mid (x, \xi), (x, \xi') \in T^*_H S^2 \setminus 0; \xi|_{T_x H} = \xi'|_{T_x H}\}.$$

Note that since ∂_θ is tangent to H, $(x, \xi)|_{TH} = (x, \xi')|_{TH}$ is equivalent to $I_1(x, \xi) = I_1(x, \xi')$. In order to get rid of the last two components of wave front set, we insert microlocal cutoff operators as in [**35**]. In this setting we can take them to be functions of the action operators $\widehat{I}_j$. Let ϕ_ε and ψ_ε be smooth cutoff functions on $\mathbb{R}$ such that

$$\phi_\varepsilon(x) = \begin{cases} 1 \text{ for } |x| \le \varepsilon/2 \\ 0 \text{ for } |x| > \varepsilon \end{cases} \tag{3.3.9}$$

$$\psi_\varepsilon(x) = \begin{cases} 1 \text{ for } |x| > 1 - \varepsilon/2 \\ 0 \text{ for } |x| < 1 - \varepsilon \end{cases} \tag{3.3.10}$$

Then we set $\widehat{\chi}_\varepsilon^n = \phi_\varepsilon(\frac{\widehat{I}_1}{\widehat{I}_2})$ and $\widehat{\chi}_\varepsilon^t = \psi_\varepsilon(\frac{\widehat{I}_1}{\widehat{I}_2})$. Finally set $\widehat{\chi}_\varepsilon = \widehat{\chi}_\varepsilon^n + \widehat{\chi}_\varepsilon^t$. Note that the operator $(I - \widehat{\chi}_\varepsilon)$ has no wave front set in a conic $\varepsilon/2$ neighborhood of both N^*H and T^*H. We now define

$$(\gamma_H^*\gamma_H)_{\geq\varepsilon} = (I - \hat{\chi}_{\varepsilon/2})\gamma_H^*\gamma_H(I - \hat{\chi}_\varepsilon) \tag{3.3.11}$$

$$(\gamma_H^*\gamma_H)_{\leq\varepsilon} = \hat{\chi}_{\varepsilon/2}\gamma_H^*\gamma_H\hat{\chi}_\varepsilon. \tag{3.3.12}$$

Proposition 3.3.2. *We have the decomposition*

$$\gamma_H^*\gamma_H = (\gamma_H^*\gamma_H)_{\geq\varepsilon} + (\gamma_H^*\gamma_H)_{\leq\varepsilon} + K_\varepsilon \tag{3.3.13}$$

where $\langle K_\varepsilon \varphi_{\lambda_j}, \varphi_{\lambda_j}\rangle_{L^2(S^2, dV_g)} = O_\varepsilon(\lambda_j^{-\infty})$ *and* φ_{λ_j} *are any orthonormal basis of eigenfunctions of* $-\Delta_g$.

For the proof of this, see section 9.1.1 in [**35**]. We also quote the following description of the cutoff restriction operator:

Proposition 3.3.3. *For each* $\varepsilon > 0$, $(\gamma_H^*\gamma_H)_{\geq\varepsilon}$ *is a Fourier integral operator in the class* $I^{\frac{1}{2}}(M, M; C_H)$ *where* C_H *is the homogeneous canonical relation*

$$C_H = \{(x,\xi,x,\xi') \in T_H^*S^2 \setminus 0 \times T_H^*S^2 \setminus 0 \mid I_1(x,\xi) = I_1(x,\xi')\} \tag{3.3.14}$$

In polar coordinates (r,θ,ρ,η) on T^*S^2, the set C_H is parametrized by the map

$$\iota_{C_H} : (\theta,\eta,\rho,\rho') \mapsto (r_0,\theta,\rho,\eta,r_0,\theta,\rho',\eta)$$

The half density part of the symbol of $(\gamma_H^*\gamma_H)_{\geq\varepsilon}$ pulls back under ι_{C_H} to the half density

$$(1-\chi_{\varepsilon/2})(r_0,\theta,\rho,\eta)(1-\chi_\varepsilon)(r_0,\theta,\rho',\eta)|d\theta\wedge d\eta\wedge d\rho\wedge d\rho'|^{\frac{1}{2}} \tag{3.3.15}$$

This follows from Lemma 18 in **[35]** setting $\mathrm{Op}_H(a) = \mathrm{Id}$, because the geodesic polar coordinates (r,θ) are Fermi normal coordinates along H.

3.3.1.1. The I_2 reflection map and the set $\widehat{C}_H$. Here we include more geometric preliminaries to the description of the averaged restriction operator

$$\bar{V}_\varepsilon = (2\pi)^{-2}\int_{T^2} U^*(\mathbf{t})(\gamma_H^*\gamma_H)_{\geq\varepsilon}U(\mathbf{t})\,d\mathbf{t}. \tag{3.3.16}$$

We begin by describing the so-called I_2 reflection map along H.

Proposition 3.3.4. *Suppose $(x,\xi) \in T_H^*S^2$. If $(x,\xi) \notin T^*H$, there are is exactly one covector $(x,\xi') \in T_H^*S^2$ such that $I_2(x,\xi) = I_2(x,\xi')$, $(x,\xi) \neq (x,\xi')$ and $\xi|_{TH} = \xi'|_{TH}$. We refer to the map*

$$r_H : (x,\xi) \mapsto (x,\xi')$$

As the I_2-reflection map.

Proof. We'll show that on the set $\{I_1 = c\}$, I_2 is an invertible function of the length $q(x,\xi) = |\xi|^2_{g(x)}$. Thus, if $I_2(x,\xi) = I_2(x,\xi')$ and $I_1(x,\xi) = I_1(x,\xi')$, then $|\xi|_{g(x)} = |\xi'|_{g(x)}$ and this means that $(x,\xi') = (r_0, \theta, \pm\sqrt{|\xi|^2_{g(x)} - c^2}, c)$ in polar coordinates. The reflection map then flips the sign of the component dual to r. From [**4**], we have the formula

$$I_2(x,\xi) = \int_{r_1}^{r_2} \sqrt{|\xi|^2_{g(x)} - \frac{p_\theta(x,\xi)^2}{a(r)^2}}\, dr + p_\theta \tag{3.3.17}$$

Where r_2 and r_1 are the two solutions of $a(r) = \frac{p_\theta(x,\xi)}{|\xi|_g}$. Now $r_1 = r_2$ if and only if $(x,\xi) \in T^*H$ thus we have that $r_1 \neq r_2$ and

$$\frac{\partial}{\partial|\xi|} I_2(x,\xi) = \int_{r_1}^{r_2} \frac{|\xi|_g}{\sqrt{|\xi|^2_{g(x)} - \frac{c^2}{a(r)^2}}}\, dr > 0$$

This shows that I_2 is an increasing function of $|\xi|_g$ on $\{I_1 = c\} \subset T^*S^2 \setminus 0$.

$\square$

We know that for each $\varepsilon > 0$, the operator $(\gamma_H^*\gamma_H)_{\geq\varepsilon}$ is a Fourier integral operator with canonical relation

$$C_H = \{(x,\xi,x,\xi') \mid (x,\xi),(x,\xi') \in T^*_H S^2; \xi|_{TH} = \xi'|_{TH}\}.$$

In the study of $\bar{V}_\varepsilon$, a related set appears. Define

$$\widehat{C}_H = \{(x,\xi,x,\xi') \mid x \in H; I_1(x,\xi) = I_1(x,\xi'); I_2(x,\xi) = I_2(x,\xi')\} \tag{3.3.18}$$

It is clear from proposition 3.3.4, $\widehat{C}_H$ has the following simple description

Proposition 3.3.5. *The set $\widehat{C}_H$ is an immersed submanifold of dimension 3 which can be written as the union of the two embedded submanifolds*

$$\widehat{C}_H = \Delta_{T_H^* S^2} \bigcup graph\, r_H|_{T_H^* S^2}$$

These intersect along the set Δ_{T^*H} where $\widehat{C}_H$ fails to be embedded.

3.3.1.2. Description of the averaged restriction operator $\bar{\mathbf{V}}_\varepsilon$. The purpose of this section is to describe the averaged restriction operator

$$\bar{V}_\varepsilon = (2\pi)^{-2} \int_{T^2} U(\mathbf{t})^* (\gamma_H^* \gamma_H)_{\geq \varepsilon} U(\mathbf{t})\, d\mathbf{t} \tag{3.3.19}$$

As a Fourier integral operator and calculate its symbolic data. In order to state the proposition, we set some notation. For any set $U \subset T^*S^2 \times T^*S^2$, we define its flow-out $\mathrm{Fl}(U)$ by

$$\mathrm{Fl}(U) = \bigcup_{\mathbf{t} \in T^2} \Phi_{\mathbf{t}} \times \Phi_{\mathbf{t}}(U) = \{(\Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}}(y,\eta)) \mid (x,\xi,y,\eta) \in U\}$$

In the calculation of the symbol of $\bar{V}_\varepsilon$, there are two important submersions. Define $i_D, i_R : T^2 \times T_H^* S^2 \to T^*S^2 \times T^*S^2$ by

$$i_D(\mathbf{t}, x, \xi) = (\Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}}(x,\xi)) \tag{3.3.20}$$

$$i_R(\mathbf{t}, x, \xi) = (\Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}}(r_H(x,\xi))) \tag{3.3.21}$$

The image of these maps are the diagonal and reflection flow-outs, $\mathrm{Fl}(\Delta_{T^*_H S^2})$, $\mathrm{Fl}(\mathrm{graph}\, r_H|_{T^*_H S^2})$

Proposition 3.3.6. *Both maps i_D and i_R are smooth submersions. Over any point $(y, \eta, y', \eta') \in T^*S^2 \times T^*S^2$ in the image of either map, the fiber can be identified with the set*

$$\{(x, \xi) \in T^*_H S^2 \mid \mathcal{P}(x, \xi) = \mathcal{P}(y, \eta)\} \tag{3.3.22}$$

For $(y, \eta) \notin T^*_H S^2$, the fiber is identified with two distinct copies of H corresponding to the choice of the northern or southern pointing covector lying on the torus $\mathcal{P}(y, \eta)$.

Proof. Fix a point (y, η, y, η) in the image of i_D. Then $\Phi_{\mathbf{t}}(x, \xi) = (y, \eta)$ for some $\mathbf{t} \in T^2$ and $(x, \xi) \in T^*_H S^2$. The covector (x, ξ) lies on the level set $\mathcal{P}^{-1}(y, \eta)$ and by proposition 3.3.4 there are two covectors in this set lying over x. Since the flow of H_{I_1} translates around the equator, for each covector (x, ξ) in the set (3.3.22), there is a unique time $\mathbf{t}$ so that $\Phi_{\mathbf{t}}(x, \xi) = (y, \eta)$. In this way the fiber is identified with two copies of H

$\square$

These maps induce half densities on the flow-outs $\mathrm{Fl}(\Delta_{T^*_H S^2})$ and $\mathrm{Fl}(\mathrm{graph}\, r_H|_{T^*_H S^2})$ as follows. We let $\mu^{\frac{1}{2}}$ be the half density on $T^2 \times T^*_H S^2$ which is equal to 1 on the product basis $\partial_{\mathbf{t}} \otimes \{\partial_\theta, \partial_\rho, \partial_\eta\}$. Then the exact sequence

$$0 \to \ker di_R \to T(T^2 \times T^*_H S^2) \to T(\mathrm{Fl}(\mathrm{graph}\, r_H|_{T^*_H S^2})) \to 0$$

implies that $\mu^{\frac{1}{2}} = |d\theta|^{\frac{1}{2}} \otimes \mu^{\frac{1}{2}}/|d\theta|^{\frac{1}{2}}$, where, under the identification of the fiber of i with two copies of H, $|d\theta|$ is the volume density such that $\int_H |d\theta| = 2\pi$ and the quotient half density $\mu^{\frac{1}{2}}/|d\theta|^{\frac{1}{2}}$ assigns the value 1 to the basis $(d\Phi_{\mathbf{t}} v_i, d\Phi_{\mathbf{t}} dr_H v_i)$ where $v_i \in$

$\{H_{I_2}, \partial_\theta, \partial_\rho, \partial_\eta\}$. The same is true for the flowout of the diagonal replacing i_R with i_D. In this case the quotient density $\mu^{\frac{1}{2}}$ assigns 1 to the basis $(d\Phi_{\mathbf{t}} v_i, d\Phi_{\mathbf{t}} v_i)$.

Proposition 3.3.7. *The operator*

$$\bar{V}_\varepsilon = (2\pi)^{-2} \int_{T^2} U^*(\mathbf{t})(\gamma_H^* \gamma_H)_{\geq \varepsilon} U(\mathbf{t})\, d\mathbf{t}$$

is a Fourier integral operator in the class $I^0(S^2 \times S^2; C_{\bar{V}})$. Its canonical relation is

$$C_{\bar{V}} = Fl(\widehat{C}_H) = Fl(\Delta_{T_H^* S^2}) \bigcup Fl(graph\, r_H|_{T_H^* S^2})$$

The half density symbol of $\bar{V}_\varepsilon$ is equal to

$$\sigma(\bar{V}_\varepsilon)(\Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}}(x,\xi')) = \frac{1}{\pi}(1-\chi_\varepsilon)(x,\xi) \left(\frac{\omega_2(x,\xi)}{\sqrt{1 - \frac{I_1^2(x,\xi)}{|\xi|_g^2 a(r_0)^2}}} \right)^{\frac{1}{2}} \frac{\mu^{\frac{1}{2}}}{|d\theta|^{\frac{1}{2}}}$$

where $\mu^{\frac{1}{2}}/|d\theta|^{\frac{1}{2}}$ is the half density induced by the fibrations of proposition 3.3.6.

In order to analyze $\bar{V}_\varepsilon$, we will view it as a composition of pullbacks and pushforwards applied to the Fourier integral operator

$$V_\varepsilon(\mathbf{t}, \mathbf{t}') = U(\mathbf{t})^* (\gamma_H^* \gamma_H)_{\geq \varepsilon} U(\mathbf{t}') \tag{3.3.23}$$

We begin by describing this operator.

Proposition 3.3.8. *The operator $V_\varepsilon(\mathbf{t}, \mathbf{t}')$ is a Fourier integral operator in the class $I^{-\frac{1}{2}}(T^2 \times T^2 \times S^2, S^2; C_V)$*

$$C_V = \{(\mathbf{t}, \mathcal{P}(x,\xi), \mathbf{t}', \mathcal{P}(x,\xi'), \Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}'}(x,\xi') \mid (x,\xi,x,\xi') \in C_H\} \tag{3.3.24}$$

The map $\iota_V : T^2 \times T^2 \times C_H \to T^*(T^2 \times T^2 \times S^2 \times S^2)$ *given by*

$$\iota_V : (\mathbf{t}, \mathbf{t}', x, \xi, x, \xi') = (\mathbf{t}, \mathcal{P}(x,\xi), \mathbf{t}', \mathcal{P}(x,\xi'), \Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}'}(x,\xi'))$$

is a Lagrangian embedding whose image is C_V*. The half density part of the principal symbol pulls back along* ι *to*

$$|d\mathbf{t} \wedge d\mathbf{t}'|^{\frac{1}{2}} \otimes \sigma((\gamma_H^* \gamma_H)_{\geq \varepsilon})$$

Proof. Viewing both $U^*(\mathbf{t})$, $U(\mathbf{t}')$ as operators $U, U^* : C^\infty(S^2) \to C^\infty(T^2 \times S^2)$ then the composition we are talking about is really

$$V_\varepsilon(\mathbf{t}, \mathbf{t}') = Id \otimes U^*(\mathbf{t}) \circ Id \otimes (\gamma_H^* \gamma_H)_{\geq \varepsilon} \circ U(\mathbf{t}')$$

The compositions are all transverse provided that C_H and C_U intersect transversely in the sense that the maps $\pi_i : C_H \to T^*S^2$ are transverse to the projections $\rho_i : C_U \to T^*S^2$ onto either factor. This follows from the fact that C_U is essentially a canonical graph. It implies the orders add to give the stated order and one can check easily that the composite canonical relation and symbol is what was stated in the proposition. $\square$

Now we describe the pullback under the time diagonal map. Let $\Delta : T^2 \times S^2 \times S^2 \to T^2 \times T^2 \times S^2 \times S^2$ be the map $\Delta : (\mathbf{t}, x, y) \mapsto (\mathbf{t}, \mathbf{t}, x, y)$.

Proposition 3.3.9. *The kernel of the operator* $V_\varepsilon(\mathbf{t}) = U^*(\mathbf{t})(\gamma_H^* \gamma_H)_{\geq \varepsilon} U(\mathbf{t})$ *is in the class* $I^{-1}(T^2 \times S^2 \times S^2; \Delta^* C_V)$ *Where* $\Delta^* C_V$ *is the pullback of* C_V*, the image of the Lagrangian embedding* $i_{\Delta^* C_V} : T^2 \times C_H \to T^*(T^2 \times S^2 \times S^2)$ *given by*

$$\iota_{\Delta^* C_V} : (\mathbf{t}, x, \xi, x, \xi) \mapsto (\mathbf{t}, \mathcal{P}(x,\xi) - \mathcal{P}(x,\xi'), \Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}}(x,\xi')) \tag{3.3.25}$$

The half density symbol of $V_\varepsilon(\mathbf{t})$ pulls back under $\iota_{\Delta^* C_V}$ to $|d\mathbf{t}|^{\frac{1}{2}} \otimes \sigma((\gamma_H^* \gamma_H)_{\geq\varepsilon}$.

Proof. Recall that the pullback of Lagrangian distributions is well-defined under a transversality condition. Namely, $V_\varepsilon(\mathbf{t}) = \Delta^* V(\mathbf{t}, \mathbf{t}')$ is a Lagrangian distribution as long as the maps $\pi|_{C_V} \to T^2 \times T^2 \times S^2 \times S^2$ and Δ are transverse, which is easily verified. Letting $N^*\Delta \subset T^*(T^2 \times S^2 \times S^2) \times T^*(T^2 \times T^2 \times S^2 \times S^2)$ be the co-normal bundle to the graph of Δ and $\pi : N^*\Delta \to T^*(T^2 \times T^2 \times S^2 \times S^2)$, projection onto the factor on the right, this implies that the pullback diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d] & C_V \arrow[d, "\iota"] \\ N^*\Delta \arrow[r, "\pi"'] & T^*(T^2 \times T^2 \times S^2 \times S^2) \end{tikzcd}$$

is transverse. The left projection of F into $T^*(T^2 \times S^2 \times S^2)$ is then the set

$$\Delta^* C_V = \{\mathbf{t}, \mathcal{P}(x,\xi) - \mathcal{P}(x,\xi'), \Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}}(x,\xi')\} \tag{3.3.26}$$

Which inherits a canonical half density determined by the symbol of $V_\varepsilon(\mathbf{t}, \mathbf{t}')$ on C_V, the canonical half density on $N^*\Delta \cong T^2 \times T^*S^2 \times T^*S^2$ and the symplectic half density on $T^*(T^2 \times T^2 \times S^2 \times S^2)$. This is the symbol of $V_\varepsilon(\mathbf{t})$. $\square$

Next, let $\pi : T^2 \times S^2 \times S^2 \to S^2 \times S^2$ be the projection onto the rightmost factors, $\pi(\mathbf{t}, x, y) = (x, y)$. Let let $\pi_* : C^\infty(T^2 \times S^2 \times S^2) \to C^\infty(S^2 \times S^2)$ be the pushforward map defined on smooth functions by

$$\pi_* u(\mathbf{t}, x, y) = (2\pi)^{-2} \int_{T^2} u(\mathbf{t}, x, y)\, d\mathbf{t}$$

Lemma 3.3.10. *Let $N^*_\pi \subset T^*(T^2 \times S^2 \times S^2) \times T^*(S^2 \times S^2)$ denote the co-normal bundle to the graph of π and $\rho_L : N^*_\pi \to T^*(T^2 \times S^2 \times S^2)$ denote the left projection. The pushforward diagram*

$$\begin{array}{ccc} F & \longrightarrow & \Delta^* C_V \\ \big\downarrow & & \big\downarrow \iota \\ N^*\pi & \xrightarrow[\rho_L]{} & T^*(T^2 \times S^2 \times S^2) \end{array}$$

is clean away from the singular set $i_{\Delta^* C_V}(T^2 \times T^* H) \subset \Delta^* C_V$.

Proof. Recall that above diagram is clean if the fiber product F is a submanifold of $\Delta^* C_V \times N^*\pi$ and the linearization

$$\begin{array}{ccc} TF & \longrightarrow & T(\Delta^* C_V) \\ \big\downarrow & & \big\downarrow d\iota \\ T(N^*\pi) & \xrightarrow[d\rho_L]{} & T(T^*(T^2 \times S^2 \times S^2)) \end{array}$$

is also a fiber product. Note that the fiber F is the set

$$F = \{(\mathbf{t}, 0, \Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}}(x,\xi'), \mathbf{t}, 0, \Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}}(x,\xi'), \Phi_{\mathbf{t}}(x,\xi), \Phi_{\mathbf{t}}(x,\xi') \mid (x,\xi,x,\xi') \in \widehat{C}_H\}$$

The natural parametrization $i_F : T^2 \times \widehat{C}_H \to F$ is an embedding on the smooth parts of $\widehat{C}_H$. The image $i_F(T^2 \times T^* H)$ of the non-smooth part corresponds to the singular set $i_{\Delta^* C_V}(T^2 \times T^* H)$. Hence we see that F is a submanifold of dimension 5 away from this set. To prove that the diagram is clean, we have to verify that TF is given by the kernel of the map $\tau : T(\Delta^* C_V \times N^*_\pi) \to T(T^*(T^2 \times S^2 \times S^2))$ given by $\tau(u,v,w) = v - u$. Suppose that $u = di_{\Delta^* C_V}(\alpha, v, v') \in d\rho_L T(N^*_\pi)$. Then we have $(v,v') \in C_H$ with $dPv - dPv' = 0$. But this implies that $(v,v') \in T(\widehat{C}_H)$ and the tangent vector

$(u, u, w) \in \ker \tau \subset T(\Delta^* C_V \times N_\pi^*)$ is actually equal to $di_F(\alpha, v, v')$, i.e. it is tangent to F.

$\square$

Now since the pushforward diagram is clean, the right projection $\rho_R : F \to T^*(S^2 \times S^2)$ is a smooth submersion whose image

$$\rho_R(F) = C_{\bar{V}} = \mathrm{Fl}(\Delta_{T_H^* S^2}) \bigcup \mathrm{Fl}(\mathrm{graph}\, r_H|_{T_H^* S^2})$$

is a Lagrangian submanifold of $T^*S^2 \times T^*S^2$. We now describe how the half densities on N_π^* and $\Delta^* C_V$ determine a half density on the image $\rho_R(F) = C_{\bar{V}}$. More precisely, at each point $p \in F$, the clean diagram determines an element $\mu \otimes \nu^{\frac{1}{2}} \in |\ker d(\rho_R)_p| \otimes |T_{\rho_R(p)} C_{\bar{V}}|^{\frac{1}{2}}$. The half density at the point $q \in C_{\bar{V}}$ is then given by integrating the density over the fiber of ρ_R over q:

$$\left(\int_{\rho_R^{-1}(q)} \mu \right) \nu^{\frac{1}{2}} \tag{3.3.27}$$

First consider the sequence of maps

$$0 \to T_p F \to T_{i_F(p)}(\Delta^* C_V \times N_\pi^*) \to \mathrm{im}\, \tau \to 0$$

Where τ is the map above. Because the diagram is clean, this sequence is exact. We suppose that $p = i_F(\mathbf{t}, x, \xi, x, \xi')$. We will make use of several different bases which we pause to notate here. First, let $\mathcal{B} = (H_{I_2}, \partial_\theta, \partial_\rho, \partial_\eta) \in T(T^*S^2)$. We will write $di_{N_\pi^*}(\partial_{\mathbf{t}} \otimes \mathcal{B})$ denote the basis on $T(N_\pi^*)$ obtained by pushing forward the product basis on $T^2 \times T^*S^2 \times T^*S^2$ determined by $\partial_{\mathbf{t}}$ and $\mathcal{B}$. We also let $\mathcal{B}'$ denote the basis

$(\partial_\theta, \partial_\theta), (\partial_\eta, \partial_\eta), (\partial_\rho, 0), (0, \partial_\rho) \in TC_H$ and similarly, $di_{\Delta^* C_V}(\partial_{\mathbf{t}} \otimes \mathcal{B}')$ denote the basis on $T(\Delta^* C_V)$ obtained by pushing forward the product basis on $T^2 \times C_H$.

Now, since both smooth branches of $\widehat{C}_H$ are graphs over $T^*_H S^2$, we have a natural half density $\mu^{\frac{1}{2}} \in |T(T^2 \times \widehat{C}_H)|^{\frac{1}{2}}$ which pulls back to $|d\mathbf{t}|^{\frac{1}{2}} \otimes |d\theta \wedge d\eta \wedge d\rho|^{\frac{1}{2}}$ on $T^2 \times T^*_H S^2$. We let $\mathcal{B}$ be a basis of $T_p F$ such that $\mu^{\frac{1}{2}}(\mathcal{B}) = 1$. We complete this to a basis of $T(\Delta^* C_V \times N^*_\pi)$ by adding the 10 vectors $\mathbf{0} \otimes di_{N^*_\pi}(\partial_{\mathbf{t}} \otimes \mathcal{B})$ in addition to the vector $(0, d\mathcal{P}\partial_\rho, 0, d\Phi_{\mathbf{t}}\partial_\rho, \mathbf{0})$. We claim that the change of basis matrix between this completed basis and the product basis $di_{\Delta^* C_V}(\partial_{\mathbf{t}} \otimes \mathcal{B}') \otimes \mathbf{0}$, $\mathbf{0} \otimes di_{N^*_\pi}(\partial_{\mathbf{t}} \otimes \mathcal{B})$ has determinant equal to ± 1.

Lemma 3.3.11. *Let $|\Omega|^{\frac{1}{2}}$ denote the symplectic half density on T^*S^2. Then*

$$\Omega^{\frac{1}{2}}(\mathcal{B}) = \left| \frac{\partial I_2}{\partial \rho} \right|^{\frac{1}{2}}$$

Proof. Since (r, θ, ρ, η) are canonical coordinates if we write H_{I_2} in terms of the basis $\partial_r, \partial_\theta, \partial_\rho, \partial_\eta$, the coefficient of ∂_r is $\frac{\partial I_2}{\partial \rho}$. Hence the change of basis from this symplectic basis to $\mathcal{B}$ has determinant $|\frac{\partial I_2}{\partial \rho}|$ $\square$

Now let $\sigma \in |T(\Delta^* C_V \times N^*_\pi)|^{\frac{1}{2}}$ denote the tensor product of the natural half density on N^*_π and the symbol of $V_\varepsilon(\mathbf{t})$ on $\Delta^* C_V$. Then in light of the lemma, σ on the completed basis above is equal to

$$(1 - \chi_\varepsilon)(x, \xi) \left| \frac{\partial I_2}{\partial \rho}(x, \xi) \right| \tag{3.3.28}$$

This means that the exact sequence, together with our reference half density $\mu^{\frac{1}{2}}$ determines the half density $\nu^{\frac{1}{2}}$ on $\operatorname{im} \tau$ which assigns the value (3.3.28) to the 11 vectors

$di_{N_\pi^*}(\mathbf{t}\otimes\mathcal{B})$, $(0,-d\mathcal{P}\partial_\rho,0,-d\Phi_{\mathbf{t}}\partial_\rho)$. We complete this to a basis of $T(T^*(T^2\times S^2\times S^2))$ by adding the vector $(0,\partial_{\tau_1},0,0)$. Then the symplectic half density on this basis is equal to $|\partial I_2/\partial\rho|^{\frac{3}{2}}$. Hence, using the exact sequence

$$0\to\operatorname{im}\tau\to T(T^*(T^2\times S^2\times S^2))\to\operatorname{coker}\tau\to 0$$

We get the negative half density on $\operatorname{coker}\tau$ which assigns the value $(1-\chi_\varepsilon)(x,\xi)|\partial I_2/\partial\rho|^{-\frac{1}{2}}$ to the residue class of $(0,\partial_{\tau_1},0,0)$.To finish, we use the exact sequence associated the submersion ρ_R:

$$0\to\ker d(\rho_R)_p\to T_pF\to T_{\rho_R(p)}C_V\to 0$$

Note that this is the exact sequence determined by either i_D or i_R of proposition 3.3.6 depending on whether (x,ξ,x,ξ') is the diagonal or reflection branch of $\widehat{C}_H$. Now $\operatorname{coker}\tau$ is symplectic dual to $\ker d\rho_R$. This allows us to identify the minus half density on $\operatorname{coker}\tau$ with the half density

$$(1-\chi_\varepsilon)(x,\xi)\left|\frac{\partial I_2}{\partial\rho}\right|^{-\frac{1}{2}}|d\theta|^{\frac{1}{2}}$$

The symbol of $\bar{V}_\varepsilon$ on the diagonal branch is therefore equal to

$$\sigma(\bar{V}_\varepsilon)(\Phi_{\mathbf{t}}(x,\xi),\Phi_{\mathbf{t}}(x,\xi))=(2\pi)^{-2}\left(\int_{i_D^{-1}(\Phi_{\mathbf{t}}(x,\xi),\Phi_{\mathbf{t}}(x,\xi))}(1-\chi_\varepsilon)(y,\eta)\left|\frac{\partial I_2}{\partial\rho}(y,\eta)\right|^{-\frac{1}{2}}|d\theta|\right)\frac{\mu^{\frac{1}{2}}}{|d\theta|^{\frac{1}{2}}}$$

and on the reflection branch we have

$$\sigma(\bar{V}_\varepsilon)(\Phi_{\mathbf{t}}(x,\xi),\Phi_{\mathbf{t}}(r_H(x,\xi)))=(2\pi)^{-2}\left(\int_{i_R^{-1}(\Phi_{\mathbf{t}}(x,\xi),\Phi_{\mathbf{t}}(x,\xi))}(1-\chi_\varepsilon)(y,\eta)\left|\frac{\partial I_2}{\partial\rho}(y,\eta)\right|^{-\frac{1}{2}}|d\theta|\right)\frac{\mu^{\frac{1}{2}}}{|d\theta|^{\frac{1}{2}}}$$

The proof is then completed by the following proposition:

Proposition 3.3.12. *For* $(x,\xi) \in T_H^*S^2$ *in the support of the cutoff* $1-\chi_\varepsilon(x,\xi)$*, we have*

$$\frac{\partial I_2}{\partial \rho}(x,\xi) = \frac{\sqrt{1-\frac{I_1^2(x,\xi)}{|\xi|_g^2 a(r_0)^2}}}{\omega_2(x,\xi)} \tag{3.3.29}$$

where ω_2 *is the second component of the frequency vector* $\omega_2 = \frac{\partial K}{\partial I_2}$*.*

PROOF. We have $I_2 = G(|\xi|_g, p_\theta)$. Since p_θ does not depend on ρ,

$$\frac{\partial I_2}{\partial \rho} = \frac{\partial I_2}{\partial |\xi|_g}\frac{\partial |\xi|_g}{\partial \rho}$$

Now for $(x,\xi) \in T_H^*S^2$, we have $|\xi|_g = \sqrt{\rho^2 + \frac{p_\theta^2}{a(r_0)^2}}$. So $\frac{\partial I_2}{\partial |\xi|_g} = \omega_2^{-1}(x,\xi)$ and

$$\frac{\partial |\xi|_g}{\partial \rho} = \frac{\sqrt{|\xi|_g^2 - \frac{p_\theta^2}{a(r_0)^2}}}{|\xi|_g}$$

$\square$

Since the symbol of the cutoff, χ_ε and all of the quanities appearing in (3.3.29) are functions of I_1 and I_2, they are constant on the fibers of i_D and i_R. Hence the integrals appearing above can be simplified to

$$\sigma(\bar{V}_\varepsilon)(\Phi_{\mathbf{t}}(x,\xi),\Phi_{\mathbf{t}}(x,\xi)) = \frac{1}{\pi}(1-\chi_\varepsilon)(x,\xi)\left(\frac{\omega_2(x,\xi)}{\sqrt{1-\frac{I_1^2(x,\xi)}{|\xi|_g^2 a(r_0)^2}}}\right)^{\frac{1}{2}} \frac{\mu^{\frac{1}{2}}}{|d\theta|^{\frac{1}{2}}}$$

$$\sigma(\bar{V}_\varepsilon)(\Phi_{\mathbf{t}}(x,\xi),\Phi_{\mathbf{t}}(r_H(x,\xi))) = \frac{1}{\pi}(1-\chi_\varepsilon)(x,\xi)\left(\frac{\omega_2(x,\xi)}{\sqrt{1-\frac{I_1^2(x,\xi)}{|\xi|_g^2 a(r_0)^2}}}\right)^{\frac{1}{2}} \frac{\mu^{\frac{1}{2}}}{|d\theta|^{\frac{1}{2}}}$$

This completes the proof of proposition 3.3.7. We now want to show that $\bar{V}_\varepsilon$ can be written as the sum of a pseudo-differential operator and a Fourier integral operator.

Proposition 3.3.13. *We have a decomposition $\bar{V}_\varepsilon = P_\varepsilon + F_\varepsilon$ where P_ε is an order zero pseudo-differential operator with scalar symbol equal to*

$$\sigma(P_\varepsilon)(y,\eta) = \frac{1}{\pi}(1-\chi_\varepsilon)(y,\eta)\frac{\omega_2(y,\eta)}{\sqrt{1-\frac{p_\theta^2(y,\eta)}{|\eta|_y^2 a(r_0)^2}}}|dy\wedge d\eta|^{\frac{1}{2}}$$

$F_\varepsilon \in I^0(S^2\times S^2; Fl(graph\, r_H|_{T_H^*S^2}))$. The symbol of F_ε is the half density

$$\sigma(F_\varepsilon)(\Phi_{\mathbf{t}}(x,\xi),\Phi_{\mathbf{t}}(r_H(x,\xi))) = \frac{1}{\pi}(1-\chi_\varepsilon)(x,\xi)\left(\frac{\omega_2(x,\xi)}{\sqrt{1-\frac{I_1^2(x,\xi)}{|\xi|_g^2 a(r_0)^2}}}\right)^{\frac{1}{2}}\frac{\mu^{\frac{1}{2}}}{|d\theta|^{\frac{1}{2}}}$$

where $\mu^{\frac{1}{2}}/|d\theta|^{\frac{1}{2}}$ is the half density on the flow-out of the reflection graph determined in proposition 3.3.6.

Proof. Note that the two flow-out sets $\mathrm{Fl}(\Delta_{T_H^*S^2})\bigcup \mathrm{Fl}(\mathrm{graph}\, r_H|_{T_H^*S^2}$ are disjoint when (x,ξ) is restricted to the support of a the cutoff $1-\chi_\varepsilon$. Since V_ε only has wave front set in the flow-outs of this region, we can let $\Psi\in C_c^\infty(T^*S^2\times T^*S^2)$ be a smooth cutoff function such that $\psi = 1$ in a neighborhood of the diagonal flow-out and has support disjoint from the reflection flow-out. Then we have

$$\bar{V}_\varepsilon = \widehat{\psi}\bar{V}_\varepsilon + (I-\widehat{\psi})\bar{V}_\varepsilon$$

The diagonal flow-out is inside $\Delta_{T^*S^2}$ so the first term is a pseudo-differential operator. The symbol is unchanged due to the fact that ψ and $1-\psi$ are equal to 1 on neighborhoods of the diagonal, reflected flow-outs. On the diagonal branch of the flow-out, we also have the natural symplectic half density $|dy\wedge d\eta\wedge dy\wedge d\eta|^{\frac{1}{2}}$. It is easy to check that (see lemma 3.3.11)

$$\frac{\mu^{\frac{1}{2}}}{|d\theta|^{\frac{1}{2}}} = \left|\frac{\partial I_2}{\partial\rho}\right|^{-\frac{1}{2}}|dy\wedge d\eta\wedge dy\wedge d\eta|^{\frac{1}{2}}$$

This accounts for the difference between the symbol of P_ε stated here and the symbol of $\bar{V}_\varepsilon$ on the diagonal branch.

$\square$

3.3.2. Preliminaries for the Trace Formula

To begin with, we need a description of the operator $f(D_\theta/\ell)$.

Proposition 3.3.14. *Let $f \in C_c^\infty(\mathbb{R})$. The operator $f\left(\frac{D_\theta}{\ell}\right)$ is a semi-classical pseudo-differential operator in the class $\Psi_{\ell^{-1}}^{-\infty}(S^2)$ with principal symbol equal to $f(p_\theta(y,\eta))$.*

Proof. Note that by Fourier inversion, we can write

$$f\left(\frac{D_\theta}{\ell}\right) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{f}(t) e^{i\frac{t}{\ell}D_\theta}\, dt \tag{3.3.30}$$

Becauase the flow of D_θ is just linear translation in the polar coordinates (r,θ,ρ,η), we can write

$$(\exp i\frac{t}{\ell}D_\theta)(r,\theta,r',\theta') = (2\pi)^{-2}\int_{\mathbb{R}^2} e^{i[(r-r')\rho+(\theta-\theta')\eta]}e^{i\frac{t}{\ell}\eta}\, d\rho\, d\eta$$

Now change variables $\rho' = \rho/\ell$, $\eta' = \eta/\ell$. Then

$$(\exp i\frac{t}{\ell}D_\theta)(r,\theta,r',\theta') = \frac{\ell^2}{(2\pi)^2}\int_{\mathbb{R}^2} e^{i\ell[(r-r')\rho+(\theta-\theta')\eta]}e^{it\eta'}\, d\rho'\, d\eta'$$

Inserting this expression into (3.3.30) and integrating in t finishes the proof.

$\square$

We also need a description of Π_ℓ as a semi-classical Fourier integral operator. For details, see for instance theorem 1 of [**38**]. Although this is written for the cluster projection of a Zoll Laplacian, the same argument applies to the operator $\widehat{I}_2$ considered here.

Proposition 3.3.15. *For $A \in \Psi^0$ a homogeneous order zero pseudo-differential operator, $A\Pi_\ell$ is a semi-classical Fourier integral operator of order $\frac{1}{2}$ associated to the canonical relation*

$$C_\Pi = \{(x,\xi,y,\eta) \in \Sigma \times \Sigma \mid \exists t \in [0,2\pi) \exp tH_{I_2}(x,\xi) = (y,\eta)\}$$

Where $\Sigma = \{I_2 = 1\}$. Along the parametrizing map $\iota_\Pi : S^1 \times \Sigma \to T^*S^2 \times T^*S^2$

$$\iota_\Pi : (t,x,\xi) \mapsto (x,\xi,\exp tH_{I_2}(x,\xi))$$

The half density symbol pulls back to

$$\iota_\Pi^* \sigma(A\Pi_\ell) = \ell^{\frac{1}{2}} e^{-i\ell t}|dt|^{\frac{1}{2}} \otimes \sigma(A)|d\mu_L|^{\frac{1}{2}}$$

Where $d\mu_L$ is Liouville measure on the energy surface Σ and $\sigma(A)$ is the scalar symbol of A with respect to the canonical symplectic half density on $N^*\Delta$.

3.3.3. Weak* limit of the phase space empirical measures

Let $B \in \Psi^0(S^2)$ and $\bar{B}$ be the average (3.3.4) of B with respect to the torus action $U(\mathbf{t})$. Then the un-normalized version of $\nu_\ell(B)$ tested against $f \in C_c^\infty(-1,1)$ is

$$\sum_{m=-\ell}^{\ell} \langle B\varphi_m^\ell, \varphi_m^\ell \rangle f\left(\frac{m}{\ell}\right) = \operatorname{Trace} f\left(\frac{D_\theta}{\ell}\right) \bar{B}\Pi_\ell$$

The right hand side is the trace of a semi-classical Fourier integral operator and by standard symbol calculus it has the leading order asymptotics

$$\sum_{m=-\ell}^{\ell} \langle B\varphi_m^\ell, \varphi_m^\ell \rangle f\left(\frac{m}{\ell}\right) = \ell \int_\Sigma f(p_\theta)\sigma(\bar{B})\, d\mu_L + O(1)$$

Similarly, the normalizing coefficient N_ℓ is

$$N_\ell = \text{Trace}\, \bar{B}\Pi_\ell = \ell \int_\Sigma \sigma(\bar{B})\, d\mu_L + O(1)$$

Finally, since $\sigma(\bar{B})$ is just the average of $\sigma(B)$ with respect to the torus action $\Phi_{\mathbf{t}}$, we have $\int_\Sigma \sigma(\bar{B})\, d\mu_L = \int_\Sigma \sigma(B)\, d\mu_L = \omega(B)$. We also write

$$\int_\Sigma f(p_\theta)\sigma(\bar{B})\, d\mu_L = \int_{-1}^{1} f(c) \int_{T_c} \sigma(\bar{B})\, d\mu_{L,c} dc = \int_{-1}^{1} f(c)\widehat{\sigma(B)}(c)\, dc$$

This completes the proof of theorem 1.1 (b) when f is compactly supported. The full statement follows from the fact that $\widehat{\sigma(B)}(c)$ is an L^1 function on $[-1,1]$.

3.3.4. Weak* limit of the L^2 restriction empirical measures

To begin with, we need to express the un-normalized version of (3.1.3) as the trace:

Proposition 3.3.16. *Let $f \in C_c^\infty(-1,1)$. For each $\varepsilon > 0$,*

$$\sum_{m=-\ell}^{\ell} ||\varphi_m^\ell||^2_{L^2(H)} f\left(\frac{m}{\ell}\right) = \mathit{Trace}\, f\left(\frac{D_\theta}{\ell}\right) \bar{V}_\varepsilon \Pi_\ell + R(\varepsilon, \ell) \tag{3.3.31}$$

where

$$\limsup_{\ell \to \infty} \frac{|R(\varepsilon, \ell)|}{\ell} = O(\varepsilon)$$

Proof. Note that by proposition 3.3.2, we have

$$
\begin{aligned}
&\sum_{m=-\ell}^{\ell} ||\varphi_m^\ell||^2_{L^2(H)} f\left(\frac{m}{\ell}\right) = \sum_{m=-\ell}^{\ell} \langle (\gamma_H^* \gamma_H)_{\geq \varepsilon} \varphi_m^\ell, \varphi_m^\ell \rangle f\left(\frac{m}{\ell}\right) + \\
&\sum_{m=-\ell}^{\ell} \langle (\gamma_H^* \gamma_H)_{\leq \varepsilon} \varphi_m^\ell, \varphi_m^\ell \rangle f\left(\frac{m}{\ell}\right) + \sum_{m=-\ell}^{\ell} \langle K_\varepsilon \varphi_m^\ell, \varphi_m^\ell \rangle f\left(\frac{m}{\ell}\right)
\end{aligned} \tag{3.3.32}
$$

The first term on the right hand side is just the trace appearing in the proposition. Further, since $|\langle K_\varepsilon \varphi_m^\ell, \varphi_m^\ell \rangle| = O_\varepsilon(\ell^{-\infty})$, we just need to show that

$$
\limsup \frac{1}{\ell} \left| \sum_{m=-\ell}^{\ell} \langle (\gamma_H^* \gamma_H)_{\leq \varepsilon} \varphi_m^\ell, \varphi_m^\ell \rangle f\left(\frac{m}{\ell}\right) \right| = O(\varepsilon) \tag{3.3.33}
$$

As in the discussion on page 37 of [**35**], we can bound the sum

$$
\frac{1}{\ell} \left| \sum_{m=-\ell}^{\ell} \langle (\gamma_H^* \gamma_H)_{\leq \varepsilon} \varphi_m^\ell, \varphi_m^\ell \rangle f\left(\frac{m}{\ell}\right) \right|
$$

By a sum of terms of the form

$$
\frac{1}{\ell} \sum_{m=-\ell}^{\ell} ||\gamma_H \widehat{\chi}_\varepsilon^j \varphi_m^\ell||^2_{L^2}(H)
$$

where $\widehat{\chi}_\varepsilon^j$ is either the tangential or the normal cutoff operator. In both cases, the symbol of the operator appearing is supported inside a set of volume $O(\varepsilon)$ inside Σ. By the pointwise Weyl law,

$$
\limsup_{\ell \to \infty} \frac{1}{\ell} \sum_{-\ell}^{\ell} |\widehat{\chi}_\varepsilon^j \varphi_m^\ell(x)|^2 = O(\varepsilon)
$$

and integrating this along H preserves this bound.

$\square$

Proposition 3.3.17. *For each $\varepsilon > 0$,*

$$Trace\, f\left(\frac{D_\theta}{\ell}\right)\bar{V}_\varepsilon\Pi_\ell = 4\pi\ell\left(\int_{-1}^{1} f(c)(1-\chi_\varepsilon)(c)\frac{\omega_2(c,1)}{\sqrt{1-\frac{c^2}{K(c,1)^2a(r_0)^2}}}\,dc\right) + O_\varepsilon(1)$$

Proof. By proposition 3.3.13, we have $\bar{V}_\varepsilon = P_\varepsilon + F_\varepsilon$. From propositions 3.3.14,3.3.15, and 3.3.13, the contribution of the P_ε term in the trace is equal to

$$\ell\left(\int_\Sigma f(p_\theta)\sigma(P_\varepsilon)\,d\mu_L\right) + O_\varepsilon(1)$$

Since the symbol of P_ε is a function of I_1 and I_2, it is constant on each torus T_c and the leading term is equal thus equal to

$$(2\pi)^2\ell\int_{-1}^{1} f(c)\sigma(P_\varepsilon)(c,1)\,dc$$

which is the stated term in the proposition. To finish the proof, we need to show that the contribution to the trace from the F_ε piece is of size $O_\varepsilon(1)$. For this, note that $f\left(\frac{D_\theta}{\ell}\right)F_\varepsilon\Pi_\ell$ is a semi-classical Fourier integral operator of order $\frac{1}{2}$ associated to the canonical relation

$$C_{R\Pi} = \{(x,\xi,y,\eta) \mid (x,\xi) = \Phi_{\mathbf{t}}(r_H(x',\xi')) \text{ and } (\Phi_{\mathbf{t}}(x',\xi'),y,\eta) \in C_\Pi\}$$

The trace is controlled by the symbol on the intersection $C_{R\Pi}\cap\Delta_{T^*S^2}$. This is equal to the set

$$\{(\Phi_{\mathbf{t}}(x',\xi'),\Phi_{\mathbf{t}}(r_H(x',\xi')) \in C_\Pi \mid \mathbf{t}\in T^2, (x',\xi')\in T^*_H S^2\}$$

And this is equivalent to the statement that (x',ξ') and $r_H(x',\xi')$ lie along the same I_2 bicharacteristic. But if $(x',\xi')\notin T^*H$, this would mean that the projection of the I_2 bicharacteristic to S^2 has a self-intersection, which is impossible. Thus it must be that $(x', \xi') = r_H(x', \xi') \in T^*H$. Due to the cutoff χ_ε, the symbol of F_ε vanishes on the aforementioned set. Hence the order ℓ term in the trace vanishes as claimed.

$\square$

Proposition 3.3.18. *The normalizing factor* $M_\ell = \sum_{m=-\ell}^{\ell} ||\varphi_m^\ell||^2_{L^2(H)}$ *satisfies*

$$\lim_{\ell\to\infty} \frac{M_\ell}{\ell} = 4\pi \int_{-1}^{1} \frac{\omega_2(c,1)}{\sqrt{1 - \frac{c^2}{K(c,1)^2 a(r_0)^2}}}\, dc$$

Proof. In the same fashion as the proof of proposition 5.3, we can write

$$M_\ell = \operatorname{Trace} \bar{V}_\varepsilon \Pi_\ell + R'(\varepsilon, \ell)$$

$$\operatorname{Trace} \bar{V}_\varepsilon \Pi_\ell = \ell \int_{-1}^{1} (1 - \chi_\varepsilon)(c) \frac{\omega_2(c,1)}{\sqrt{1 - \frac{c^2}{K(c,1)^2 a(r_0)^2}}}\, dc + O_\varepsilon(1)$$

where $\limsup_{\ell\to\infty} |R'(\varepsilon, \ell)|/\ell = O(\varepsilon)$. Since

$$\int_{-1}^{1} (1 - \chi_\varepsilon)(c) \frac{\omega_2(c,1)}{\sqrt{1 - \frac{c^2}{K(c,1)^2 a(r_0)^2}}}\, dc \to \int_{-1}^{1} \frac{\omega_2(c,1)}{\sqrt{1 - \frac{c^2}{K(c,1)^2 a(r_0)^2}}}\, dc$$

as $\varepsilon \to 0$, the statement follows.

$\square$

Now in light of propositions 5.1,5.2,and 5.3, for $f \in C_c^\infty(-1,1)$,

$$\langle \mu_\ell, f \rangle = \frac{1}{M_\ell} \sum_{m=-\ell}^{\ell} ||\varphi_m^\ell||^2_{L^2(H)} f\left(\frac{m}{\ell}\right) = 4\pi \frac{\ell}{M_\ell} \left(\int_{-1}^{1} f(c)(1 - \chi_\varepsilon)(c) \frac{\omega_2(c,1)}{\sqrt{1 - \frac{c^2}{K(c,1)^2 a(r_0)^2}}}\, dc \right) + R''(\varepsilon, \ell)$$

where $\limsup |R''(\varepsilon, \ell)| = O(\varepsilon)$. Taking $\ell \to \infty$ and then $\varepsilon \to 0$ finishes the proof of theorem 1.1 (a) when f is compactly supported. We can freely upgrade this statement to $f \in C^0([-1,1])$ because

$$\frac{\omega_2(c,1)}{\sqrt{1-\frac{c^2}{K(c,1)^2 a(r_0)^2}}}$$

is an L^1 function of c on $[-1,1]$.

CHAPTER 4

Scaling Asymptotics for Ladder Sequences of Spherical Harmonics

4.1. Introduction

Let (S^2, g_{can}) be the round sphere with standard polar coordinates $(\phi, \theta) \in (0, \pi) \times (0, 2\pi)$ where θ is the polar angle measured relative to fixed meridian geodesic, and ϕ is the azimuthal angle from the north pole. Let

$$Y_N^m(\phi, \theta) = \sqrt{\frac{2N+1}{4\pi} \frac{(N-m)!}{(N+m)!}} P_N^m(\cos\phi) e^{im\theta} \tag{4.1.1}$$

be the standard L^2 normalized spherical harmonics. Suppose that we choose sequences of integers $0 \leq m_k \leq N_k$ so that $m_k, N_k \to \infty$ while the ratio $c = m_k/(N_k + \frac{1}{2})$ In this section we show that along such sequences, the functions $Y_{N_k}^{m_k}$ have Airy function type asymptotics in an $(N_k + \frac{1}{2})^{-\frac{2}{3}}$ size neighborhood of the caustic latitude circles determined by $\sin\phi_\pm = c$, where $0 < \phi_- < \pi/2 < \phi_+ < \pi$.

Theorem 4.1.1. *For integers $0 \leq m \leq N$, let Y_N^m be the standard spherical harmonics (1.0.4) on (S^2, g_{can}) and let $x = (\phi, \theta)$ be geodesic polar coordinates from the north pole. Suppose $0 \leq m_k \leq N_k$ are sequences of integers such that $m_k/(N_k + \frac{1}{2}) = c$ for all k. Then there exists an $\varepsilon > 0$ such that if $x = (\phi, \theta)$ with $c < \sin\phi_- < c + \varepsilon$, with $h_k = \left(N_k + \frac{1}{2}\right)^{-1}$,*

$$\begin{aligned} Y_{N_k}^{m_k}(x)\sqrt{dV_g(x)} \sim & Ai\left(-h_k^{-\frac{2}{3}}\rho(x)\right)\sum_{n=0}^{\infty} u_{0,n}(x) h_k^{-\frac{1}{6}+n} + \\ & Ai'\left(-h_k^{-\frac{2}{3}}\rho(x)\right)\sum_{n=0} u_{1,n}(x) h_k^{\frac{1}{6}+n} \end{aligned} \tag{4.1.2}$$

The argument of the Airy function and its derivative is

$$\rho(x) = \left(\frac{4}{3}\int_{\gamma_x}\alpha\right)^{\frac{2}{3}} \tag{4.1.3}$$

Here, γ_x is the geodesic arc joining the two pre-images $\pi^{-1}(x) \in T_c$ and α is the canonical 1-form on T^*S^2. The arc is oriented so as to make the integral positive. The $u_{i,j}$ are smooth half densities on S^2 and the leading order coefficient $u_{0,0}$ is

$$u_{0,0}(x) = \left(\frac{4\rho(x)}{\sin^2\phi - c^2}\right)^{\frac{1}{4}} e^{im_k\theta}\sqrt{dV_g} = (2\pi)\rho(x)^{\frac{1}{4}}e^{im_k\theta}\pi_*\sqrt{d\mu_{L,c}} \tag{4.1.4}$$

where $d\mu_{L,c}$ is the normalized joint flow invariant density on T_c and $\pi : T^*S \to S^2$ is the natural projection.

We recall that the torus T_c is the Lagrangian submanifold $S^*S^2 \cap \{p_\theta = c\} \subset T^*S^2 \setminus 0$, where as before, $p_\theta(x,\xi) = \langle \xi, \partial_\theta\rangle$ is the Clairaut integral, the symbol of D_θ. The Hamiltonian flow of p_θ commutes with the geodesic flow and defines the torus action which for $\mathbf{t} = (t_1, t_2) \in T^2$ is given by

$$\Phi_{\mathbf{t}} : (x,\xi) \mapsto \exp t_1 H_{p_\theta} \circ \exp t_2 H_{|\xi|_g}$$

This action preserves T_c and is free and transitive on it for every $c \in [0,1]$. The measure $d\mu_{c,L}$ pulls back to the normalized measure $(2\pi)^{-2}(dt_1 \wedge dt_2)$ under the embedding given by the orbit of any fixed point $(x_0, \xi_0) \in T_c$. To prove the theorem, we construct an approximation to the Legendre functions $P_{N_k}^{m_k}$. In section 4.2, we conjugate the associated Legendre operator

$$L_m := -\partial_x(1-x^2)\partial_x + \frac{m^2}{1-x^2} + \frac{1}{4} \tag{4.1.5}$$

on the interval $[-1,1]$ to a Schrödinger operator on $I = (0,\pi)$ and construct a global WKB quasi-mode (approximate eigenfunction) for this operator in such a way that it is a locally uniform approximation to $P_N^m(\cos\phi)$, $\phi \in (0,\pi)$ as $m, N \to \infty$ with the ratio $c = m/(N+\frac{1}{2})$ fixed. Section 4.3 contains the derivation of the Airy expansion of the quasi-mode and in section 4.4 we explain how the quantities appearing in the expansion have interpretations in terms of the geometry of the sphere.

4.1.1. Background

4.1.1.1. The Legendre functions. To establish notation and collect basic facts we quote the following classical results about the Legendre functions and refer to the standard references [**28**],[**20**] for more detail. We note that these functions are called 'Ferrer's functions' or 'Legendre functions on the cut' by some authors. For each pair of integers (m, N) with $0 \le m \le N$, let $P_N^m(x)$ be the following function defined for $x \in [-1,1]$:

$$P_N^m(x) := \left(\left(N + \frac{1}{2}\right)\frac{(N-m)!}{(N+m)!}\right)^{\frac{1}{2}} \frac{1}{2^N N!}(1-x^2)^{\frac{m}{2}}\partial_x^{N+m}(x^2-1)^N. \tag{4.1.6}$$

We refer to $P_N^m(x)$ as the normalized Legendre function of degree N and order m. They are real-valued, smooth on $(-1,1)$, and satisfy

$$(4.1.7) \qquad (1-x^2)\partial_x^2 P_N^m(x) - 2x\partial_x P_N^m(x) + \left(\left(N+\frac{1}{2}\right)^2 - \frac{m^2}{1-x^2} - \frac{1}{4}\right) P_N^m(x) = 0,$$

$$(4.1.8) \qquad \int_{-1}^{1} P_N^m(x)^2\, dx = 1.$$

Proposition 4.1.2. *For $m \in \mathbb{Z}_{\geq 0}$, define the (positive) Legendre operator L_m,*

$$(4.1.9) \qquad L_m := -\partial_x(1-x^2)\partial_x + \frac{m^2}{1-x^2} + \frac{1}{4}$$

As an unbounded operator on $L^2[-1,1]$ with domain $C_c^\infty([-1,1],dx)$, L_m has only discrete spectrum consisting of simple eigenvalues

$$Spec(L_m) = \left\{\left(N+\frac{1}{2}\right)^2 \mid N \in \mathbb{N}, N \geq m\right\}.$$

Each eigenspace is the complex span of $P_N^m(x)$ and the set $\{P_N^m\}_{N=m}^\infty$ is an orthonormal basis of $L^2([-1,1],dx)$.

For a proof, see **[31]**. The formula

$$P'_{N+1}(x) = xP'_N(x) + (N+1)P_N(x)$$

together with $P_N^0(1) = 1$ implies that for all $0 \leq m \leq N$, $P_N^m(x)$ is positive near $x = 1$. Depending on the relative of parity of m and N, $P_N^m(x)$ is either odd or even,

$P_N^m(-x) = (-1)^{m+N} P_N^m(x)$. We will use these properties to match the quasi-mode with P_N^m.

4.1.1.2. Review of Oscillatory Functions Associated to Lagrangian Manifolds. This section contains a review of the basic theory of oscillatory integrals which we will use in the construction of the quasi-mode in section 4.2.

Let (M^n, g) be a Riemannian manifold. The theory reviewed here depends upon working with smooth half densities rather than functions. Fix a smooth, positive density ν on M. We may then identify functions with half densities via the isomorphism

$$f(x) \cong f(x)\sqrt{\nu}$$

Let $\Lambda \subset T^*M$ be a compact Lagrangian submanifold. In order to define the space $\mathcal{O}^*(M, \Lambda)$ of oscillatory half densities associated to Λ, we fix a locally finite open cover $\{U_j\}$ of Λ such that for each j, there exists a phase function $\psi_j(x, \theta) \in C^\infty(V_j \times \mathbb{R}^{N_j}, \mathbb{R})$ defined on some open subsets $V_j \subset M$ which are small enough so that the maps

$$i_{\psi_j} : (x, \theta) \ni C_{\psi_j} \mapsto (x, d_x \psi_j(x, \theta))$$

are embeddings onto $U_j \subset \Lambda$. Here C_{ψ_j} is the zero set of $d_\theta \psi_j$ which is assumed to be an n dimensional submanifold. We further fix a partition of unity χ_j subordinate to this cover.

Definition 4.1.3. The space $\mathcal{O}^\mu(M, \Lambda)$ is the space of all half densities which can be written in the form

$$(4.1.10) \qquad u(x,h) = \left(\sum_j (2\pi h)^{-\frac{N_j}{2}} \int_{\mathbb{R}^{N_j}} a_j(x,\theta,h) e^{\frac{i}{h}\psi_j(x,\theta)}\, d\theta \right) \sqrt{\nu}$$

$$(4.1.11) \qquad a_j(x,\theta) \sim \sum_{n=0}^{\infty} a_{j,n}(x,\theta) h^{\mu+n}$$

where each $a_j(x,\theta)$ is a smooth function with compact support. We write $\mathcal{O}^\infty(M,\Lambda) = \cap_{\mu\in\mathbb{R}} \mathcal{O}^\mu(M,\Lambda)$ and when h is restricted to take values in a particular sequence h_k we will signify this with the notation $\mathcal{O}^\mu(M,\Lambda,h_k)$. Associated to each $u(x,h) \in \mathcal{O}^\mu(M,\Lambda)$ is a geometric object $\sigma(u)$ called its principal symbol, which is a section of a certain line bundle over Λ. To define it, we first recall that the Maslov bundle $\mathbb{L} \to \Lambda$ is a flat complex line bundle which can be described concretely using the choice of $\{U_j, \psi_j\}$. On $U_i \cap U_j$, define the locally constant functions

$$m_{ij}(\lambda) = \frac{1}{2}\left(\operatorname{Sgn} \partial_\theta^2 \psi_j(\lambda) - \operatorname{Sgn} \partial_\theta^2 \psi_i(\lambda)\right)$$

where $\partial_\theta^2 \psi$ is the hessian with respect to the fiber variables. The functions $\exp i\frac{\pi}{2} m_{ij}(\lambda)$ are the transition functions of the Maslov bundle on $U_i \cap U_j$. The choice of phase functions determines a canonical section, s, of $\mathbb{L}$ by

$$s_j(\lambda) = \exp i\frac{\pi}{4} \operatorname{Sgn} d_\theta^2 \psi_j(\lambda) \qquad \lambda \in U_j.$$

Let Ψ_j be the lift of ψ_j to U_j via the map i_{ψ_j} and $\Omega^{\frac{1}{2}} \to \Lambda$ be the half density bundle over Λ. Fix a smooth positive density ρ_0 on Λ and define the space of symbols of order μ, $S^\mu(\Lambda)$, to be the set of all smooth sections of $\Omega^{\frac{1}{2}} \otimes \mathbb{L} \to \Lambda$ which may be written in the form

$$h^\mu \left(\sum_j \exp i \frac{\Psi_j(\lambda)}{h} f_j(\lambda) s_j(\lambda) + O(h) \right) \sqrt{\rho_0} \tag{4.1.12}$$

where f_j are smooth functions on Λ with $\operatorname{supp} f_j(\lambda) \subset U_j$. The principal symbol map $\sigma : \mathcal{O}^\mu(M, \Lambda) \to S^\mu(\Lambda)/S^{\mu+1}(\Lambda)$ is defined so that when $u(x,h)$ is written in the form (4.1.10) then

$$[\sigma(u)](\lambda) = h^\mu \left(\sum_j \exp i \frac{\Psi_j(\lambda)}{h} a_{j,0}(\lambda) g_j(\lambda) s_j(\lambda) \right) \sqrt{\rho_0}. \tag{4.1.13}$$

Here, the g_j are smooth functions on U_j defined by

$$g_j \sqrt{\rho_0} = (i_{\psi_j}^{-1})^* \sqrt{d_{C_{\psi_j}}} \qquad \lambda \in U_j$$

where $d_{C_{\psi_j}}$ is the canonical δ-density on the critical set C_{ψ_j} determined by the density $\nu \otimes |d\theta|$ on $V_j \times \mathbb{R}^{N_j}$. Next, we define a map which takes a symbol to an oscillatory half density,

$$\mathcal{Q} : S^\mu(\Lambda) \to \mathcal{O}^\mu(M, \Lambda).$$

Suppose that $\sigma \in S^\mu$ is written in the form (4.1.12) (which is always possible using χ_j). Define $\mathcal{Q}(\sigma)$ to be the smooth half density (4.1.10) with amplitudes a_j chosen so that $(i_{\psi_j}^{-1})^* a_j g_j = f_j$. Then $\mathcal{Q}$ is a right inverse for the principal symbol map. It depends on the choices (U_j, ψ_j, χ_j) while the principal symbol map does not. Finally, suppose that P is an order zero semi-classical pseudo-differential operator with principal symbol p_0 and with sub-principal symbol equal to zero. If $p_0 = 0$ on Λ and ρ is a density on Λ

invariant under the Hamiltonian flow $t \mapsto \exp tX_{p_0}$ of p_0, then for any $u \in \mathcal{O}^{\mu}(M, \Lambda)$, $Pu \in \mathcal{O}^{\mu+1}(M, \Lambda)$ and if $u(x, h)$ has principal symbol

$$\sigma(u) = \left(\sum_j \exp i \frac{\Psi_j(\lambda)}{h_k} f_j(\lambda) s_j(\lambda) \right) \sqrt{\rho}, \tag{4.1.14}$$

then the order $\mu + 1$ symbol of Pu is

$$\sigma(Pu) = \left(\sum_j \exp i \frac{\Psi_j(\lambda)}{h_k} \frac{2}{i} X_{p_0} f_j(\lambda) s_j(\lambda) \right) \sqrt{\rho}. \tag{4.1.15}$$

4.2. Maslov-WKB Quasi-modes for the Legendre Functions

We begin by conjugating the Legendre operator on $[-1, 1]$ to a Schrödinger operator on $I = (0, \pi)$. The following proposition is a straightforward calculation.

Proposition 4.2.1. *Let U be the unitary map $U : L^2((-1, 1), dx) \to L^2((0, \pi), d\phi)$*

$$(Uf)(\phi) = f(\cos\phi)\sqrt{\sin\phi}$$

Let $0 < c < 1$ and define the operator $H_{h,c}$ for $f \in C^\infty((0, \pi))$

$$H_{h,c} f(\phi) := -h^2 f''(\phi) + \left(\frac{c^2}{\sin^2\phi} - \frac{h^2}{4\sin^2\phi} \right) f(\phi) \tag{4.2.1}$$

Suppose that $m(h)$ is an integer such that $c = m(h)h \in (0, 1)$ for all h. Then

$$h^2 U L_{m(h)} U^* = H_{h,c}.$$

For the remainder of this section we fix once and for all some $c \in (0,1)$ and a rational ladder sequence, that is, integers $0 \leq m_k \leq N_k$ such that for all k, $m_k/(N_k + \frac{1}{2}) = c$. Putting $h_k = (N_k + \frac{1}{2})^{-1}$, it follows from propositions 4.1.2 and 4.2.1 that the spectrum of $H_{h_k,c}$ is

$$\mathrm{Spec}(H_{h_k,c}) = \left\{ h_k^2 \left(N + \frac{1}{2} \right)^2 \mid N \geq m_k \right\}.$$

In particular, 1 is an eigenvalue of $H_{h_k,c}$ for all k. Moreover $\ker (H_{h_k,c} - 1)$ is one dimensional and spanned by $u_{h_k}(\phi) := UP_{N_k}^{m_k}$. It follows that there exists $\delta > 0$ so that $H_{h_k,c}$ has the spectral gap,

$$\inf_{\lambda \in \mathrm{Spec}(H_{h_k,c}) \setminus \{1\}} |1 - \lambda| \geq \delta h_k \tag{4.2.2}$$

4.2.1. Construction of a global h^∞ quasi-mode for $H_{h_k,c}$

We say that a smooth function v_h on $I = (0, \pi)$ is a quasi-mode of order h^∞ for $H_{h,c}$ with quasi-eigenvalue $E(h)$ if

$$|| \left(H_{h,c} - E(h) \right) v_h ||_{L^2(I)} = O(h^\infty) \tag{4.2.3}$$

where $E(h)$ has the semi-classical expansion $E(h) \sim E_0 + \sum_{j=1}^{\infty} h^j E_j$. Let (ϕ, τ) be coordinates for $T^*\mathbb{R}$, $p : T^*\mathbb{R} \to \mathbb{R}$ the natural projection, and

$$f(\phi, \tau) = \tau^2 + \frac{c^2}{\sin^2 \phi}$$

be the principal symbol of $H_{h,c}$. The energy curve

$$\Sigma = \{f(\phi,\tau) = 1\} \tag{4.2.4}$$

is a smooth, closed curve, symmetric about $\tau \mapsto -\tau$, intersecting $\{\tau = 0\}$ at $\phi_\pm = \pi/2 \pm \phi_0$ where $\phi_\pm$ are the two solutions of $\sin\phi = c$, $\phi \in (0,\pi)$. We follow the well-known procedure of WKB-Maslov quantization in order to construct a quasi-mode v_{h_k} approximating $u_{h_k} = UP^{m_k}_{N_k}$ locally uniformly on I. For the remainder of the section we identify smooth functions on I with smooth half densities on I by

$$f(\phi) \cong f(\phi)\,|d\phi|^{\frac{1}{2}}.$$

In this way we may speak of oscillatory functions instead of half densities and we do this without further comment. The rest of this section contains the proof of the following:

Proposition 4.2.2. *There exists a smooth, real-valued function $v_{h_k}(\phi) \in \mathcal{O}^0(I,\Sigma,h_k)$ with $||v_{h_k}||_{L^2(I)} = 1$ and a sequence of real numbers E_j so that if $E(h_k) \sim 1 + h_k^2 E_2 + h_k^3 E_3 + \cdots$, then*

$$||(H_{h_k,c} - E(h_k))v_{h_k}||_{L^2(I)} = O(h_k^\infty). \tag{4.2.5}$$

Moreover, for any fixed $\phi \in (\phi_-,\phi_+)$,

$$v_{h_k}(\phi) = \begin{cases} \sqrt{\frac{2\sin\phi}{\pi}}\ \frac{\cos\left(\frac{1}{h_k}\int_{\gamma_\phi}\alpha + \frac{\pi}{4}\right)}{\left(\sin^2\phi - c^2\right)^{\frac{1}{4}}} + O(h)_{L^2} & N_k - m_k \ \textit{odd} \\[2ex] -\sqrt{\frac{2\sin\phi}{\pi}}\ \frac{\sin\left(\frac{1}{h_k}\int_{\gamma_\phi}\alpha + \frac{\pi}{4}\right)}{\left(\sin^2\phi - c^2\right)^{\frac{1}{4}}} + O(h)_{L^2} & N_k - m_k \ \textit{even} \end{cases} \tag{4.2.6}$$

and there exists an $\varepsilon > 0$ such that if $\phi < \phi_- + \varepsilon$, then

$$v_{h_k}(\phi) = (2\pi h_k)^{-\frac{1}{2}} \int a(\tau, h) e^{\frac{i}{h_k}(\phi\tau - G_2(\phi))}\, d\tau + O(h^\infty)_{L^2}, \tag{4.2.7}$$

where $a(\tau, h) \sim \sum_j a_j(\tau) h^j$, $a_0(\tau) = \frac{1}{\sqrt{\pi}} V'(G_2'(\tau))^{-\frac{1}{2}}$, and $G_2(\tau)$ satisfies $G_4(0) = 0$, $(G_2'(\tau), \tau) \in \Sigma$ on the support of a.

The existence of $O(h^\infty)$ quasi-modes is well known, see for instance [**4**], [**7**], [**10**]. The solvability of (4.2.5) up to error $O(h^\infty)$ requires (Σ, h_k) to have the following three properties:

Proposition 4.2.3. *Let $\alpha = \tau d\phi|_\Sigma$, $[\mathfrak{m}] \in H^1(\Sigma, \mathbb{Z})$ be the Maslov class, and X_f be the Hamiltonian vector field of f.*

(a) For all k large enough,

$$\frac{1}{2\pi h_k}[\alpha] - \frac{1}{4}[\mathfrak{m}] \in H^1(\Sigma, \mathbb{Z}) \tag{4.2.8}$$

(b) There exists a positive density ρ_0 invariant under the flow of X_f

(c) For each smooth function r_0 on Σ satisfying $\int_\Sigma r_0\, \rho_0 = 0$, there exists a smooth function r_1 so that $dr_1(X_f) = r_0$.

Proof. (a) Since Σ is a curve, we can check this by integration. We define the Maslov class below, but to check this it suffices to know that $\int_\Sigma [\mathfrak{m}] = 2$ when Σ is oriented counter-clockwise. Since the integral of α is the area enclosed by Σ and Σ is symmetric across the lines $\phi = \pi/2$, $\tau = 0$, the integral is four times the area of the upper right quadrant,

$$\int_\Sigma \tau\, d\phi = 4c \int_0^{\sqrt{1-c^2}} \frac{\tau^2\, d\tau}{(1-\tau^2)\sqrt{1-c^2-\tau^2}} = 2\pi(1-c)$$

Therefore

$$\frac{1}{2\pi h_k}\int_\Sigma \alpha - \frac{1}{2} = h_k^{-1} - \frac{1}{2} - c h_k^{-1} = N_k - m_k \in \mathbb{Z}.$$

(b) The map

$$i : [0,\pi) \to \Sigma \qquad i(t) = \exp t X_f(\phi_+, 0)$$

is a surjective Lagrangian immersion. To see this, one only needs to note that the period of the Hamiltonian flow through $(\phi_+, 0)$ is π. This follows from the fact the curve $\exp \frac{t}{2} X_f(\phi_+, 0)$ can be identified with a geodesic on S^2 (see section 4.4). The density ρ_0 defined by $i^*\rho_0 = \pi^{-1}|dt|$ is clearly positive and invariant.

(c) Pulling back under i, we may assume $r_0(t)$ is smooth on $[0,\pi)$, $\int_0^\pi r(t)\,|dt| = 0$, and $\lim_{t\to\pi} r_0(t) = r_0(0)$. Then the function $r_1(t) = \int_0^t r_0(s)\,|ds|$ solves the equation.

□

4.2.1.1. Explicit choice of phases and canonical operator. Let $\{U_j\}_{j=1}^4$ be the open cover of Σ pictured below. The sets U_1, U_3 are symmetric about $(\phi,\tau) \mapsto (\phi, -\tau)$ and U_2, U_4 are symmetric with respect to reflection over $\phi = \frac{\pi}{2}$.

Let χ_j be a partition of unity subordinate to this cover so that $\chi_1(\phi,\tau) = \chi_3(\phi,-\tau)$ and $\chi_2(\phi,\tau) = \chi_4(\pi-\phi,\tau)$. We choose local phase functions parametrizing this open cover as follows. For $j = 2,4$ we put

$$\psi_j(\phi,\tau) = \phi\tau - G_j(\tau), \tag{4.2.9}$$

where G_j are chosen so that $G_j(0) = 0$ and $(G_j'(\tau),\tau) \in \Sigma$ on the τ-projection of U_j. For $\phi \in p(U_1) = p(U_3)$, let

$$\psi_1(\phi) = \int_{\gamma_\phi} \alpha \qquad \psi_3(\phi) = \int_{-\gamma_\phi} \alpha = -\psi_1(\phi) \tag{4.2.10}$$

Here, γ_ϕ is the arc joining the turning point $(\phi_+,0)$ to the point $(\phi,\tau) \in U_1$ and $-\gamma_\phi$ is the arc joining $(\phi_+,0)$ to $(\phi,\tau) \in U_3$. Since the lifts Ψ_j of the phases to Σ are primitives of α, they differ by a constant $\Psi_i - \Psi_j := C_{ij}$ on each $U_i \cap U_j$. It is easy to see for this choice of phases that $C_{12} = C_{23} := C$ and $C_{34} = C_{41} = 0$. Note that this means $\int_\Sigma \alpha = 2C$ where the integral is in the counter-clockwise direction. As described in section 1.3, this choice of phases shows that the co-cycle which defines the Maslov class $[\mathfrak{m}]$ is $m_{21} = m_{32} = m_{43} = m_{14} = \frac{1}{2}$.

Proposition 4.2.4. *Define constants β_j as follows*

$$\begin{cases} \beta_1 = -\beta_3 = \frac{\pi}{4} & N_k - m_k \ \textit{odd} \\ \beta_1 = -\beta_3 = \frac{3\pi}{4} & N_k - m_k \ \textit{even} \end{cases} \qquad \begin{cases} \beta_2 = \beta_4 = 0 & N_k - m_k \ \textit{odd} \\ \beta_2 = 0, \beta_4 = \pi & N_k - m_k \ \textit{even} \end{cases}$$

Then the local expressions

$$S_j(\lambda) = \exp\, i\left(\frac{\Psi_j(\lambda)}{h_k} + \beta_j\right) s_{\psi_j}(\lambda) \qquad \lambda \in U_j \tag{4.2.11}$$

define a global section of the Maslov bundle over Σ.

Proof. For each $\lambda \in U_i \cap U_j$, recall that we have

$$s_{\psi_j}(\lambda) = s_{\psi_i}(\lambda) \exp i\frac{\pi}{2} m_{ij}. \tag{4.2.12}$$

Therefore, the above expression defines a global section if and only if

$$\frac{\Psi_j - \Psi_i}{h_k} + \frac{\pi}{2} m_{ij} + \beta_j - \beta_i = 0 \mod 2\pi \tag{4.2.13}$$

The quantization condition (4.2.8) implies that

$$\frac{\Psi_j - \Psi_i}{h_k} - \frac{\pi}{2} = \pi(N_k - m_k) \tag{4.2.14}$$

for $j = 1$, $i = 2$ and $j = 2$, $i = 3$. Using this together with $\Psi_1 = \Psi_4$ and $\Psi_3 = \Psi_4$ on the intersection of their domains, we easily verify the values of the β_j, are determined except for a $\pm$ sign ambiguity and this is removed by requiring $\beta_2 = 0$. $\square$

4.2.1.2. Conclusion of the proof of proposition 2.3. Let ρ_0 be the positive invariant density on Σ as in proposition 4.2.3. Define the symbol $\sigma_0 \in S^0(\Sigma)$ by

$$\sigma_0 = \left(\sum_j \exp i \frac{\Psi_j(\lambda)}{h_k} \chi_j(\lambda) s_j(\lambda) \right) \sqrt{\rho_0}. \tag{4.2.15}$$

We inductively find a sequence of smooth functions $r_j(\lambda)$ on Σ and complex numbers E_j so that for each $n \geq 0$,

(4.2.16)

$$\left(H_{h_k,c} - (1 + h^2 E_1 + \cdots + h^{n+1} E_n)\right) (\mathcal{Q}(\sigma_0) + h\mathcal{Q}(r_1\sigma_0) + \cdots + h^n \mathcal{Q}(r_n\sigma_0)) \in \mathcal{O}^{n+2}$$

With $r_0 = E_0 = 1$, the $n = 0$ case follows from formula (4.1.15) and $\mathcal{L}_{X_f}\sigma_0 = 0$. Supposing it holds for $n \geq 0$, let

$$U_n = \mathcal{Q}\left(\left(1 + \sum_{j=1}^{n} r_j \right) \sigma_0 \right) \in \mathcal{O}^{n+2}$$

$$\mathcal{E}_n = 1 + \sum_{j=1}^{n} h^{j+1} E_j$$

Then with E_{n+1} and r_{n+1} to be determined, the function

$$\left(H_{h_k,c} - \mathcal{E}_n - h^{n+2} E_{n+1}\right) \left(U_n + h^{n+1} \mathcal{Q}(r_{n+1}\sigma_0)\right) \tag{4.2.17}$$

is in $\mathcal{O}^{n+2}$ and its principal symbol is the same as the principal symbol of

$$U_n + h^{n+2} E_{n+1} \mathcal{Q}(\sigma_0) + h^{n+1} (H_{h_k,c} - 1) \mathcal{Q}(r_{n+1}\sigma_0)$$

which vanishes if and only if

$$\frac{2}{i}dr_{n+1}(X_f) + E_{n+1} + u_n = 0 \tag{4.2.18}$$

where $h^n u_n \sigma_0 = \sigma(U_n)$. If $E_{n+1} = -\int_\Sigma u_n\,\rho_0$, then proposition 4.2.3 implies that there is a smooth r_{n+1} which solves this equation. Now letting $r \sim 1+\sum_{j=1}^\infty r_j h^n \sigma_0$, $v_{h_k} = \mathcal{Q}(r\sigma_0)$ satisfies $(H_{h_k,c} - E(h))v_{h_k} \in \mathcal{O}^\infty(I, \Sigma, h_k)$. Finally, to verify the pointwise asymptotics, we write $K_j = i^*_{\psi_j}\chi_j$ and observe that

$$\mathcal{Q}(\sigma_0) = \sum_{j=1,3} K_j(\phi)a_j(\phi)e^{i\left(\frac{\psi_j}{h_k}+\beta_j\right)} + \sum_{j=2,4}(2\pi h_k)^{-\frac{1}{2}}\int K_j(\tau)a_j(\tau)e^{i\left(\frac{\psi_j}{h_k}+\beta_j\right)}\,d\tau \tag{4.2.19}$$

$$a_j(\phi) = \frac{1}{\sqrt{2\pi}}\frac{1}{(1-V(\phi))^{\frac{1}{4}}} = \frac{1}{\sqrt{2\pi}}\frac{\sqrt{\sin\phi}}{(\sin^2\phi - c^2)^{\frac{1}{4}}} \quad j = 1,3 \tag{4.2.20}$$

$$a_j(\tau) = \frac{1}{\sqrt{\pi}}\frac{1}{|V'(G_j'(\tau))|^{\frac{1}{2}}} \quad j = 2,4 \tag{4.2.21}$$

Notice that $|G_j''(\tau)|^{-\frac{1}{2}} = \frac{1}{\sqrt{2}}\frac{|V'(G_j'(\tau))|^{\frac{1}{2}}}{|\tau|^{\frac{1}{2}}}$, so if we apply stationary phase to $j = 2,4$ terms at some fixed $\phi \in (\phi_-, \phi_+)$, the amplitudes match those in the $j = 1,3$ terms. Proposition 4.2.4 implies that the phases match as well, so using the fact that the K_j are a partition of unity when lifted to Σ, we get (4.2.6). The statement (4.2.7) is obvious since the $j = 2$ term does not have any critical points away from the projection of U_2 and the $j = 1,3$ are supported away from $(\phi_-, 0)$. Now take the real part of v_{h_k}. It satisfies the equation (4.2.5) with $E(h)$ replaced by its real part. The principal symbol of $\overline{v}_{h_k}$ is

$$\sigma(\overline{v}_{h_k})(\phi,\tau) = \overline{\sigma(v_{h_k})}(\phi,-\tau) = \sigma_0$$

so $||\mathrm{Re}(v_{h_k})||_{L^2(I)} = 1 + O(h)$. It follows that L^2 normalizing $\mathrm{Re}\, v_{h_k}$ only multiplies the lower order terms in the full symbol by a constant. And therefore the expression for the leading part of $||\mathrm{Re}\, v_{h_k}||^{-1}_{L^2(I)}\mathrm{Re}\, v_{h_k}$ is the same as (4.2.6).

4.2.2. Comparison of the quasi-mode to the mode

Here we show that the quasi-mode v_{h_k} of proposition 4.2.2 is locally uniformly close to the true mode $u_{h_k} = UP^{m_k}_{N_k}$.

Proposition 4.2.5. *Let $v_{h_k} \in \mathcal{O}^0(I,\Sigma,h_k)$ be as in proposition 4.2.2 and u_{h_k} be the L^2 normalized, real-valued function satisfying $H_{h_k,c}u_{h_k} = u_{h_k}$. Let Π denote orthogonal projection onto* $\ker H_{h_k,c} - 1$. *Then*

$$||v_{h_k} - \Pi v_{h_k}||_{L^2(I)} = O(h_k^\infty) \tag{4.2.22}$$

Proof. From the spectral gap (4.2.2), it follows that the lower bound

$$||(H_{h_k,c} - 1)u||_{L^2(I)} \geq \delta h_k ||u||_{L^2(I)}$$

holds for $u \in (\ker H_{h_k,c} - 1)^\perp$. The estimate

$$||(H_{h_k,c} - E(h))v_{h_k}||_{L^2(I)} = O(h_k^\infty)$$

implies that there is an eigenvalue of H_{h_k} in an $O(h_k^N)$ neighborhood of $E(h)$ for all large N. Since $E(h) = 1 + O(h^2)$ and the eigenvalues of $H_{h_k,c}$ are separated by $O(h)$ distances, this means that $E(h) = 1 + O(h_k^\infty)$. Therefore

$$(4.2.23) \qquad ||(H_{h_k,c} - 1)(v_{h_k} - \Pi v_{h_k})||_{L^2(I)} = O(h_k^\infty)$$

which proves the estimate in view of the lower bound above.

□

Proposition 4.2.6. *For each $\delta > 0$, with $I_\delta = (\delta, \pi - \delta)$,*

$$(4.2.24) \qquad ||v_{h_k} - u_{h_k}||_{L^\infty(I_\delta)} = O_\delta(h_k^\infty)$$

Proof. Writing $\partial_\phi^2 = -h_k^{-2}(H_{h_k,c} - V)$. we have

$$||\partial_\phi^2(v_{h_k} - \Pi v_{h_k})||_{L^2(I_\delta)} \leq h_k^{-2}\left(||(H_{h_k,c}(v_{h_k} - \Pi v_{h_k})||_{L^2(I_\delta)} + ||V(v_{h_k} - \Pi v_{h_k})||_{L^2(I_\delta)}\right)$$

From proposition 4.2.5, $||H_{h_k,c}(v_{h_k} - \Pi v_{h_k})||_{L^2(I)} = O(h_k^\infty)$ and since V is bounded on I_δ depending on δ, the right hand side is $O_\delta(h_k^\infty)$. Applying the Sobolev estimate

$$||f||_{L^\infty} \leq C||f'||_{L^2}$$

twice on the interval I_δ together with the above inequality yields

$$(4.2.25) \qquad ||v_{h_k} - \Pi v_{h_k}||_{L^\infty(I_\delta)} = O_\delta(h_k^\infty).$$

Similarly, we have

$$(4.2.26) \qquad ||u_{h_k}||_{L^\infty(I_\delta)} \leq C||u''_{h_k}||_{L^2(I_\delta)} = h_k^{-2}||\left(H_{h_k,c} - V\right)u_{h_k}||_{L^2(I_\delta)} = O_\delta(h_k^{-2})$$

so

$$(4.2.27) \quad ||v_{h_k} - u_{h_k}||_{L^\infty(I_\delta)} \leq ||v_{h_k} - \Pi v_{h_k}||_{L^\infty(I_\delta)} + ||(\zeta(h_k) - 1)u_{h_k}||_{L^\infty(I_\delta)} = O_\delta(h_k^\infty)$$

Where we have written $\Pi v_{h_k} = \zeta(h_k)u_{h_k}$ and $\zeta(h_k) = 1 + O(h_k^\infty)$ since v_{h_k} is real valued and positive in a neighborhood of $\phi = \phi_-$.

$\square$

4.3. Airy Expansion of the Quasi-mode

The goal of this section is to prove the following Airy expansion for v_{h_k} in a neighborhood of the turning points $\phi_\pm$.

Proposition 4.3.1. *Let v_{h_k} be the quasi-mode in proposition 4.2.2. There exists $\varepsilon > 0$ such that for $\phi_- < \phi < \phi_- + \varepsilon$, $v_{h_k}(\phi)$ has the full asymptotic expansion*

$$(4.3.1) \quad v_{h_k}(\phi) \sim Ai\left(-h^{-\frac{2}{3}}\rho(\phi)\right) h_k^{-\frac{1}{6}} \sum_{n=0}^{\infty} u_{0,n}(\phi)h^n + Ai'\left(-h^{\frac{2}{3}}\rho(\phi)\right) h^{\frac{1}{6}} \sum_{n=0}^{\infty} u_{1,n}(\phi)h^n$$

The leading part of the expansion is

$$(4.3.2) \qquad v_{h_k}(\phi) \sim \sqrt{\sin\phi}\, h_k^{-\frac{1}{6}} \left(\frac{4\rho(\phi)}{\sin^2\phi - c^2}\right)^{\frac{1}{4}} Ai\left(-h_k^{-\frac{2}{3}}\rho(\phi)\right) + O(h_k^{\frac{1}{6}})$$

Here, the argument of the Airy function is

$$\rho(\phi) = \left(\frac{3}{4}\int_{\gamma_\phi} \alpha\right)^{\frac{2}{3}} \tag{4.3.3}$$

where γ_ϕ is the arc on Σ passing through $(\phi_-, 0)$ from (ϕ, τ_-) to (ϕ, τ_+).

To prove this we write

$$v_{h_k}(\phi) = (2\pi h_k)^{-\frac{1}{2}} \int a(\tau, h) \exp i\left(\frac{\psi_2(\phi,\tau)}{h_k} + \beta_2\right) + O(h_k^\infty) \tag{4.3.4}$$

For ϕ in a neighborhood of the turning point $(\phi_-, 0)$. The expansion is a consequence of the following proposition from Hörmander:

Proposition 4.3.2 (Ho1, Theorem 7.7.18)**.** *Let $f(t,x)$ be a real-valued smooth function defined in a neighborhood $(0,0) \in V \subset \mathbb{R}^2$. Suppose that $\partial_t f(0,0) = \partial_t^2 f(0,0) = 0$ and that $\partial_t^3 f(0,0) \neq 0$. Then there exists smooth, real-valued functions $a(x), b(x)$ and smooth compactly supported functions $u_{0,n}(x)$, $u_{1,n}(x)$ such that*

$$\begin{aligned} e^{-\frac{i}{h}b(x)} \int u(t,x) e^{\frac{i}{h}f(t,x)}\, dt \sim & Ai(h^{-\frac{2}{3}}a(x))h^{\frac{1}{3}} \sum_0^\infty u_{0,n}(x)h^n \\ & + Ai'(h^{-\frac{2}{3}}a(x))h^{\frac{2}{3}} \sum_{n=0}^\infty u_{1,n}(x)h^n \end{aligned} \tag{4.3.5}$$

For a smooth, compactly supported amplitude $u(t,x)$ supported sufficiently close to $(0,0)$.

4.3.1. Proof of Proposition 4.3.1

As explained in [**22**], page 234 the functions $a(x)$ and $b(x)$ can be calculated by putting the phase function into the following cubic normal form:

Proposition 4.3.3 (Ho1, Theorem 7.5.13)**.** *Let $f(t,x)$ be a real valued smooth defined in a neighborhood $(0,0) \in V \subset \mathbb{R}^2$ such that $\partial_t f(0,0) = \partial_t^2 f(0,0) = 0$ and $\partial_t^3 f(0,0) \neq 0$. Then there exists a real-valued smooth function $T(t,x)$ in a neighborhood of $(0,0)$ with $T(0,0) = 0$, $\partial_t T(0,0) > 0$ and smooth functions $a(x), b(x)$ such that*

$$f(t,x) = \frac{T^3(t,x)}{3} + a(x)T(t,x) + b(x). \tag{4.3.6}$$

We apply this theorem to the phase

$$\psi_2(\phi,\tau) = \phi\tau - G_2(\tau)$$

It has a degenerate critical point at the turning point$(\phi_-,0)$. Indeed, by differentiating the Eikonal equation,

$$\tau^2 + \frac{c^2}{\sin^2 G_2'(\tau)} = 1 \tag{4.3.7}$$

We see that $\partial_\tau^2 \psi_2(\phi_-,0) = -G_2''(0) = 0$ and $\partial_\tau^3 \psi_2(\phi_-,0) = -G_2'''(0) = \frac{4c}{\sqrt{1-c^2}} \neq 0$. The functions $a(x)$ and $b(x)$ are calculated in the next proposition.

Proposition 4.3.4. *There exists a smooth function $T(\phi,\tau)$ in a neighborhood of $(\phi_-,0)$ as in proposition 4.3.3 such that*

$$\psi_2(\phi,\tau) = \frac{T^3(\phi,\tau)}{3} + a(\phi)T(\phi,\tau) + b(\phi) \tag{4.3.8}$$

If $\phi_- < \phi_- + \varepsilon$, then

$$a(\phi) = -\left(\frac{3}{4}\int_{\gamma_\phi} \alpha\right)^{2/3} \tag{4.3.9}$$

$$b(\phi) = \beta_2 = 0 \tag{4.3.10}$$

Where γ_ϕ is the arc on Σ defined in proposition 4.3.1.

Proof. Existence follows from proposition 4.3.3. Put $\rho(\phi) = -a(\phi)$. Take the τ-derivative of (4.3.8) and observe that $\partial_\tau \psi_2(\phi, \tau) = \phi - G_2'(\tau) = 0$ if and only if $T^2(\phi, \tau) = \rho(\phi)$. For a fixed $\phi \in (\phi_-, \phi_- + \varepsilon)$, let $\tau_\pm$ be the two τ-critical points of ψ_2, the τ-coordinates of the two points $(\phi, \tau_\pm) \in \Sigma$ lying over ϕ,

$$\tau_\pm(\phi) = \pm\sqrt{1 - \frac{c^2}{\sin^2\phi}} \tag{4.3.11}$$

Since $T^2(\phi, \tau_\pm(\phi)) = \rho(\phi)$, we can write $T(\phi, \tau_+(\phi)) = -\sqrt{\rho(\phi)}$ and $T(\phi, \tau_-(\phi)) = \sqrt{\rho(\phi)}$. These imply that $\psi_2(\phi, \tau_\pm) = \mp\frac{\rho(\phi)^{3/2}}{3} - \pm\rho^{3/2}(\phi) + b(\phi)$ which means

$$\frac{4}{3}\rho^{3/2}(\phi) = \psi_2(\phi, \tau_+) - \psi_2(\phi, \tau_-) \qquad 2b(\phi) = \psi_2(\phi, \tau_+) + \psi_2(\phi, \tau_-) \tag{4.3.12}$$

The formulas then follow since ψ_2 is odd in τ and $\Psi_2(\tau) = \psi_2(G_2'(\tau), \tau)$ is a primitive for $\alpha|_{U_2}$.

$\square$

Now let $\chi(\phi)$ be a bump function equal to one on $(\phi_- - \frac{\varepsilon}{2}, \phi_- + \frac{\varepsilon}{2})$ and supported in $(\phi_- - \varepsilon, \phi_- + \varepsilon)$. The amplitude $\chi(\phi)a_2(\tau, h)$ appearing in (4.3.4) will then have no critical points outside of a τ neighborhood $B_r(0)$ of $\tau = 0$, $r = o(\varepsilon)$. Split up the integral by inserting a τ cutoff $\eta(\tau)$, $\chi(\phi)a_4(\tau, h) = \chi(\phi)\eta(\tau)a(\tau, h) + \chi(\phi)(1 - \eta(\tau)a_2(\tau, h)$ supported on $B_r(0)$, equal to 1 on $B_{r/2}(0)$. If ε is small enough, the first term is supported close enough to $\tau = 0$ to apply proposition 4.3.2, and the second term is $O(h_k^\infty)$. Finally, we calculate the leading order amplitude $u_{0,0}(\phi)$ appearing in the expansion. The leading term is

$$v_{h_k}(\phi) \sim (-1)^{m_k + N_k}(2\pi)^{\frac{1}{2}} h_k^{-\frac{1}{6}} u_{0,0}(\phi) Ai(-h_k^{-\frac{2}{3}}\rho(\phi)). \tag{4.3.13}$$

We compare this to the standard expansion of the Airy function for large $t > 0$ (see [**22**] pg. 215)

$$Ai(-t) \sim \frac{1}{\sqrt{\pi} t^{1/4}} \cos(\frac{2}{3} t^{\frac{3}{2}} - \frac{\pi}{4}) \tag{4.3.14}$$

this means that when $h^{-\frac{2}{3}}\rho(\phi) >> 0$,

$$v_{h_k}(\phi) \sim u_{0,0}(\phi) \left(\pi^{-\frac{1}{2}} \rho(\phi)^{-\frac{1}{4}} \sin\left(h_k^{-1} \int_{\gamma_\phi} \alpha + \frac{\pi}{4} \right) \right) \tag{4.3.15}$$

But this must match the leading term in proposition 4.2.2 which forces

$$u_{0,0}(\phi) = \sqrt{\sin\phi} \left(\frac{4\rho(\phi)}{\sin^2\phi - c^2} \right)^{\frac{1}{4}} \tag{4.3.16}$$

4.4. Geometric Interpretation of the Expansion

Recall that the generator of rotations, $D_\theta = -i\partial_\theta$ commutes with the Laplacian on S^2. The Clairaut integral, $p_\theta(x,\xi) = \langle \xi, \partial_\theta \rangle_x$ is the symbol of the D_θ so $\{p_\theta, q\} = 0$ where $q(x,\xi) = |\xi|_x$. Together they generate a homogeneous Hamiltonian torus action, $\Phi(t,s)$ on T^*S^2,

$$\Phi(t,s,x,\xi) = \exp sX_{p_\theta} \circ \exp tX_q(x,\xi) \tag{4.4.1}$$

which acts transitively on the level sets of the moment map,

$$\mu : T^*S^2 \to \Gamma \subset \mathbb{R}^2 \qquad \mu(x,\xi) = (p_\theta(x,\xi), q(x,\xi)) \tag{4.4.2}$$

whose image is the cone $\Gamma = \{(x,y) \mid y \geq 0, |x| \leq y\}$. Since μ is homogeneous, we need only consider level sets with $q = 1$. For $c \in [-1,1]$ set $T_c = \mu^{-1}(1,c) = S^*S^2 \cap \{p_\theta = c\}$. For $c \in (-1,1)$, T_c is Lagrangian torus inside of S^*S^2. When $c = \pm 1$, T_c is just the lift of the standard equator γ_e to S^*S^2. In terms of the dual polar coordinates $(\phi, \tau, \theta, \eta)$ on T^*S^2,

$$T_c = \{(\phi, \tau, \theta, \eta) \mid \tau^2 + \frac{c^2}{\sin^2 \phi} = 1; \eta = c\} \tag{4.4.3}$$

Therefore its projection to S^2 is $\pi(T_c) = \{(\phi, \theta) \mid \sin\phi \geq c\}$. The projection is an annular band consisting of all geodesics which make the fixed angle $\arccos c$ with the equator. The energy curve associated with the associated Legendre functions is just the (ϕ, τ) cross-section of T_c. For x in the interior of $\pi(T_c)$, let γ_x be the geodesic arc connecting the two points lying above x in T_c, from (x, ξ_-) to (x, ξ_+), where the sign corresponds to the sign of τ. It is clear that $\int_{\gamma_x} d\theta = 0$ since there is no change in the θ coordinate across the arc. But the canonical 1-form is $\alpha = \tau d\phi + \eta d\theta$. Since $\eta = c$ is constant on T_c the second term contributes nothing to the integral over γ_x, and the first term is clearly equal to the integral (4.3.3) in the Legendre function expansion. The density

$$d\mu_c = \frac{|d\phi| \otimes |d\theta|}{(2\pi)^2 |\tau|} \tag{4.4.4}$$

is invariant under the joint flow on T_c and

$$\pi_* d\mu_c = \frac{1}{(2\pi)^2} \frac{\sqrt{\sin\phi}\ |d\phi| \otimes |d\theta|}{(\sin^2\phi - c^2)^{\frac{1}{2}}}$$

which verifies the formula (4.1.4). The reason for the Airy bump at the caustic latitude circles is the presence of a fold singularity for the projection $\pi|_{T_c} : T_c \to S^2$. Recall that a smooth map $f : X^n \to Y^n$ between n-dimensional manifolds is said to have a fold singularity with fold locus S if there exists a codimension one submanifold $S \subset X$ such that

(1) S is equal to the set of critical points of the map f, i.e. $S = \{x \in X \mid df_x \text{ is not surjective } \}$
(2) For each $s \in S$, the kernel of df_s is transverse to the tangent space $T_s S$.

Proposition 4.4.1. *The projection $\pi|_{T_c} \to S^2$ is a folding map with fold locus $S = S_+ \cup S_-$,*

$$S_\pm = \{(\phi_\pm, \theta, 0, c) \mid \theta \in [0, 2\pi)\}$$

where $\phi_\pm$ *are the two solutions of* $\sin\phi = c$*. The images of* $S_\pm$ *are the latitude circles which bound* $\pi(T_c)$*.*

Proof. Writing (ρ, η) as dual coordinates to (ϕ, θ), T_c is cut out by the equations $\eta = c$ and $\rho^2 + \frac{c^2}{\sin^2\phi} = 1$. differentiating the second equation gives

$$\rho d\rho - c^2 \frac{\cos\phi}{\sin^3\phi} d\phi = 0$$

so writing $x = (\phi, \theta)$, $\xi = (\rho, \eta)$,

$$T_{(x,\xi)}T_c = \{\alpha\partial_\phi + \beta\partial_\theta + \gamma\partial_\rho \mid \rho\gamma = c^2 \frac{\cos\phi}{\sin^3\phi}\alpha\}$$

So for $v \in T_(x, \xi)T_c$, $d\pi v = 0$ if and only if $\alpha = \beta = 0$. But then $\rho\gamma = 0$. If $\rho = 0$, then $v = 0$, so the kernel of $d\pi$ is non-trivial only when $\rho = 0$, and this means that $(x, \xi) \in S$. At such points, the kernel of $d\pi$ is the span of ∂_ρ, which is transverse to $T_{(x,\xi)}S = \mathbb{R}\partial_\theta$.

$\square$

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