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Essays on Heterogeneous Beliefs in Financial Markets

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ABSTRACT

Essays on Heterogeneous Beliefs in Financial Markets

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In this thesis, I investigate how the disagreements among market participants can affect markets in various settings. In the first chapter, I study how market participants with heterogeneous beliefs and non-commitment can create and manage counterparty risk in a sequentially and bilaterally traded market. I find that the equilibrium price may not always reflect counterparty risk due to risk-management efforts by market participants. Even when there is no default in equilibrium, market participants cannot attain the best allocations since risk-management is costly. In the second chapter, I study disagreements among market participants under more general belief structures. Here, I employ the collateral equilibrium framework to study the how the disagreements can affect equilibrium pricing of assets and derivatives. I provide sufficient conditions for bubble to exist in equilibrium prices. Moreover, I find that certain types of disagreements can also generate volatility smirks in options. In chapter three, I study asynchronized trading among market participants in presence of a growing asset bubble. Market participants disagree on the starting date of an exogenous asset bubble and decide when to exit the market. I also

introduce a large market participant alongside numerous infinitesimal market participants to study their interactions and the mechanism of the bursting an asset bubble. I find results in contrast to those in the currency attack literature. The market participants in this setting stand to benefit from a growing asset bubble whereas the market participants in the currency attack literature only benefit if an attack is successful.

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CHAPTER 1

Counterparty Risk in the Over-the-Counter Derivatives Market: Heterogeneous Insurers with Non-commitment

1.1. Introduction

Central to any OTC derivative market is the bilateral nature of the trades that involves counterparty risk, which is the risk that trading counterparties may default on their future obligations. Counterparty risk entered the spotlight when major players in the OTC derivative market, e.g. Lehman Brothers and AIG, either declared bankruptcy or were bailed out by the government during the 2007-08 global financial crisis. The financial crisis raised an important question of whether the OTC derivatives market participants can adequately manage counterparty risk themselves, without regulations such as the mandated central clearing of OTC derivatives.

How do market participants manage counterparty risk? Empirically, market participants have been shown to manage counterparty risk through counterparty selection (Du et al., 2016) and hedging[1] (Gündüz, 2016). However, in the growing theory literature on counterparty risk, there has been little focus on counterparty risk management strategies besides margins. In particular, good and bad insurers do not coexist in existing models.[2]

[1]Hedging refers to purchasing credit default swaps on the counterparties.

[2]Papers studying counterparty risk, e.g. Biais et al. (2016), Stephens and Thompson (2014), typically model derivative contracts as insurance and study one-sided risk-taking of sellers or insurers. So does this chapter. Bad insurers are insurers who take on risks that generate a negative externality on others. Good insurers hedge their positions. In Biais et al. (2016), insurers who sell insurance are homogeneous.

Thus, in these models, market participants seeking to buy insurance have no choice but to contract with bad insurers. The contribution of this chapter is to study a novel setting in which good and bad insurers coexist. This setting is necessary for studying how market participants can manage counterparty risk. Though good and bad insurers coexist in this model, the roles are determined endogenously.

I model OTC derivative contract as insurance. The model features a risk-averse hedger who seeks insurance against her future risky endowment. The hedger can buy insurance from two insurers, who are heterogeneous in beliefs about the hedger's endowment. The *optimist* is more optimistic about the hedger's endowment than the *pessimist*. Because of their difference in beliefs, the optimist and the pessimist may wish to speculate with each other after selling insurance to the hedger. Ex-ante, the insurers cannot commit to not speculating. This is the source of counterparty risk, as the insurers' speculation with each other may devalue the hedger's claim. Because the insurers speculate with *each other*, the good insurer is bound to have enough money to insure the hedger.[3] However, which insurer is good may very well depend on which insurer has sold insurance to the hedger. For example, after the hedger buys insurance from the optimist, the optimist may want to sell the same insurance to the pessimist and potentially devalue the hedger's insurance. Here the pessimist is good. Realizing this, the hedger may want to purchase insurance from the pessimist. However, when the hedger buys insurance from the pessimist, the pessimist may also want to sell the same insurance to the optimist and possibly devalue the hedger's insurance. Given the hedger purchases insurance from the pessimist, the optimist

Thus, insurers are either all bad or all good. In Stephens and Thompson (2014), insurers are all bad but take on varying amount of risks that harm others.

[3]Suppose at least one insurer has enough money to insure the hedger.

becomes good. Thus, the good insurer who does not devalue the hedger's insurance may prove to be ever elusive.

The model builds on two important characteristics of the OTC derivatives market. First, agents may be heterogeneous in beliefs. The heterogeneous beliefs can be a stand-in for heterogeneity in agents' asset positions. For example, agents may have offsetting exposures so they can insure each other. However, if the agents were to sell insurance to a hedger, the agents may change the insurance they sell to each other. Second, there is non-commitment. Agents can always trade with other agents between the time they sign a contract and the maturity of that contract. For example, the typical maturity of a credit default swap (CDS) is five years. So in these five years, a CDS seller may have the incentive to engage in activities that devalue the CDS she has sold. Imagine a firm buying a five-year CDS contract from AIG before the financial crisis. There is no way[4] for the firm to prevent AIG from selling CDS contracts to the point of near-bankruptcy.

A key insight of my analysis is that when there is a bad insurer who devalues her existing contracts, there is always a good insurer who hedges her existing contracts. When the insurers speculate with each other, they shift wealth across states. However, because the insurers speculate with *each other*, both insurers cannot shift wealth out of the same state. If the optimist shifts her wealth out of the state in which the hedger requires insurance payment, the pessimist cannot also shift her wealth out of that same state. Moreover, in that state, the pessimist holds the optimist's endowment as well. So, the pessimist can now fully insure the hedger even if the pessimist's own endowment were not enough. In this case, the optimist is the bad insurer while the pessimist is the good

[4]Assume the firm cannot require 100% initial margin and the variation margin is subject to valuation disputes.

insurer. This intuition holds even when the identities of the good and bad insurers are determined endogenously. Since the good and bad insurers coexist, the hedger is not limited to buying insurance from only the bad insurer. In the worst case, the hedger can always buy insurance from both insurers to ensure delivery of payment on at least one of the contracts. Under certain conditions, the hedger can do better and only buy insurance from one insurer.

In equilibrium, the hedger can manage counterparty risk by choosing either trade size or counterparty. When the hedger chooses to contract with both insurers, exactly one insurer defaults. Nevertheless, the hedger pays the full price for both contracts in order to ensure the contracts are incentive compatible. So while the contracts traded in equilibrium look risk-free, it doesn't mean there is no counterparty risk. When the hedger chooses to contract with only one insurer, she buys either cheap partial insurance from the optimist or expensive full insurance from the pessimist. In this case, the hedger buys only risk-free contracts. However, having only risk-free contracts traded in equilibrium does not mean there is no counterparty risk. In this case, the hedger instead chooses the suboptimal risk-free contracts precisely because better contracts are risky with counterparty risk. In general, when no risky contract is traded in equilibrium, it is possible that the counterparty risk is so severe that no one wants to trade any risky contracts.

Recent regulations mandate central clearing of standardized OTC derivatives in an effort to reduce counterparty risk. Central clearing is an important change to the OTC derivative market. I study the effect of central clearing in this setting. I focus on the loss-sharing ability of central clearing as opposed to the ability of central clearing to enforce margin requirement. The agents in the model do not have any cash upfront, so margin requirements do not apply. I find that central clearing improves the hedger's welfare beyond what the hedger can achieve by trying to manage counterparty risk without central clearing. The key difference is that central clearing can reduce the speculations between the insurers while the hedger cannot.

The rest of the chapter is organized in the following manner. Section 1.2 reviews the related literature. Section 2.3 defines the baseline model when hedging is infeasible and proceeds to present the benchmark. Section 1.4 analyzes the equilibrium under different parameters. Section 1.5 then analyzes the effect of central clearing. Section 1.6 examines hedging and hedging costs. Section 1.7 discusses the assumptions of the model, and finally, Section 1.8 concludes this chapter. The proofs and additional analysis can be found in Appendix A.

1.2. Related Literature

This chapter is a study of sequential trading under non-commitment. It is closely related to Coase (1972), Bizer and DeMarzo (1992), Bisin and Rampini (2006) and subsequent papers. Nevertheless, the mechanism of non-commitment in this chapter differs from that of the others. In Coase (1972) (Bizer and DeMarzo, 1992; Bisin and Rampini, 2006), the seller (borrower) cannot commit to not selling to (borrowing from) other buyers (lenders) in subsequent periods, respectively. In this chapter, however, sellers cannot commit to not trading with other sellers in the subsequent period. Thus, in this model, sellers with bad incentives coexist with sellers with good incentives. Moreover, the natural insurance providers are exactly the sellers with bad incentives. Though buyers can overcome the non-commitment problem of sellers by trading through sellers with good incentives, the first-best allocation cannot be achieved. Moreover, the allocation in equilibrium is sensitive to the wealth of both types of sellers.

This chapter is also closely related to the theory literature on counterparty risk, e.g. Thompson (2010), Stephens and Thompson (2014), Biais et al. (2016). This chapter is closest to Stephens and Thompson (2014) and Biais et al. (2016). Stephens and Thompson (2014) study the case when insurance buyers have varying degrees of aversion to default, modeled with heterogeneous non-pecuniary default costs. While this chapter and Stephens and Thompson (2014) both study the trade-offs between price and risk, the focuses are different. Stephens and Thompson (2014) focus on insurance buyer's incentive to avoid bad insurance seller; I take the insurance buyer's incentives as given and study how the insurance buyer manages counterparty risk. Biais et al. (2016) consider insurance sellers' hedging incentives, which can be distorted by bad news, moral hazard, and limited liability. The insurance sellers who trade with insurance buyers in their model are homogeneous while insurance sellers in this chapter are heterogeneous. The heterogeneity of the insurers in this chapter allows the hedger more flexibility in terms of counterparty choice. As a result, the hedger's optimal contract in this chapter features interesting counterparty risk management strategies, with novel empirical implications.

In the literature on financial intermediation, this chapter is closest to Babus and Hu (2017). In both Babus and Hu (2017) and this chapter, financial intermediation arises endogenously due to non-commitment. Agents in Babus and Hu (2017) solve the non-commitment problem using information network and repeated games. This chapter differs in that the hedger can contract with multiple insurers. Moreover, equilibrium in this chapter may feature default.

This chapter is also related to the theory literature on central clearing, e.g. Pirrong (2011), Duffie and Zhu (2011), Acharya and Bisin (2014), Stephens and Thompson (2014). Pirrong (2011) provides an extensive overview of central clearing. Duffie and Zhu (2011) discuss benefit of single central counterparty. Acharya and Bisin (2014) discuss the ability of central clearing to increase market transparency. Stephens and Thompson (2014) focus on loss-sharing ability of central clearing as I do. However, Stephens and Thompson (2014) focus on ex-ante contribution by insurers to cover potential loss of the central counterparty, while I focus on ex-post loss-sharing.

This chapter complements the search theory literature on OTC market, e.g. Duffie et al. (2007), Lagos et al. (2011). This chapter's focus is the strategic risk-taking and counterparty risk. While matching in search models are typically random, the hedger in this chapter chooses her counterparties to manage counterparty risk. This chapter also complements Chang and Zhang (2015), which studies endogenous network formation. While Chang and Zhang (2015) focuses on network formation with exogenous risk, this chapter focuses on endogenous risk in a network in which all agents are connected to each other.

This chapter is also related to the empirical literature on counterparty risk in OTC derivative market, e.g. Arora et al. (2012), Du et al. (2016), Gündüz (2016). My results are similar to the price implication in Arora et al. (2012), counterparty selection in Du et al. (2016), and hedging of OTC derivative contract in Gündüz (2016). I offer novel empirical implications.

1.3. Model

I model OTC derivatives as insurance contracts. There are three dates, $t = 0, 1, 2$, one hedger, and two insurers with heterogeneous beliefs. The hedger wishes to purchase insurance from the insurers. At $t = 0$, the hedger makes take-it-or-leave-it offers to insurers and insurers can choose whether to accept the offers. At $t = 1$, insurers trade with each other. At $t = 0, 1$, contracts are agreed upon but no money changes hands. At $t = 2$, money changes hands as payments are made.

1.3.1. Agents and Beliefs

Hedger H is risk-averse with twice-differentiable strictly concave utility function u, and is endowed with one unit of risky asset with a random payoff R in $t = 2$. For simplicity, I normalize $R \in \{0, 1\}$; I refer to the state in which $R = s$ as state s at $t = 2$. H has the belief that state 1 happens with probability π and state 0 happens with probability $1 - \pi$. I assume H has all the bargaining power when trading with insurers. This assumption is sufficient but not necessary. As long as H has some bargaining power, I get similar results.

The two insurers are risk-neutral. One insurer is more optimistic about R with the belief that state 1 happens with probability $\pi' > \pi$. I shall refer to this insurer as optimist O. The other insurer, pessimist P, shares H's belief that state 1 happens with probability π. It is central to this model that O is more optimistic about R than P. This assumption gives us the non-commitment friction that is at the heart of the model. The belief of H relative to the beliefs of insurers is of no consequence. I choose P having the same belief as H to ensure H is willing to purchase insurance from both O and P. I shall discuss the implications of different assumptions about beliefs in section 1.7.1.

Both insurers are endowed with cash, or constant endowments, at $t = 2$. O is endowed with w_O while P is endowed w_P. I make the following assumption to ensure O has enough wealth to insure H.

Assumption 1.1. $w_O \geq \frac{\pi'}{1-\pi'}$.

$\frac{1-\pi'}{\pi'}$ is the price of insurance that makes O break-even. Since H only has endowment of 1 in state 1, H can only purchase up to $1/\frac{1-\pi'}{\pi'} = \frac{\pi'}{1-\pi'}$ units of insurance. Thus, as long as O has wealth higher than $\frac{\pi'}{1-\pi'}$, O can fully insure H at price $\frac{1-\pi'}{\pi'}$. This bound is sufficient but not necessary since $\frac{\pi'}{1-\pi'}$ is the upper bound on how much insurance H can purchase.

When O trades with P at $t = 1$, I assume O has all the bargaining power. This is for modeling convenience. Changing the bargaining power between O and P has no material effect on the model. Since all endowments arrive at $t = 2$, all agents maximize expected utility of consumption at $t = 2$. Moreover, all endowments and beliefs are common knowledge. At $t = 1$, everything, including contracts and decisions at $t = 0$, are common knowledge. Since many objects defined in this chapter are functions of realization of R, I refer to $x(s)$ as the value x takes in state $s \in \{0, 1\}$ for any object x, respectively.

1.3.2. Contracts and Trading

At $t = 0$, H makes a take-it-or-leave-it offer[5] to insurer $i \in \{O, P\}$ with contract

$$\tau_{H,i} \equiv (\tau_{H,i}(0), \tau_{H,i}(1)) \in \mathbb{R}_+ \times [-1, 0]. \tag{1.1}$$

Contract terms $\tau_{H,i}(0)$ and $\tau_{H,i}(1)$ specify transfer from i to H in states 0 and 1, respectively. Positive value represents transfer from i to H while negative value represents transfer from H to i. I restrict the attention to $\tau_{H,i} \in \mathbb{R}_+ \times [-1, 0]$, since H can only credibly promise payment in state 1 and H would never consider a contract $\tau_{H,i} \in \mathbb{R}_{--} \times [-1, 0]$.

At t=1, O makes a take-it-or-leave-it offer[6] to P with contract

$$\tau_{O,P} \equiv (\tau_{O,P}(0), \tau_{O,P}(1)) \in \mathbb{R} \times \mathbb{R} \tag{1.2}$$

The terms are defined similarly. Positive value represents transfer from P to O while negative value represents from O to P. For now, there are no restrictions on $\tau_{O,P}$ as there is on $\tau_{H,i}$. In Assumption 1.3, I assume P is wealthy enough so that O always wants to sell insurance to P, i.e. $\tau_{O,P} \in \mathbb{R}_- \times \mathbb{R}_+$. Later, I will relax the assumption and study the problem without restrict the direction of O's contract with P.

I assume $\tau_{O,P}$ is senior to $\tau_{H,O}$ and $\tau_{H,P}$ in the sense that O and P's claims in $\tau_{O,P}$ are paid out before H's claim is paid out from $\tau_{H,O}$ and $\tau_{H,P}$. Moreover, since everything is common knowledge, O and P can only credibly promise each other their wealth plus any transfer from H. Thus, O and P have commitment with each other. The seniority

[5]As long as H has some bargaining power, the same intuition goes through. I do not consider the case when H has no bargaining power since my focus is on strategic behavior of H. See section 1.7.1.

[6]See section 1.7.1.

assumption and insurers' commitment to each other resemble the usage of collateral. I discuss this model in relation to collateral usage in section 1.7.2.

For easy comparison between contracts, I define the price of any contract in $t = 0, 1$ as

$$q(\tau_{i,j}) \equiv \left| \frac{\tau_{i,j}(1)}{\tau_{i,j}(0)} \right|. \tag{1.3}$$

This price represents the amount contract buyer (seller) pays (receives) in state 1 per unit of wealth she receives (pays) in state 0, respectively. Moreover, I denote the standardized contract with price q as

$$\tau_q \equiv (1, -q). \tag{1.4}$$

This helps simplify the notation.

1.3.3. Insurers' Problems

First, I state P's value function. Then, I state O's problem. Given contracts $\tau_{H,P}$ and $\tau_{O,P}$, P's value function is

$$U_P\left(\tau_{O,P}, \tau_{H,P}\right) \equiv \mathbb{E}_P\left[\left(w_P - \tau_{O,P} - \tau_{H,P}\right)^+\right]$$

Whenever P is indifferent between accepting or not accepting any contract, I assume P accepts the contract. Given (1.1), H only buys insurance. Moreover, there is commitment between O and P. Thus, P's time 2 wealth, i.e. $w_P - \tau_{O,P} - \tau_{H,P}$, can only be negative in state 0. This is useful. For example, suppose P's time 2 wealth in state 0 is positive. Then, the $()^+$ operator from P's value function can be removed.

Now I define O's problem. Given contract $\tau_{H,O}$, O solves at $t = 1$

$$U_O\left(\tau_{H,O}, \tau_{H,P}\right) \equiv \max_{\tau_{O,P}} \hat{U}_O(\tau_{O,P}|\tau_{H,O}, \tau_{H,P}) \equiv \mathbb{E}_O\left[\left(w_O + \tau_{O,P} - \tau_{H,O}\right)^+\right] \tag{1.5}$$

subject to P's individual rationality constraint

$$U_P\left(\tau_{O,P}, \tau_{H,P}\right) \geq U_P\left((0,0), \tau_{H,P}\right) \tag{IR-P}$$

and budget constraints for both insurers

$$-\tau_{O,P} \leq w_O - \left(\tau_{H,O}\right)^-, \tag{BC-O}$$

$$\tau_{O,P} \leq w_P - \left(\tau_{H,P}\right)^-. \tag{BC-P}$$

Given common knowledge, O can credibly promise to P as much as O's wealth as well as any promises from H to O, i.e. $-\left(\tau_{H,O}\right)^-$. This is represented by O's budget constraint. This is where I assume O has commitment to P. Moreover, O can default on $\tau_{H,O}$. So, only the promises from H to O enter into (BC-O). Similarly, only the promises from H to P enter into (BC-P). Thus, the budget constraints also assumes seniority of $\tau_{O,P}$.

1.3.4. Additional Assumptions

O may be indifferent between several contracts that O can offer to P. I make the following assumption.

Assumption 1.2 (Tie-breaking). *Given $\tau_{H,O}$ and $\tau_{H,P}$, suppose there are 2 contracts τ^1 and τ^2 such that both contracts satisfy* (IR-P), (BC-P), (BC-O), *and* $\hat{U}_O(\tau^1|\cdot) = \hat{U}_O(\tau^2|\cdot)$. *O prefers τ^i such that $U_P(\tau^i, \tau_{H,P}) \leq U_P(\tau^j, \tau_{H,P})$ for $i \neq j \in \{1, 2\}$.*

The above assumption states that when O is indifferent between offering two contracts, O would choose the one that gives P less expected utility. This assumption may seem to contradict Pareto Optimality. However, whenever O is in this situation, she must be defaulting on $\tau_{H,O}$ by offering at least one of the two contracts. When O increases P's expected utility, O is simultaneously decreasing H's expect utility. Thus, Assumption 1.2 does not violate Pareto Optimality. Moreover, Assumption 1.2 helps H by making O choose paying H over paying P whenever O is indifferent.

I now make an assumption on the endowment of the pessimist. This assumption helps put structure on the contract between the optimist and the pessimist, simplifying the problems for the baseline results. Later in section 1.6, I relax the assumption for more general results.

Assumption 1.3. $w_P > \frac{1}{h^{-1}(\pi) - h^{-1}(\pi')} > h(\pi)$.

The term $h(\pi) \equiv \frac{\pi}{1-\pi} \equiv 1/h^{-1}(\pi)$ is the hazard rate. Intuitively, as P's wealth increases relative to H's endowment in state 1, the cost of O buying insurance from P increases since O gives up more of her valuable[7] state 1 wealth for state 0 wealth. Given contract $\tau_{H,P}$ between H and P, the benefit O receives from buying insurance from P is constant with respect to w_P. Thus, as w_P increases above the threshold in Assumption 1.3, the cost of O buying insurance from P outweighs the benefit. Thus, in this case, the

[7] O is more optimistic about state 1.

incentive for O to sell insurance to P is so strong that it is infeasible for H to change O's incentives. H simply expects O to sell insurance to P and cannot do anything to stop it. Therefore, H cannot impact P's problem except directly through contract $\tau_{H,P}$. Together with Assumption 1.2, Assumption 1.3 simplifies the agents' problems.[8]

1.3.5. Hedger's Problem

Since I will relax Assumption 1.3 later, I will state H's problem in general.[9] At $t = 0$, H solves

$$\max_{\tau_{H,O},\tau_{H,P}} U_H(\tau_{H,O}, \tau_{H,P}) = \mathbb{E}_H \left[u(R + \tau'_{H,O} + \tau'_{H,P})\right]$$

subject to individual rationality constraints

$$\text{(IR-HO)} \qquad U_O\left(\tau_{H,O}, \tau_{H,P}\right) \geq U_O\left((0,0), \tau_{H,P}\right),$$

$$\text{(IR-HP)} \qquad \mathbb{E}_P[(w_P - \tau_{H,P})^+] \geq \mathbb{E}_P[w_P],$$

and budget constraints for insurers

$$\text{(BC-HO)} \qquad \tau'_{H,O} = \min\left(\tau_{H,O}, w_O + \tau^*_{O,P}[\tau_{H,O}, \tau_{H,P}]\right),$$

$$\text{(BC-HP)} \qquad \tau'_{H,P} = \min\left(\tau_{H,P}, w_P - \tau^*_{O,P}[\tau_{H,O}, \tau_{H,P}] - \tau_{H,P}\right),$$

where $\tau^*_{O,P}[\tau_{H,O}, \tau_{H,P}] \in \arg\max_\tau \mathbb{E}_O\left[(w_O + \tau_{O,P} - \tau_{H,O})^+\right]$ is the solution to O's problem given contracts H offers to O and P. Note that $\tau^*_{O,P}$ may not be unique. In that case, I assume H can force O to pick the $\tau^*_{O,P}$ that is better for H. Such selection is also Pareto

[8]See sections A.1.1 and A.1.2.

[9]See section A.1.4 for details on how 1.3 simplifies H's problem.

Optimal. The right-hand-side of (IR-HO) doesn't have $()^+$ is because of (BC-HO). I shall refer to $\tau'_{H,O}$ and $\tau'_{H,P}$ as the recovery contracts of the corresponding contracts.

One may wonder whether H can choose risk-free $\tau'_{H,O}$ and $\tau'_{H,P}$ directly rather than choosing risky $\tau_{H,O}$ and $\tau_{H,P}$. In general, H cannot choose $\tau'_{H,O}$ or $\tau'_{H,P}$ directly since $\tau^*_{O,P}[\tau_{H,O}, \tau_{H,P}]$ may differ from $\tau^*_{O,P}[\tau'_{H,O}, \tau_{H,P}]$ and $\tau^*_{O,P}[\tau_{H,O}, \tau'_{H,P}]$. However, by Assumption 1.3 and Lemma A.2, I know $\tau_{H,P} < w_P < w_P - \tau^*_{O,P}[\tau_{H,O}, \tau'_{H,P}]$ if (IR-HP) binds. Thus, (BC-HP) becomes redundant and $\tau_{H,P}$ is always risk-free. This simplifies H's problem as shown in section A.1.4. Let us first consider a useful benchmark that gives us the highest utility H can attain.

1.3.6. Benchmark: $w_P = 0$

Since there is no pessimist, the problem reduces to a two-agent contracting problem in which H has all the bargaining power.

Proposition 1.1. *There is a unique solution $\tau^B_{H,O}$ s.t. $\tau^B_{H,O} \propto \tau_{h^{-1}(\pi')}$. Given Assumption 1.1, $\tau^B_{H,O}$ is either interior or $\tau^B_{H,O}(1) = -1$. In either case, $\tau^B_{H,O} \leq w_O$.*

Since H has all the bargaining power, she would extract all the surplus from O. Thus, the price of the optimal contract between H and H is $h^{-1}(\pi')$. This provides a useful benchmark. Given the price $h^{-1}(\pi')$, H would choose to purchase $\tau^B_{H,O}(0)$ units of the contract. I define counterparty risk as the difference between H's expected utility with equilibrium contract and H's expected utility with benchmark contract $\tau^B_{H,O}$. When the equilibrium contract is $\tau^B_{H,O}$, there is no counterparty risk by definition.

1.4. Equilibrium

Equilibrium is defined as Subgame Perfect Nash Equilibrium with contracts $\{\tau^*_{H,O}, \tau^*_{H,P}, \tau^*_{O,P}\}$ such that they solve H and O's problems and all offered contracts are accepted. I shall divide the parameter space into 2 scenarios to highlight the effect of the commitment problem. In the first scenario, O has more wealth relative to H and P. In that case, the commitment problem has no effect on H as P's budget constraint binds before O can sell enough insurance to default on H's contract. In the second scenario, O's wealth is lower compared to the first scenario. In this case, the commitment problem becomes worse for H as O's wealth decreases.

1.4.1. Scenario 1: Wealthy O, No Counterparty Risk

I first study the case when O is wealthy enough so that O's commitment problem does not affect H. Formally,

Assumption 1.4. $w_O \geq \tau^B_O(0) + h(\pi)w_P$.

This assumptions states that O has enough wealth to trade with both P and H without default. The first term $\tau^B_O(0)$ is the optimal amount of insurance H purchases when faced with a price of $h^{-1}(\pi')$. Recall O can sell insurance to P for a price of $h^{-1}(\pi)$. Thus, the second term $h(\pi)w_P$ represent how much wealth O needs to exhaust P's endowment when O speculates with P. Assumption 1.4 only restricts O's wealth relative to H and P's wealth. It does not impose any condition on the relative wealth between H and P. Thus, Assumption 1.4 does not conflict with earlier assumptions. Under this condition, I have the following Proposition.

Proposition 1.2. *Given Assumption 1.4, there is a unique equilibrium with* $\tau^{(0)}_{H,O} = \tau^B_{H,O}$, $\tau^{(0)}_{H,P} = (0,0)$, *and* $\tau^{(0)}_{O,P} = -h(\pi)w_P\tau_{h^{-1}(\pi)}$.

Since H can purchase cheaper insurance from O, H strictly prefers to do so. In this case, the benchmark contract is available and so H has no appetite for more insurance from P, especially since P only sells insurance a higher price. Thus, when O is wealthy relative to H and P, O does not default on contract with H. However, O only fulfills promises to H because P's endowment constraints P from buying more insurance. Thus, the commitment problem of O does not affect H. In this case, there is no counterparty risk.

1.4.2. Scenario 2: Less Wealthy O, with Counterparty Risk

Here O has less wealth than in scenario 1.

Assumption 1.5. $w_O < \tau^B_O(0) + h(\pi)w_P$.

The inequality states that when O trades to the limit with P, the benchmark contract between H and O is no longer feasible. This assumption is the complement of Assumption 1.4 in terms of the parameter space.

Proposition 1.3. *There is a unique equilibrium. There are 3 cases depending on* w_O.

(1) $\tau^{(1)}_{H,O} = w_O + \tau^{(1)}_{O,P}(0)\tau_{h^{-1}(\pi')}$, $\tau^{(1)}_{H,P} = (0,0)$, *and* $\tau^{(1)}_{O,P} = -h(\pi)w_P\tau_{h^{-1}(\pi)}$

(2) $\tau^{(2)}_{H,O} = (0,0)$, $\tau^{(2)}_{H,P} \propto \tau_{h^{-1}(\pi)}$, *and* $\tau^{(2)}_{O,P} = -h(\pi)(w_P - \tau^{(2)}_{H,P}(1))\tau_{h^{-1}(\pi)}$,

(3) $\tau^{(2)}_{H,O} = (0,0)$, $\tau^{(2)}_{H,P} \propto \tau_{h^{-1}(\pi)}$, *and* $\tau^{(3)}_{O,P} = -w_O\tau_{h^{-1}(\pi)}$,

where $\tau^{(2)}_{H,P}(0)$ *is the optimal amount of insurance* H *purchases given price* $h(\pi)$. *Equilibrium is in case 2 and 3 when* $U_H(\tau^{(2)}_{H,O}, \tau^{(2)}_{H,P}) \geq U_H(\tau^{(1)}_{H,O}, \tau^{(1)}_{H,P})$. *There exist unique*

$w_O^* \geq h(\pi)w_P$ and w_O^{**}, such that case 1 is the equilibrium for $w_O > w_O^*$, case 2 is the equilibrium for $w_O^{**} < w_O \leq w_O^*$, and case 3 is the equilibrium otherwise.

When w_O is in the interval defined in Assumption 1.5, there are 3 possible cases. When O's wealth is high, H only buys insurance from O. Though H purchases the insurance at a low price of $h^{-1}(\pi')$, the quantity H can purchase is constrained by O's commitment problem. In such case, H can only purchase partial insurance. H can also simultaneously buy insurance from P but H chooses not to since buying insurance from P increases O's commitment problem and devalues H's existing contract with O. When H buys insurance from both O and P, H is essentially competing against herself for O's wealth. Thus, H only buys cheap partial insurance from O. In this case, there is no pricing effect of the commitment problem, since H deals with the problem by decreasing the quantity purchased. This implies that when counterparty risk is not priced in the data, it doesn't mean the commitment problem has no effect. The effect may just not be in the price. H's welfare can still very much be improved as shown in section 1.5.

In case 2 and 3, H is better off buying full insurance at a higher price than buying partial insurance at a lower price. When H buys more the expensive insurance from P, part of the insurance may be sold by O through P. Even though P may have enough wealth[10] to insure H, each unit of insurance H buys induces O to sell one more unit of insurance to P. This happens in case 2 until O runs out of wealth and then the equilibrium moves to case 3. This result is similar to Du et al. (2016), who document that CDS market participants are less likely to trade with counterparties who have credit risk correlated with the CDS's underlying asset. However, in this model, O's ex-ante

[10]This is true by Assumption 1.3.

endowment is not correlated with R. O has incentive to take on risks correlated with R only after selling insurance to H. Even so, in case 2 and 3, H would choose not to contract with O. Proposition 1.3 gives a more refined result than the observations from Du et al. (2016). It provides a new empirical prediction.

Notice H offers the same contracts in cases 2 and 3, regardless of the contract between O and P. This is because P has enough wealth to insure H by Assumption 1.3. If that were not the case, things get more complicated as shown in section 1.6. In all 3 cases, H cannot offer the benchmark contract. In case 1, the price is same as in the price in the benchmark contract but the quantity is less. In case 2 and 3, the price is higher than the price of the benchmark contract.

1.5. Central Clearing

After the recent financial crisis, both U.S. and Euro-zone has pushed for central clearing of standardized OTC derivatives to reduce counterparty risk. Central clearing is implemented through the use of central counterparty (CCP), which stands between trades and guarantees payment. The CCP replaces each existing contract with two new contracts. The two new contracts are equivalent to the old contract. However, buyer and seller of the old contract now both trade with the CCP instead. This way, the CCP can reduce counterparty risk through collateral requirements and loss-sharing. Since the agents do not have money in $t = 0$, collateral requirements do not apply here. I shall focus on the loss-sharing.

1.5.1. Loss sharing

The CCP does not have any endowment. Agents trading directly with the CCP are called clearing members. Agents trading with the clearing members are called clients. When a clearing member defaults on a client's trade, the CCP spreads the loss to all other clearing members by withholding their payments until the client's obligations are paid in full. In this setting, the CCP maximizes H's welfare by designating H as the client and the insurers as the clearing members.

While it is possible to model the CCP literally by creating 2 new contracts for each existing contract, it is not necessary to do so. For the purpose of modeling loss-sharing, the CCP can simply deduct any defaulted amount from all the non-defaulting clearing members.

Given Lemma A.2, I only need to consider when O defaults on $\tau_{H,O}$. Recall $\tau'_{H,O}$ as defined in the hedger's problem. Suppose O defaults on contract with H, i.e. $\tau_{H,O} > \tau'_{H,O}$. the CCP deducts the difference $\tau_{H,O} - \tau'_{H,O}$ from O's contract with P, i.e. $\tau_{O,P}$. The effective contract P receives from O is thus

$$-\tau'_{O,P} = -\tau_{O,P} - (\tau_{H,O} - \tau'_{H,O})$$

with the restriction that $\tau'_{O,P} \geq 0$. I need to rewrite O's problem in terms of $\tau'_{O,P}$. For simplicity, I assume $\tau_{H,O} \leq w_O$. This simplifies the notation. At $t = 1$, O now solves

$$\max_{\tau'_{O,P}} \mathbb{E}_O[w_O + \tau'_{O,P} - \tau_{H,O}]$$

subject to P's individual rationality constraint

$$\text{(IR-P-CCP)} \qquad \mathbb{E}_P[\tau'_{O,P}] = 0,$$

and budget constraints, or loss-sharing constraints

$$\text{(BC-O-CCP)} \qquad -\tau'_{O,P} \leq w_O - \tau_{H,O},$$

$$\text{(BC-P-CCP)} \qquad \tau'_{O,P} \leq w_P - \tau_{H,P}.$$

With loss-sharing, O cannot credibly promise $\tau_{O,P}$ to P, O can only promise $\tau'_{O,P}$. There is still commitment between O and P. However, loss-sharing makes H's claim more senior to $\tau_{O,P}$. Given this problem, I have the following Proposition.

Proposition 1.4. *There is a unique equilibrium with $\tau^{CCP}_{H,O}$, which weakly improves H's welfare compared to corresponding cases in section 1.4.2. In some cases, the price of $\tau^{CCP}_{H,O}$ is between $h^{-1}(\pi)$ and $h^{-1}(\pi')$. $\tau^{CCP}_{H,O}$ is worse than the benchmark contract for H. Nevertheless, $\tau^{CCP}_{H,O}$ is Pareto Optimal, since H's utility cannot be increased without sacrificing O's utility.*

Loss-sharing weakly improves H's welfare. With loss-sharing, H can be guaranteed payment if O accepts H's contract. However, since O can reject contract from H and sell insurance to P, H competes with P in price. Thus, the benchmark contract cannot be attained by H in equilibrium. With loss-sharing, H can purchase blocks of insurance from O at different prices. For example, in case 1 from 1.4.2, H can only purchase partial insurance at price $h^{-1}(\pi')$ without the CCP. With the CCP, H can purchase additional insurance from O at a higher price of $h^{-1}(\pi)$. Thus, the average price H pays for insurance is between the high price and the low price.

1.5.2. Voluntary Central Clearing

In this case, both O and P are indifferent between participating and not participating in central clearing. Even if P has some bargaining power, H can always compensate O and P enough so that both O and P would be willing to enter central clearing. In that case, central clearing is still welfare improving for H.

1.6. Hedging

In this section, I relax Assumption 1.3 and allow O to have the option to *purchase* insurance from P. In this case, there is incentive for O and P to speculate in either direction. Thus, H may have to hedge by trading with both O and P. First, I find the partial equilibrium.

Proposition 1.5. *Given* $\tau_{H,P}$ *such that* (IC-O-B) *holds for* $\tau_{H,O} = (0,0)$*, there is a unique* $\tau^{(4)}_{H,O}[\tau_{H,P}]$ *that maximizes* H*'s objective function.* $\tau^{(4)}_{H,O} \propto \tau_{h^{-1}(\pi')}$*. Either* $\tau'_{H,O} = \tau^{(4)}_{H,O}$ *and* $\tau'_{H,P} = (0, \tau_{H,P}(1))$ *or* $\tau'_{H,O} = (w_O - (w_P - \tau_{H,P}(1))h^{-1}(\pi))^+$ *and* $\tau'_{H,P} = \tau_{H,P}$*.*

Above I characterize the solutions to H's problem given $\tau_{H,P}$ that induces O to sell insurance to P when O does not trade with H. Given such a $\tau_{H,P}$, H chooses either to hedge $\tau_{H,O}$ by giving money to P for free or to hedge $\tau_{H,P}$ by offering O a contract that H knows O will default on. Thus, it's possible for H to hedge her contract with O by inducing O to buy insurance from P. When P defaults on $\tau_{H,P}$, $\tau_{H,P}(1)$ is the cost of hedging $\tau_{H,O}$. When O defaults on $\tau_{H,O}$, $\tau_{H,O}(1) - \tau'_{H,O}(0)h^{-1}(\pi')$ is the cost of hedging $\tau_{H,O}$.

Hedging $\tau_{H,O}$ may be expensive. Below I provide a lower bound and an upper bound on cost of hedging.

Proposition 1.6. *Hedging cost for $\tau_{H,O}$ has a lower bound of*

$$\min\left[(w_O + w_P)\left(h^{-1}(\pi) - h^{-1}(\pi')\right),\ h^{-1}(\pi)\left(1 + h^{-1}(\pi)\right)\left(h(\pi') - h(\pi)\right) w_P,\ h^{-1}(\pi) w_P\right]$$

and an upper bound of $h^{-1}(\pi)w_P$. Hedging cost for $\tau_{H,P}$ has a lower bound of 0 and an upper bound of $\left[\min\left(w_O, h(\pi)w_P\right)\left(h^{-1}(\pi) - h^{-1}(\pi')\right) - h^{-1}(\pi')w_P\right]^+$.

Hedging cost of $\tau_{H,O}$ increases with w_P. When w_P increases, so does counterparty risk. Thus, hedging cost co-moves with counterparty risk. Depending on w_O, w_P, π' and π, hedging $\tau_{H,O}$ may be expensive. Hedging $\tau_{H,P}$ is not as expensive since H can always pick a $\tau_{H,P}$ so that no hedging is needed, i.e. both (IC-O-B) and (IC-O-S) binds with equality at $\tau_{H,O} = (0, 0)$. Hedging is cheaper when the gains from O speculating with P, i.e. $h^{-1}(\pi) - h^{-1}(\pi')$ is small. When cost of hedging is small enough, H may choose to hedge in equilibrium.

1.7. Discussions

1.7.1. Different Beliefs, Bargaining Power

As long as there is an insurer who is more optimistic than the other insurer, I get similar results. If both insurers are more optimistic than the hedger, I also get the similar results. If the pessimist's belief is below the hedger's certainty equivalent, the hedger prefers buying no insurance to buying insurance from the pessimist. Similarly, if both insurers' beliefs are below the hedger's certainty equivalent, the hedger will choose not to purchase insurance.

In the model, O has all the bargaining power when trading with P. I can give all the bargaining power to P and I would get similar results. When P has all the bargaining powers, P can extract all the surplus when trading with O. However, P will still only accept contract from H with a price no lower than $h^{-1}(\pi)$. Thus, in cases similar to the ones in section 1.4.2, H may still prefer to purchase partial insurance from O at price of $h^{-1}(\pi') < h^{-1}(\pi)$. When both O and P have some bargaining power and when H doesn't have all the bargaining power, I get similar results. In this case, O and P still cannot commit to not speculating with each other.

1.7.2. Collateral

The seniority assumption can be replaced by usage of costly collateral. Imagine O and P have endowment at $t = 0$ and H can ask for collateral. However, suppose O and P can manage collateral without a cost while H incurs a cost when holding collateral. In the case, when the hold cost of collateral for H is too high, H would prefer the equilibrium with no collateral. In that case, O and P can still post collateral to each other and thus have seniority in each other's claim.

1.8. Conclusion

I study how the hedger manages counterparty risk when insurers with heterogeneous beliefs cannot commit to not speculating with each other. When the insurers are wealthy relative to the hedger, the hedger cannot change the direction of the insurers' speculations. In that case, the hedger chooses between cheaper partial insurance and more expensive full insurance. The hedger does not trade with both insurers since her contract with one insurer devalues her contract with the other insurer. When the hedger chooses cheaper partial insurance, she manages counterparty risk through the rationing of quantities purchased. In that case, the price of the insurance does not reflect counterparty risk. This is consistent with the empirical findings in Arora et al. (2012) whereby the effect of counterparty risk on the price of OTC derivative contracts is small. When counterparty risk is not priced, it does not mean that there is no counterparty risk. Counterparty risk may still appear as costs for the hedger in other dimensions.

When the hedger chooses the more expensive full insurance, the hedger chooses to trade with the pessimist. This is similar to the counterparty selection in Du et al. (2016). However, insurers in this model do not have existing risky assets when selling insurance to the hedger. Thus, this model predicts that even if an insurer does not have existing credit risk correlated with the endowment of the hedger, the hedger may still choose not to contract with that insurer. This is a new empirical prediction.

I also provide an upper and lower bound on the cost of hedging insurance contracts. When gains from speculating are small, hedging becomes cheaper. When hedging is cheap enough, furthermore, the hedger may choose to hedge by trading with both insurers. Given specific utility function, this model can predict when the hedger will choose to trade with both insurers. This prediction connects Du et al. (2016) and Gündüz (2016).

Finally, I examine the effect of central clearing on the hedger's welfare. I focus on the ability of the central counterparty to share losses across its clearing members. In this case, central clearing increases the hedger's welfare. However, since the hedger has to compete with the pessimist in price, the price of the equilibrium contract depends on the bargaining power between the optimist and the pessimist. Both insurers, this chapter argues, are indifferent between participating and not participating in central clearing. Thus, even if participation is voluntary, both insurers would participate.

CHAPTER 2

Leverage, Bubble and Option

2.1. Introduction

This chapter extends Simsek (2013) to study more general belief structure in collateral equilibrium. Market participants have heterogeneous beliefs and disagree about future payoffs. The majority of the literature in heterogeneous beliefs has focused on disagreement about the mean. However, in practice, market participants may disagree about future volatility as well as the tail risks. This chapter explores more general beliefs structures that allow the agents to disagree in more than one dimension.

Heterogeneous beliefs can cause that the market participants to speculate on their beliefs and take on risky portfolios with high leverage. This is especially true when the market participants are presumedly risk-neutral. Many stock market crashes in the past were attributed to this kind of speculation by optimistic investors. For example, the hallmark of the financial crisis in 2007 is optimism and high leverage. In another example, the rapid growth in the Chinese stock market in 2015 before the eventual crash in 2016 is also said to have been fueled by optimistic investors borrowing money (often from family and friends) to invest in the stock market. This chapter studies how more general heterogeneous beliefs interact with leverage and how they affect the prices. In particular, I find the sufficient conditions in terms of disagreements in beliefs that can generate bubble in asset prices.

A natural setting to study heterogeneous beliefs, leverage, and bubble is the collateral equilibrium framework introduced by Geanakoplos (1997) and employed in Simsek (2013). I follow Simsek (2013) to model an economy with an infinite number of states that are determined by realization of payoffs from an asset A. There are two types of risk-neutral agents in the economy, one unit mass each. They are called optimists and pessimists for their belief about the mean of A's payoff. However, the optimists can disagree with the pessimists in other aspects as well. For example, the optimists may believe in a greater variance in the distribution of A's payoffs. In addition to the asset A, the agents can also trade simple debt contracts which have to be collateralized by the asset A. For each unit of simple debt contract an agent sells, the agent has to hold one unit of asset A. The simple debt contract is essentially a way for the agents to leverage up their position by borrowing to purchase the asset A. Thus, leverage is endogenous in this model. A third investment option for the agents is to simply hold cash, which has a gross return of 1.

Despite the more general belief structure in the model, a unique Nash Equilibrium exists if the pessimists are wealthy enough to always hold cash. In this unique equilibrium, I find that a bubble can exist if the optimists believe in larger future volatility. I define bubble as the equilibrium price of the asset that exceeds the optimists' valuation. Here, the bubble exists for reasons different from both Harrison and Kreps (1978) and Fostel and Geanakoplos (2015). Using the simple debt contracts, the asset can be split into pieces held by different agent. Thus, the asset as a whole is priced by more than one type of agents in this model. Therefore, the price of the asset may exceed the optimists' valuation if each agent holds the portion of the asset that she is most optimistic about. This is different from resale value or collateral value. As a result, this a model can generate a bubble in the equilibrium asset price with a simple one-period model.

Moreover, since leverage is endogenous in this model, I find interesting and counter-intuitive results. Since the pessimists' large endowment prevent them from taking on leverage, I will only discuss the results for the optimists. First, the more wealth the optimists have, the lower their equilibrium leverage. Essentially, leverage is a poor man's tool. While one can achieve high return with leverage, one mainly uses leverage to obtain positions that are out of her reach with her available endowment. As the optimists have more endowment while the supply of asset A remains constant, the need for leverage decreases. Thus, endogenous leverage decreases. The second interesting result is that the equilibrium price of asset A decrease with leverage. This is counterintuitive as people often blame speculators for driving up the price with their leverage. This result, by no means, claims that the price is low. In fact, the equilibrium price can still be higher than the optimists' valuation while decreasing with leverage. The reason for these results is again due to endogenous leverage. Since only the poor uses high leverage, higher leverage means the optimists are poorer. Thus, the pessimists, who do not use leverage, will obtain a larger share of the payoffs from asset A. So, the market as a whole has lower leverage and thus lower prices.

Since the asset and the simple debt contracts replicate option payoffs, I also examine implications in option pricing. In the case when the risk-neutral investors' beliefs are normally distributed with different variances, the model can generate the option smirk. The drivers of the smirk in this chapter are different from that in Buraschi and Jiltsov (2006). In Buraschi and Jiltsov (2006), the option smirk is driven by the investors' disagreement in the mean of the payoff distribution. In this model, even when investors agree on the mean, if they disagree on variance of the distribution, the volatility smirk arises. The main drivers of the smirk are the underpricing of options with high strike price and the overpricing of the asset itself. The intuition is simple. If options with high strike prices are underpriced, there will be the smirk. If the asset is overpriced, the true distribution is skewed to the left relative the distribution used to back out implied volatility. Thus, volatility would be higher for the options with lower strike prices and lower for the options with higher strike prices.

The chapter proceeds as follows. Section 2.2 discusses the literature review. Section 2.3 presents the model. Section 2.4 present the equilibrium results. Section 2.5 studies implications in option pricing. Section 2.6 provides extensions and discussions on alternative assumptions. Finally, Section 2.7 concludes. The proofs and additional analysis can be found in Appendix B.

2.2. Literature Review

This chapter extends the literature in collateral equilibrium. This chapter is closely related to Fostel and Geanakoplos (2015) and Simsek (2013). More specifically, this chapter extends the framework from Simsek (2013) to study more general belief structures. The more general beliefs structures yield similar yet different results. Similar to Fostel and Geanakoplos (2015), in the more general belief structure, the equilibrium asset price contains a positive collateral value. At the same time in this model, the optimists, who hold the asset in the equilibrium, also enjoy a discount due to their disagreement with the pessimists. This discount counteracts the effect of the positive collateral value. Thus, I provide sufficient conditions in which the positive collateral value overcomes the effect of the discount to result in a bubble in the asset price. The existence of a bubble is in contrast to results from Simsek (2013). Moreover, I find that leverage decreases the asset price in contrast to Fostel and Geanakoplos (2015).

This chapter is also closely related to the literature on heterogeneous beliefs and bubbles (e.g. Harrison and Kreps, 1978; Scheinkman and Xiong, 2003; Buraschi and Jiltsov, 2006). In this literature, the models are typically dynamic and the agents disagree over the process for the mean. The dynamic disagreement is able to generate bubble in asset prices due to resale value of the asset and optimists becoming pessimists in the future. The model in this chapter is static and the agents disagree over the entire payoff distribution rather than just over the mean. Thus, the optimists who are overall optimists about the asset may in fact be more pessimistic than the pessimist over either the left tail or the right tail of the payoff distribution. In fact, the disagreement over the tail is enough to generate a bubble in the asset price. The static nature of the bubble means the bubble can grow or crash easily depending on how disagreement changes through periods. One can imagine an overlapping generation type formulation to extend this model to multiple periods to study dynamics of the bubble.

This chapter is also closely related to the literature on asset price bubbles (e.g. Harrison and Kreps, 1978; Banerjee, 1992; Abreu and Brunnermeier, 2003). The bubble here is a result of collateral value of the asset. It is in contrast to the resale value in Harrison and Kreps (1978), herding in Banerjee (1992), and the asynchronized trading in Abreu and Brunnermeier (2003).

This chapter has option pricing implications. Thus, this chapter is also closely related to the option pricing literature (e.g. Merton, 1976; Buraschi and Jiltsov, 2006). In this chapter, option pricing is non-standard as the asset price includes an additional collateral component. Moreover, the disagreement between the agents result in a discount in the asset prices. This positive collateral component and the discount compete to cause the asset to be either overpriced or underpriced. Thus, using the price as an input to compute the option prices is incorrect. Moreoever, the discount in the asset price translate directly to the option prices, causing the options to be undervalued. Thus, this model can generate a volatility smirk. The driver of the volatility smirk in this model is the discount in the asset price, which is a result of disagreement between the agents over the tail of the payoff distribution. This is in addition to the disagreement between the agents over the mean as in Buraschi and Jiltsov (2006). Under the more general belief structure, this chapter identifies more ways the agents can disagree to cause implied volatility to smirk.

2.3. Model

This is a one period model with time $t \in \{0, 1\}$. There is one unit of asset A in the economy with a payoff $s \in [\underline{s}, \bar{s}]$ at $t = 1$. For convenience, I restrict $\underline{s}$ and $\bar{s}$ to be finite. Since the payoff of the asset is the only state variable, s also represents states of the world at $t = 1$. Agents cannot short-sell asset A.

In addition to A, the agents in the model can trade simple debt contracts with each other. In a simple debt contract, the seller promises to repay D at $t = 1$ for upfront payment $\pi(D)$ by the buyer at $t = 0$. As there is no commitment or enforcement in repaying the debt, each unit of simple debt contract must be collateralized by one unit of

A . Thus, for promised repayment D, the simple debt contract pays out

$$\min(s, D)$$

at $t = 1$. If s, the realized payoff from A, is not enough to cover promised repayment D, the buyer simply keeps[1] realized payoff s and let the borrower default with no further recourse. In this case, the seller of the simple debt contract is the borrower and the buyer is the lender. Since simple debt contracts are solely defined by promised repayment D. I call the simple debt contract D if the seller promises to repay D. Moreover, since promising to repay $D > \bar{s}$ is the same as promising to repay $D = \bar{s}$, I can restrict my attention to $D \in [\underline{s}, \bar{s}]$. The risk-free rate is normalized to 0.

Before proceeding, I would like iron out some wrinkles in the definitions. Since selling the simple debt contract $D = \bar{s}$ (collateralized by the asset) is the same as selling the asset and thus canceling out the seller's asset position, the simple debt contract $\bar{s}$ can create potential problems. One immediate problem is that simple debt contracts cannot be collateralized by other simple debt contracts. Thus, if an agent hoards all of asset A and then use it as collateral to sell simple debt contract $\bar{s}$, no other agents can sell simple debt contracts since they do not have any asset. This scenario is clearly suboptimal. So, I make the following assumption.

Assumption 2.1. *For every unit of simple debt contract $\bar{s}$ an agent sells, her position in asset A is reduced by a unit.*

[1]For each unit of debt contract sold, the seller must hand over one unit of A as collateral at $t = 0$.

Note that Assumption 2.1 does not relax the short-sell constraint, since simple debt contract $\bar{s}$ still needs be collateralized with asset A.

2.3.1. Agents

There are two types of agents, one unit mass each. The types are defined by the agent's beliefs about asset A. Type o agents are optimist with the optimistic belief that state s occurs with probability $f_o(s)$ for all $s \in [\underline{s}, \bar{s}]$. Type p agents are pessimist with the pessimist belief $f_p(s)$ for all $s \in [\underline{s}, \bar{s}]$. For simplicity, I require $f_o(s)$ and $f_p(s)$ to be continuous over $[\underline{s}, \bar{s}]$. The continuity ensures the functions' respective integral, i.e. cumulative distribution functions $F_o(s)$ and $F_p(s)$, to be well defined. In Assumption 2.2, I define optimism.

Assumption 2.2 (optimism)**.** $\mathbb{E}_o[s] > \mathbb{E}_p[s]$

This is the natural definition of optimism as the risk-neutral optimists value A more. If A were the only thing trade, the risk-neutral agents would only care about the mean. However, the agents can also trade simple debt contracts with each other. Since the payoff of simple debt contracts is not a linear function of s, the agents also care about the entire distributions f_o and f_p. For this reason, Simsek (2013) uses a stronger condition for optimism

$$\frac{f_p(s)}{1 - F_p(s)} > \frac{f_o(s)}{1 - F_o(s)}, \tag{2.1}$$

for all $s \in [\underline{s}, \bar{s}]$. Though this condition is convenient as it guarantees uniqueness in solution, it is perhaps too strong as it rules out bubble in asset prices. It is possible to weaken (2.1) and still achieve uniqueness without restricting equilibrium prices too much.[2] For now, Assumption 2.2 suffices in defining optimism.

Each agent is endowed with cash at $t = 0$. There are no asset endowments. One can think of the unit of asset being held by some unmodeled agents who sell their assets at $t = 0$ for consumption. The optimists are endowed with n_o in cash while the pessimists are endowed with n_p in cash. One can also think of n_o and n_p as the mass of optimists and pessimists, respectively, while each agent has one unit of cash endowment at $t = 0$. I assume

Assumption 2.3 (Wealthy Pessimists)**.** *n_p is large enough so that pessimists always hold some cash in any equilibrium.*

This assumption only matters when the optimists don't also hold cash in equilibrium. When only the pessimists hold cash in equilibrium, Assumption 2.3 implies the optimists have all the bargaining power when determining the equilibrium price of the simple debt contracts. In such case, although there is no bargaining in the model, the equilibrium price of the debt contract would coincide with the bargaining price. Assumption 2.3 also implies the pessimists weakly prefers[3] holding cash to holding asset A and using A as collateral to borrow. Thus, I can restrict my attention to any equilibrium in which only the optimists hold the asset and borrow. Turns out Assumption 2.3 is not as restrictive as it seems. Under Assumption 2.6 in section 2.3.3, Assumption 2.3 is redundant[4].

[2]See section 2.3.3.
[3]See section B.1.2 for more detail.
[4]See section 2.6.1.

2.3.2. Agents' Problem

At $t = 0$, the agents choose their asset, cash, and debt positions. Agents of type i chooses asset positions α_i. The agents cannot short-sell, so

$$\alpha_i \geq 0, \tag{2.2}$$

for all $i \in \{o, p\}$. The agents cannot carry negative cash balance, since the only form of borrowing in the model is through simple debt contracts. So, the agents' cash balance c_i is subjected to

$$c_i \geq 0, \tag{2.3}$$

for all $i \in \{o, p\}$. The agents also have to choose positions in simple debt contracts. For simple debt contract D, the agents have to choose positions $\mu_i(D)$. The agents have to do this for all $D \in [\underline{s}, \bar{s}]$. Thus, the agents' debt positions can be summarized by the function $\mu_i : [\underline{s}, \bar{s}] \to R$ mapping D to a real number. As collateral is core of this chapter, the agents' debt positions are subjected to collateral constraint

$$\alpha_i \geq \int_{D \in [\underline{s}, \bar{s}]} -\mu_i(D)^- dD, \tag{2.4}$$

for all $i \in \{o, p\}$, where $\mu_i(D)^- = \min(0, \mu_i(D))$ is the negative positions of the agents in simple debt contract D. In other words, only the sellers have to put up collateral since they promise payments in the future and may not deliver. It is useful to also define $\mu_i(D)^+ = \max(0, \mu_i(D))$. Note that $\mu_i(D) = \mu_i(D)^- + \mu_i(D)^+$.

Lastly, the agents have to balance their budgets. They are subjected to budget condition

$$n_i \geq \alpha_i q + c_i + \int_{D \in [\underline{s}, \bar{s}]} \pi(D) \mu_i(D) dD, \tag{2.5}$$

for all $i \in \{o, p\}$, where q is asset A's price and $\pi(D)$ is the price of simple debt contract D. The agents take these prices as given. The budget constraint is fairly straightforward with the left-hand-side being the endowment and the right-hand-side being the costs of obtaining positions in asset A, cash, and simple debt contracts.

The agents choose their positions and maximize their $t = 1$ wealth. Formally, type i agents solve

$$V_i(n_i) \equiv \max_{\alpha_i, c_i, \mu_i} U(\cdot | n_i) \equiv \mathbb{E}_i [\alpha_i s] + \mathbb{E}_i \left[\int_{D \in [\underline{s}, \bar{s}]} \min(s, D) \mu_i(D) dD \right] + c_i, \tag{2.6}$$

subject to constraints (2.2), (2.3), (2.4), and (2.5).

2.3.3. Belief Structure

The belief structure is the main focus of this chapter. I generalize the belief struction from Simsek (2013), which is a bit restrictive. To see this, I have the following corollary.

Corollary 2.1. *Given* (2.1) *from Simsek (2013),* $1-F_o(s) > 1-F_p(s)$ *for all* $s \in (\underline{s}, \bar{s})$.

In other words, (2.1) implies that the complementary cumulative distribution function (CCDF) of the optimists' beliefs is always higher than the CCDF of the pessismists' beliefs.

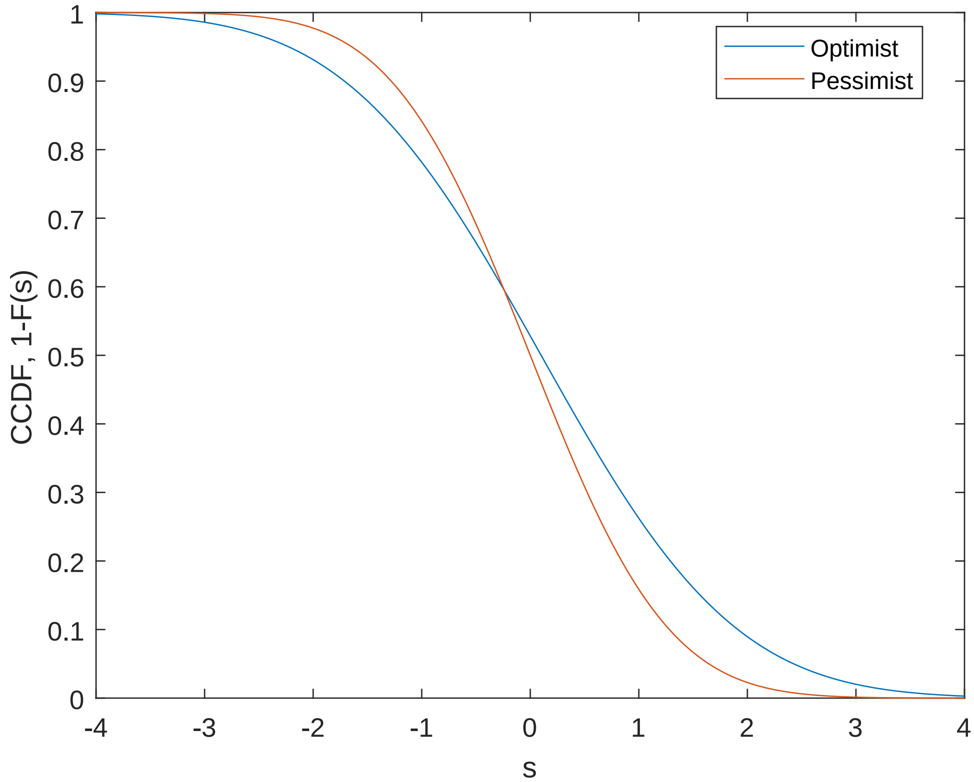

Figure 2.1. CCDFs for $f_o \sim \mathcal{N}(0.1, 2)$ and $f_p \sim \mathcal{N}(0, 1)$

This implies that the equilibrium price of A is always lower than the optimists' expected value of A.[5] Thus, (2.1) essentially rules out a bubble in the equilibrium price of A.

Although (2.1) is convenient to use as it helps satisfy the second order condition for the agents' problem and establish uniqueness in solution, it only allows certain types of disagreement between the agents. For example, if the agents have truncated normal (TN) beliefs and only disagree over the mean, (2.1) will be satisfied. However, if the agents have TN beliefs and disagree over both the mean and the variance, (2.1) will be violated, as shown in Figure 2.1. It is not uncommon that the agents would disagree over the variance of their beliefs. Thus, I will replace (2.1) with a weaker condition.

[5]This follows from (B.16) and Corollary B.3

Assumption 2.4. *The slope of* $\frac{1-F_o(s)}{1-F_p(s)}$ *crosses zero at most once and is otherwise non-zero for all* $s \in (\underline{s}, \bar{s})$.

This assumption is weaker than (2.1) and gives the agents more freedom in terms of disagreement. It's easy to see that (2.1) satisfies Assumption 2.4. Moreover, if the slope of $\frac{1-F_o(s)}{1-F_p(s)}$ does not cross zero, Assumption 2.4 will be equivalent to (2.1). Intuitively, if the slope doesn't cross zero, it's either negative or positive. In that case, Assumption 2.2 implies that the slope has to be positive, which is equivalent to (2.1).

As seen in Figure 2.1, there exists exactly one point at which the two CCDF's cross and the ratio $\frac{1-F_o(s)}{1-F_p(s)}$ is increasing. This is implied by Assumption 2.4.

Corollary 2.2 (Single Crossing and Maximum)**.** *There is a unique* $s_{sc} \in [\underline{s}, \bar{s})$ *such that* $\frac{1-F_o(s_{sc})}{1-F_p(s_{sc})} = 1$ *and* $\frac{f_p(s_{sc})}{1-F_p(s_{sc})} > \frac{f_o(s_{sc})}{1-F_o(s_{sc})}$. *There is also a unique solution* $s_m (> s_{sc})$ *to* $\arg\max_s \frac{1-F_o(s)}{1-F_p(s)}$.

Turns out, s_{sc} is the least amount the optimists are willing to borrow[6] and s_m is the most amount the optimists are willing to borrow[7]. This is useful in determine whether the optimists have too much or too little endowment. In Simsek (2013), $s_{sc} = \underline{s}$ and $s_m = \bar{s}$. Hence, Simsek (2013)'s Assumption A1, $0 < n_o < E_o[s] - \underline{s}$. For the same reason, I make an additional assumption.

Assumption 2.5. $\frac{1-F_p(s_m)}{1-F_o(s_m)} \mathbb{E}_o[s - \min(s, s_m)] < n_o < \mathbb{E}_o[s - \min(s, s_{sc})]$.

Under Assumption 2.5, the optimists don't have too much cash. Thus, they can always find some investment that is more profitable than holding cash. Assumption 2.5 also helps

[6]This means the optimists will not choose simple debt contracts with $D < s_{sc}$

[7]Similarly, this means the optimists will not choose simple debt contracts with $D > s_m$

avoid corner cases when the optimists don't have enough endowment to enter their desired investment. The $\mathbb{E}_o[s - \min(s, D)]$ term is simply the cost of borrowing D to buy asset, i.e. a leveraged position. As shown in Appendix B.1, the term $\frac{1-F_o(D)}{1-F_p(D)}$ is the return on the leveraged position with debt D.

Assumption 2.4 also has some interesting implications. On one hand, if the slope of $\frac{1-F_o(s)}{1-F_p(s)}$ crosses zero exactly once *from below*, the optimists still care more about the best states than the pessimists, but the optimists may also care more about the worst states. In other words, the optimists' belief may have a greater variance than the pessimists' belief. On the other hands, if the slope of $\frac{1-F_o(s)}{1-F_p(s)}$ crosses zero exactly once *from above*, the opposite is true in that the pessimists may believe in greater variance in the future. It is possible to remove Assumption 2.4 completely to consider any belief structure, but the problem becomes much more complex. So instead, I will discuss an example in which the belief structure violates Assumption 2.4.

I also define a slightly stronger assumption

Assumption 2.6. *The slope of* $\frac{1-F_o(s)}{1-F_p(s)}$ *crosses zero at most once and it is from below. Otherwise,* $\frac{1-F_o(s)}{1-F_p(s)}$ *is non-zero for all* $s \in (\underline{s}, \bar{s})$.

Figure 2.1 also satisfies this assumption. This stronger assumption help establishing uniqueness. Under Assumption 2.6, the optimists may believe in larger variance or a larger right tail in the payoff distribution. This is reminiscent of the capital asset pricing model and the optimists believe in a larger beta than the pessimists. There are alternatives to Assumption 2.6.[8]

[8]See discussion in section 2.6.2.

2.4. Equilibrium

First, I define the market clearing conditions.

$$\alpha_o + \alpha_p = 1 \tag{2.7}$$

$$\mu_o(D) + \mu_p(D) = 0 \qquad \forall\, D \in [0, \bar{s}] \tag{2.8}$$

Next, I define the equilibrium and present the first major result.

Definition 2.1 (Nash Equilibrium)**.** A Nash Equilibrium is a collection of prices ($q \in \mathbb{R}_{++},\ \pi : [0, \bar{s}] \to \mathbb{R}_{++}$) and portfolios (α_i, c_i, μ_i) such that the portfolios solve problem (2.6) for each $i \in \{o, p\}$ and the market clears with (2.7) and (2.8).

Proposition 2.1. *Given Assumptions 2.2, 2.3, 2.5, and 2.6, Nash Equilibrium exists and is unique up to allocations and prices of traded contracts. In the unique Nash Equilibrium,*

(1) *only the asset A and simple debt contract $D^*(n_o)$ are traded;*

(2) *optimists hold all of asset A and sells simple debt contract $D^*(n_o)$;*

(3) *$D^*(n_o)$ is strictly decreasing function in n_o;*

(4) *the equilibrium prices are $q = q_o(D^*)$*[9] *and $\pi(D^*) = \mathbb{E}_p[\min(s, D^*)]$;*

(5) *for $D \neq D^*$, the price $\pi(D)$ lies in an interval defined by* (B.24) *and* (B.25)*;*

(6) *q is strictly decreasing in D^*.*

If the optimists also believe in more future variance, there is a unique Nash Equilibrium in which the optimists sell simple debt contract D^* to help fund their purchase of asset A.

---

[9] $q_o(D^*) \equiv \mathbb{E}_p\left[\min(s, D^*)\right] + \frac{1-F_p(D^*)}{1-F_o(D^*)}\mathbb{E}_o\left[s - \min(s, D^*)\right]$. Also see (B.16) in Appendix B.1.4.

In this case, the pessimists would never purchase the asset in equilibrium since asset A's price is higher than the pessimists' valuation and they make zero profit on selling simple debt contracts.

2.4.1. Prices

The traded simple debt contract D^* is priced by the pessimists' break-even price. It is as if the optimists have all the bargaining power when selling the simple debt contracts. Note that I made no assumption about bargaining power, only about the pessimists' wealth (Assumption 2.3). One conclusion that can be drawn is that the more abundant side of the market pays the break-even price. The price of each non-traded simple debt contract lies in an interval and is not unique. They are bounded above and below by the optimists and pessimists' first order conditions. The rest of asset A, after deducing the payoffs to simple debt contract D^*, is priced by the optimists at a discount $\frac{1-F_p(D^*)}{1-F_o(D^*)}$. This discount is the effective bargaining power of the optimists. The overall return from the optimists' investment is exactly the reciprocal, $\frac{1-F_o(D^*)}{1-F_p(D^*)}(\geq 1)$.

Despite the discount, the equilibrium price of asset A may exceed both types of agents' expected values. The overpricing is due to the additional value of asset A as collateral. Specifically, the collateral value of asset A is the profit the optimists gain from selling simple debt contract D^*, or $\frac{1-F_o(D^*)}{1-F_p(D^*)}\mathbb{E}_p[\min(s, D^*)] - \mathbb{E}_o[\min(s, D^*)]$. In Simsek (2013), this collateral value is negative due to the assumption (2.1) on the beliefs. In this framework under the more general belief structure, there are cases when the collateral value is positive. Thus, overpricing may result. Below I give the sufficient condition on the beliefs to general a bubble in the price of asset A.

Proposition 2.2. *Given Assumption 2.2 and Assumption 2.3, for there to be a n_o such that some equilibrium price $q > \mathbb{E}_o[s]$, it is sufficient that the slope of $\frac{1-F_o(s)}{1-F_p(s)}$ crosses zero exactly once from below.*

By take leveraged position, the agents essentially split the asset into two pieces. Each type of agents holds one piece of the asset. If the asset can be split in a way that each holder is more optimistic about her portion of the asset, the combine price of the asset will exceed each individual's valuation. The sufficient condition is a stronger version of Assumption 2.6, where the optimists' belief about the variance is larger than the pessimists' belief. If the disagreement about the variance is large enough relative to their disagreement about the mean, overpricing can result.

Since this is not a pure exchange economy, the overpricing of the asset matters. Even though the agents each hold pieces of the asset and only pay for their pieces, together they pay the full price for the unit of asset A. Since A is help by an unmodeled third party, the more overpricing there is the more money is drained from the economy.

2.4.2. Endogenous Leverage

The promised repayment D in simple debt contracts can be thought of as leverage, since it reduces capital required to purchase the asset. Thus, leverage can be defined as D. The more the agents borrow, the higher their leverage. In this case, only the optimists use leverage. Here, the leverage is endogenous as D^* is determined in equilibrium. Thus, there are two interesting observations from Proposition 2.1.

First, in the unique Nash Equilibrium, the leverage decreases with the optimists' endowment. While higher leverage gives the optimists higher return, leverage is essentially a poor man's tool. Given the market clearing conditions, as the optimists get wealthier, there is still one unit of the asset available. Thus, the optimists cannot leverage up even if they wanted. From anecdotal evidence, this is the reason why hedge funds are not scalable even though they have high returns.

Second, in the similar vein, the unique equilibrium price of asset A is strictly decreasing in leverage. If the equilibrium leverage D^* is higher, it simply means the optimists are paying less for their portion of the asset while owning smaller portion of the asset. This mean the pessimists, who do not use leverage, own bigger portion of the asset. Thus, even though the optimists' leverage is higher, overall leverage in the market actually decreases. Though the equilibrium price decreases with leverage, the equilibrium price is not cheap. As discussed before, the equilibrium price can still be higher than both types of agents' valuations.

2.5. Option Pricing

Since Assumption 2.6 gives uniqueness, I will proceed under this assumption. The asset plus the simple debt contracts replicate option payoffs. When holding the asset and borrowing in the unique equilibrium, the optimists' payoff in each state s is

$$s - \min(s, D^*) = \max(0, s - D^*),$$

which is equivalent to the payoff from a call option written on asset A with strike price D^*. Thus from now on, I will refer to the optimists holding as the call option with strike price D^*. Since any non-traded simple debt contract does not have a unique price, I will examine implications in options pricing across equilibrium for different endowments. I

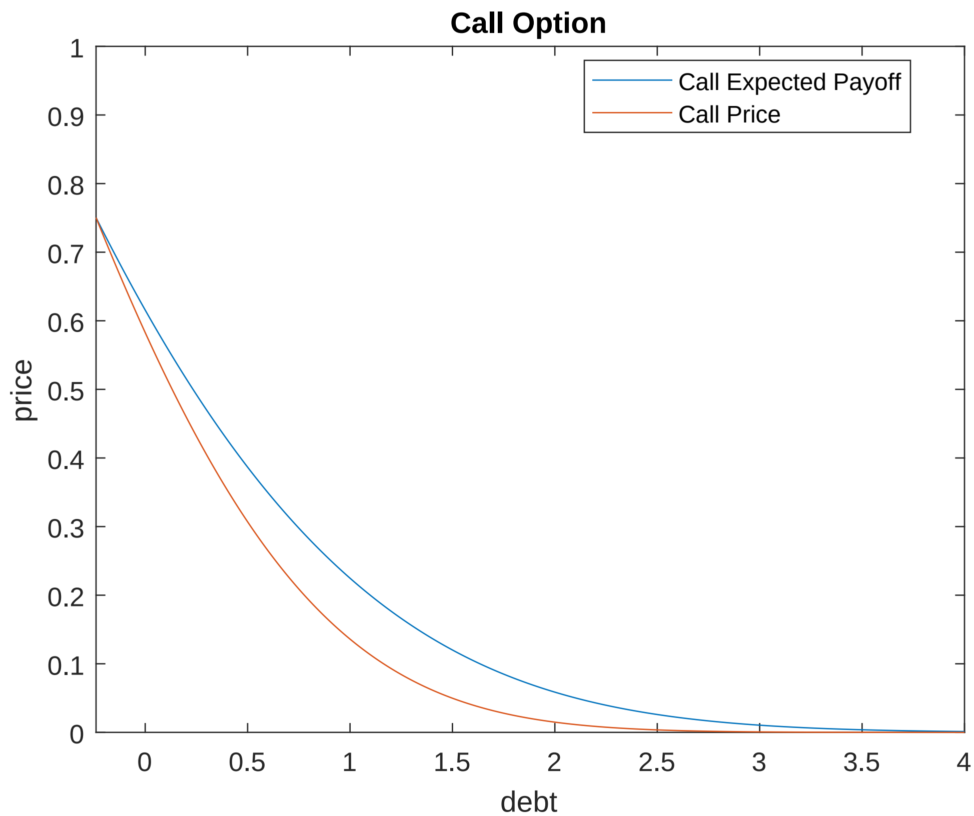

Figure 2.2. Call Option Expected Payoff vs. Price

will focus on call options since put options, obtained through the put-call-parity, tell the same story.

The cross-sectional call options prices can be determined by the following formula. Given equilibrium D^*, the call option price $C(D^*)$ is

$$C\left(D^*\right) = q_o(D^*) - \mathbb{E}_p[\min(s, D^*)] = \frac{1 - F_p(D^*)}{1 - F_o(D^*)}\mathbb{E}_o[\min(s, D^*)]. \tag{2.9}$$

The option price decreases with the equilibrium leverage D^* as the equilibrium price $q_o(D^*)$ decreases with D^*. This is shown in Figure 2.2. Regardless of the equilibrium D^*, the prices of the call options are always below the optimists' expected payoff. The

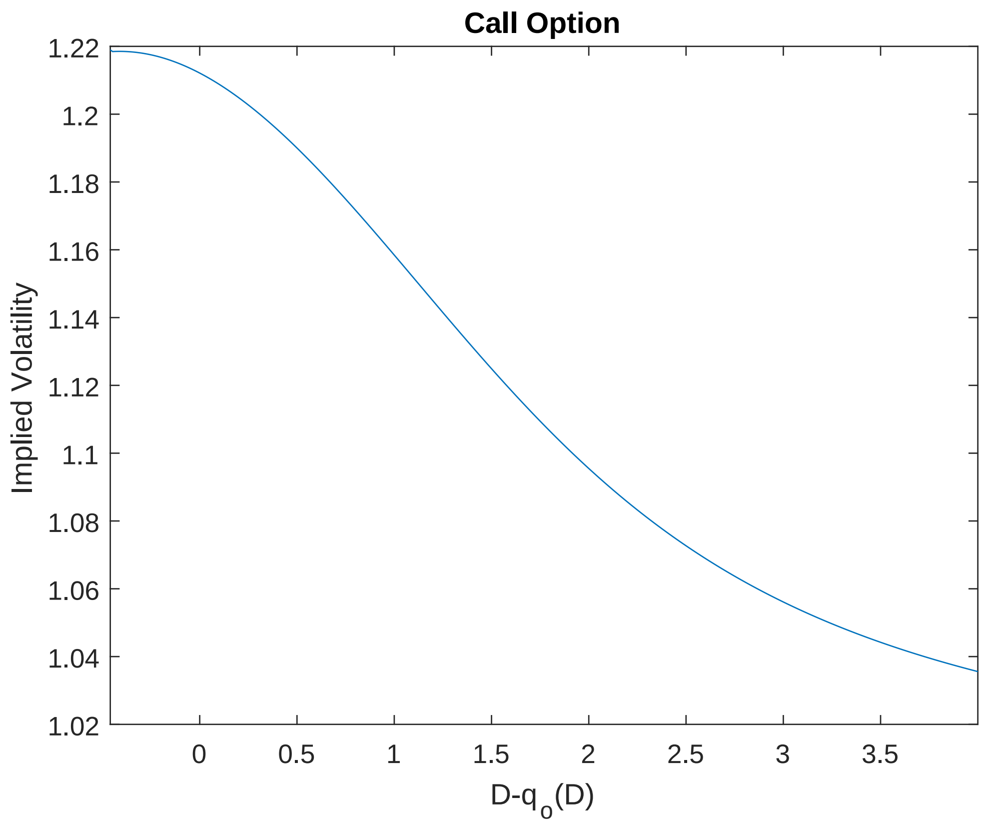

Figure 2.3. Implied Volatility

call options are underpriced due to the discount $\frac{1-F_p(D^*)}{1-F_o(D^*)} < 1$ the optimists enjoy. Under Assumption 2.6, the discount decreases with D^*. Thus, the out of money call options are more expensive. In fact, as shown in Figure 2.3, the call options are either too cheap at low strike prices or too expensive at high strike prices. Consequently, the implied volatility features a downward volatility smirk as seen in Figure 2.3. The steepness in the smirk depends on how fast the discount decreases with D^*. The fast the decrease, the steeper the smirk. In other words, if the optimists are increasingly more optimistic about the good states, the volatility smirk becomes steeper.

Note in Figure 2.3, there are no values for call options with strike prices far below the asset price. This is because these strike prices are not in any equilibrium. The agents in this model would never choose these simple debt contracts, reiterating the risky debt that investors take from Simsek (2013).

This result is different from that of Buraschi and Jiltsov (2006), which also examines heterogeneous beliefs and volatility smirk. In Buraschi and Jiltsov (2006), the main driver of the volatility smirk is the disagreement over the mean. Here however, the main driver of the volatility smirk is the increasing optimism of the optimists over the good states. In this setting, even if the agents were to agree on the mean, as long as they disagree over the variance, volatility smirk would result. Moreover, the agents in this chapter are risk-neutral in contrast to the agents in Buraschi and Jiltsov (2006) with constant-relative-risk-aversion utilities. Thus, the effect of disagreement on option pricing in this chapter is not a result of risk-aversion. It is a result of the effective bargaining power for the optimists due to their beliefs.

2.6. Discussions

2.6.1. Endowment

Assumption 2.3 simplifies the problem by making the pessimists wealthy. For the market to clear, the pessimists cannot achieve a return higher than one. In the parameter space outside of the one defined by Assumption 2.3, the pessimists may achieve a return of $\lambda_b \geq 1$, where λ_b is the Lagrange multiplier on the pessimists' budget constraint. This is equivalent to δ_b for the optimists[10]. Without Assumption 2.3, the pessimists and optimists' returns are determined jointly in equilibrium. In analysis similar to the one in section B.1.3, one can find $\frac{\delta_b}{\lambda_b} = \frac{1-F_o(D)}{1-F_p(D)}$ for simple debt contract D to be traded in equilibrium. This gives rise to the possibility that the pessimists may in fact hold the asset and borrow in equilibrium. Nevertheless, given Assumption 2.6, one can show that there is no equilibrium in which the pessimists hold the asset and borrow. Intuitively, under Assumption 2.6 as shown in Figure 2.1, the optimists value the good states of the asset more than the pessimists. If the pessimists hold the asset and borrow, the pessimists are holding the asset in the good states, which they value less. Thus, if the pessimists prefer to hold the asset in the good states, the optimists would prefer to hold the asset even more. So, there is no equilibrium in which the pessimists hold the asset and borrow as the optimists would always compete for the asset and prevent the market from clearing. Though for small pessimists' endowment n_p, the pessimists can have a return higher than one. This would depress both the asset price and the prices of simple debt contracts. Thereefore, under Assumption 2.6, Assumption 2.3 does not make qualitative differences.

2.6.2. Belief Structure

Assumption 2.6 guarantees uniqueness of the equilibrium. Under the more general Assumption 2.4, there can be multiple equilibria. In particular, the equilibrium in which the pessimists hold the asset and borrow can coexist with the equilibrium in which the optimists hold the asset and borrow. For example, when the pessimists believe in a larger future volatility, Assumption 2.6 is violated while Assumption 2.4 holds. In this case, the

[10]See section B.1.3.

pessimists may actually be more optimistic about the states with higher payoffs. Despite the multiple equilibria, a bubble in the asset price can still result if the pessimists are more optimistic about the states with higher payoffs. This hints at a more general condition for a bubble to exist. Due to the nature of simple debt contracts, as long as the pessimists are more optimistic than the optimists about either the left tail or the right of the payoff distribution, a bubble can exist. Thus, complex financial instruments are not necessary for a bubble to exist in the asset price when the agents disagree about the tail distribution.

While it is possible to generalize the belief structure even further to include cases when the CCDFs crosses multiple times and the slope of their ratio changes signs multiple times, it becomes more of an empirical exercise as multiple equilibria become even more of an issue. Moreover, for normal and log-normal distributions, disagreement over the mean and variance fall under Assumption 2.4. Thus in most cases, Assumption 2.4 suffices.

2.7. Conclusion

This chapter generalizes Simsek (2013) and provides sufficient conditions for a bubble to exist in equilibrium. Bubble exists in this setting because of positive collateral value for the asset and low effective bargaining power from the optimist. The drivers are different from Fostel and Geanakoplos (2015) and Harrison and Kreps (1978). Since the portion of the asset held by the optimists in equilibrium replicates a call option, I also examine option pricing implications. Due to the heterogeneous beliefs, the optimists enjoy a discount when purchasing their portion of the asset. This discount causes the equivalent call options to be undervalued. This is particularly true at higher strike prices. This implies a downward sloping volatility smirk. Here the disagreement over risk in the right tail of the payoff distribution is the main driver of the discount and the volatility smirk.

CHAPTER 3

Attack on the Bubble: Role of a Large Arbitrageur and Desynchronized Small Arbitrageurs

3.1. Introduction

In the past, there have been many asset bubbles. From the earliest Dutch Tulip Mania to the recent Housing Bubble in the Great Recession, all asset bubbles grew and eventually burst. In many cases, the market participants willingly ride the bubble even though they know that the bubble would eventually burst.[1] More interestingly, some market participants enter, exit, and reenter the bubble. One famous anecdote is that the great physicist Sir Isaac Newton bought into the South Sea Bubble in 1720, exited with a large profit, and reentered the bubble a few months later only to lose all of his investments. Commenting on his experience, Newton stated, "I can calculate the motions of heavenly bodies, but not the madness of people." Perhaps Newton himself was the mad/irrational one. This raises the questions of whether rational market participants would choose to ride the bubble and whether they would reenter the bubble if they have previously exited. Moreover, from a policy perspective, it is interesting to study whether a single large market participant can help discipline the smaller market participants and burst the bubble earlier like how George Soros broke the peg on the British pound. This

[1] See Temin and Voth (2004) and Brunnermeier and Nagel (2004).

chapter examines the interaction between large and small market participants to answer the above questions.

This chapter extends the setting from Abreu and Brunnermeier (2003) with a large market participant and uses a novel solution technique to study whether and how large and small market participants can coordinate to burst an asset bubble. I compare my results to the results from the currency attack literature (e.g. Corsetti et al., 2004) since the settings are similar. I find that similar to the currency attack literature, the presence of a large market participant induces the small market participants to attack the bubble more aggressively. In other words, the small market participants ride the bubble for a shorter duration. However, in contrast to the currency attack literature, I also find that the information of the large market participant and the information of the small market participants are complements. The more information the large market participant has, the longer the small market participants ride the bubble. Finally, I find that under certain conditions, the market participants would not reenter the bubble if they have previously exited.

This model is a continuous time model. There is one large arbitrageur and a unit mass of small arbitrageurs. They choose their optimal holding in an asset with an exogenous price process. After a certain time, the price starts growing faster than the fundamental value of the asset and overpricing occurs. The arbitrageurs who hold the asset sequentially become aware of its overpricing. They can choose to sell their share or to ride the price growth. When a sufficient number of arbitrageurs sell their shares, the price collapses to the fundamental value of the asset. It is in the arbitrageur's best interest, therefore, to try to time the price collapse and reap maximum profits. With this framework in mind, the role of the small arbitrageurs is to sustain the overpricing while the role of the large arbitrageur is to discipline the small arbitrageurs and accelerate the bubble bursting.

The main risk in the model is that the arbitrageurs do not know whether they are the first ones or the last ones to discover the overpricing. This is what Abreu and Brunnermeier (2002) call the synchronization risk. More specifically, the arbitrageurs can only perceive time relative to the time when they discover the overpricing. Thus, they have no concept of absolute time. Therefore, it is more convenient to consider the problem in each arbitrageur's relative time, or time relative to the arbitrageur's time of discovery of the overpricing. This is the novel solution technique that this chapter introduces. It greatly simplifies the problem since all the small arbitrageurs solve the same problem in their relative time. The usage of relative time also allows an easy addition of the large arbitrageur for analysis in this chapter.

This chapter is organized in the following manner. Section 3.2 reviews the literature. Section 3.3 presents the model. Sections 3.4 and 3.5 present the equilibrium results. Section 3.6 presents the extension. Finally, 3.7 concludes. The proofs and additional analysis can be found in Appendix C.

3.2. Literature Review

This chapter extends Abreu and Brunnermeier (2003) and is closely related to Doblas-Madrid (2012) and Sato (2015). I use the same synchronization risk that is present in all three papers. The exogenous pricing of the asset and the synchronization risk allows me to study the mechanism of the bursting of an asset bubble. Distinct from all three papers, I use a novel relative time solution technique and introduce a large arbitrageur to study the interaction between the large and small arbitrageurs.

This chapter studies asset bubbles. It is closely related to the various strands of the theoretical bubble literature. There is the overlapping generation models pioneered by Samuelson (1958), Diamond (1965), Tirole (1985), and others. There is Allen and Gorton (1993) from the delegation literature. And, there is Harrison and Kreps (1978) and Scheinkman and Xiong (2003) from the Heterogeneous-Beliefs literature. However, these papers focus on how an asset bubble can exist and be sustained in equilibrium. They do not offer insight into how a bubble may crash. Backward induction generally rules out the coexistence of a bubble and its crash.

This chapter is also related to currency attack/global games literature (e.g. Corsetti et al., 2004), which also focuses on coordination problems between market participants. I compare my results to those in the currency attack literature. There are similarities and differences. The differences highlight the difference between a currency attack and an attack on a bubble asset.

Finally, this chapter is also related to the empirical analyses on asset bubbles: Temin and Voth (2004) and Brunnermeier and Nagel (2004). These papers document market participants riding the bubble which is consistent with the predictions from this model.

3.3. Model

3.3.1. Basic Setting

The model setting follows Abreu and Brunnermeier (2003) closely. The only major difference is the addition of one large arbitrageur. Time t is continuous and lies in the interval

$[0, \bar{\tau}]$, for some finite constant $\bar{\tau}$. There are one large arbitrageur and a unit mass of small arbitrageurs in the economy. The arbitrageurs choose their positions at each time instance in a single asset with an exogenous price process. Since the price is exogenous, there is no need to worry about the supply of the asset. When the arbitrageurs buy or sell the asset, they are guaranteed execution at the exogenous price. One can think of the exogenous price as a price driven by unmodeled behavioral traders or noise traders. For convenience, I restrict short-selling. The instantaneous risk-free rate for holding cash is a constant denoted r.

3.3.1.1. The Asset. The asset has an exogenous price process that coincides with its fundamental value process until some time t_0. Formally,

$$f_t = P_t = e^{gt}, \qquad \forall\, t < t_0,$$

where f_t is the fundamental value process for the asset, P_t is the price process for the asset, and g is a constant denoting the instantaneous grow rate of the fundamental value and the price process. After t_0, the price process remains the same and grows at rate g. The fundamental value process, however, drops to

$$f_t = (1 - \beta(t - t_0))P_t, \qquad \forall\, t \geq t_0,$$

where $\beta(t - t_0) : [0, \bar{\tau}) \to [0, \bar{\beta}]$ is a strictly increasing function of $t - t_0$. To keep the fundamental value process non-negative, I assume $\beta(\bar{\tau}) = \bar{\beta} < 1$. I call $\beta(t - t_0)P_t$ the bubble component in the price, since that is how much the price exceeds the fundamental value. In other words, the overpricing of the asset starts at t_0.

The high and sustained growth in the price process can be justified in the same way as in Doblas-Madrid (2012). To make this model interesting and meaningful, I make the following assumption.

Assumption 3.1. *$g > r$; $\frac{df_t}{f_t} \leq rdt$, for $t \geq t_0$.*

In words, the fundamental value process grows faster than the risk-free rate before t_0 and slower than the risk-free rate after t_0. Therefore, the price process dominates the risk-free rate for all t while the risk-free rate weakly dominates the fundamentals value process after t_0. The time t_0 is random and exponentially distributed with the cumulative distribution function

$$\Phi(t_0) = 1 - e^{-\lambda t_0}, \qquad t_0 \in [0, \infty),$$

for some constant λ.

3.3.1.2. The Small Arbitrageurs. In this economy, there is a unit mass of identical risk-neutral small rational arbitrageurs with initial wealth of 1. They sequentially learn about the overpricing of the asset over the time interval $[t_0, t_0 + \eta]$, for some constant η. More specifically, at each $t' \in [t_0, t_0 + \eta]$, $\frac{1}{\eta}$ of the small arbitrageurs wake up and learn about the overpricing. However, the small arbitrageurs who learned about the overpricing t' do not know the current time t' or the exact time t_0 when the overpricing started. Their best (rational) guess of t_0 given t_i is the conditional cumulative distribution,

$$\Phi(t_0 | t_i, \eta) = \frac{\Phi(t_0) - \Phi(t_i - \eta)}{\Phi(t_i) - \Phi(t_i - \eta)} = \frac{e^{\lambda\eta} - e^{-\lambda(t_0 - t_i)}}{e^{\lambda\eta} - 1}, \qquad \forall\, t_0 \in [t_i - \eta, t_i],$$

where t_i is the time when small arbitrageur i realizes there is an overpricing. Note that $\Phi(t_0|t_i,\eta) = 0$ for $t_0 < t_i - \eta$ and $\Phi(t_0|t_i,\eta) = 1$ for $t_0 > t_i$.

3.3.1.3. The Large Arbitrageur. Now, I introduce a risk-neutral large arbitrageur. I first consider the case in which the large arbitrageur has initial wealth $\eta\kappa$ and learns about the overpricing at time t_0. As defined in section 3.3.1.4, κ is an important constant. I then derive the comparative statics for when the large arbitrageur has wealth $w \leq \eta\kappa$ and is uncertain about when the overpricing started.

3.3.1.4. Bubble and Burst. First I define the price collapse. The price of the asset collapses to the fundamental value if the total selling pressure, i.e. cumulative amount of the asset sold over time, exceeds some constant κ. This is useful in defining a bubble. In the absence of the large arbitrageur, a bubble is defined as in Abreu and Brunnermeier (2003); a bubble is the overvaluation of the fundamentals for a duration longer than $\eta\kappa$. $\eta\kappa$ is the time needed for at least κ small arbitrageurs to learn about the overpricing. Suppose all arbitrageurs buy the asset at $t = 0$[2]. If the overprice persists for a duration longer than $\eta\kappa$, it means that there are enough small arbitrageurs who know about the overpricing to collapse the price but decide not to do so. In other words, they are riding the bubble.

In the presence of the large arbitrageur who has wealth $w \leq \eta\kappa$ and realizes the overpricing at $t_\ell \in [t_0, t_0+\nu]$, a bubble is defined as the overvaluation of the fundamentals for a duration longer than $\eta\kappa \vee (t_\ell \wedge (\eta\kappa - w))$, i.e. the minimum time to reach a total selling pressure of $\eta\kappa$. The intuition is as the following. If weighted by wealth and time,

[2]This is true without loss of generality due to Assumption 3.1.

the large arbitrageur counts as w/η small arbitrageurs.[3] Therefore, to reach a critical mass of κ in selling pressure, duration of $\eta\kappa - w$ is needed. Thus, $t_\ell \wedge (\eta\kappa - w)$ gives the earliest time that enough large and small arbitrageurs learn about the overpricing to collapse the price. Moreover, to remedy the case in which κ small arbitrageurs learn about the overpricing before the large arbitrageur does, I take the minimum $\eta\kappa \vee (t_\ell \wedge \eta\kappa - w)$.

When the price collapses, the bubble bursts. Henceforth, I will refer to the price collapse as the bubble bursting.

3.3.2. Information Structure and Filtration

For simplicity, henceforth I refer to the cohort of the small arbitrageurs who learn about the overpricing at t_i simply as small arbitrageur t_i. For the large arbitrageur who learns about the overpricing at t_ℓ, I refer to her as large arbitrageur t_ℓ. First, I define relative time $t - t_i$ for arbitrageur t_i.[4] This is useful as the arbitrageurs do not know the absolute time when they learn about the overpricing and can only perceive relative time. For arbitrageur t_i, she learns the existence of the overpricing at t_i. Her conditional cumulative distribution is

$$\Phi(t_0 - t_i|\eta) \equiv \Phi(t_0|t_i, \eta) = \frac{e^{\lambda\eta} - e^{-\lambda(t_0 - t_i)}}{e^{\lambda\eta} - 1}, \qquad \forall\, t_0 - t_i \in [-\eta, 0].$$

[3]Time refers to when the arbitrageur learn about the overpricing. At each time, $1/\eta$ small arbitrageurs learn about the overpricing. Thus, there is a mass of $1/\eta$ of small arbitrageurs distributed across time. Since the large arbitrageur learn about the overpricing at a single time, she is comparable to $1/\eta$ of the small arbitrageurs for each unit of wealth she has.

[4]It is the same for both large and small arbitrageurs. So, here I will not distinguish them.

Here, I rewrite Φ as a function of the relative time $t_0 - t_i$. Moreover, $t_0 - t_i$ is independent of t_i.[5] Thus, I can drop the condition on t_i.

Now, let's look at the price process and fundamental value process in terms of relative time. From the perspective of small arbitrageur t_i, I define

$$P(t - t_i) \equiv e^{g(t-t_i)} = P_t / P_{t_i}.$$

Since all quantities are in terms of prices, it is possible to define P_{t_i} as the numeraire for arbitrageur t_i and replace all quantities to be defined with respect to numeraire P_{t_i} for arbitrageur t_i. Thus, the fundamental value process for $t > t_0$ can also be replaced with

$$f(t - t_i, t_0 - t_i) \equiv f_t = (1 - \beta(t - t_0))P(t - t_i) = [1 - \beta((t - t_i) - (t_0 - t_i))]P(t - t_i).$$

Each arbitrageur solves different problems before and after the time when she realizes the overpricing. The problem before the arbitrageur's realization of the overpricing is trivial since the asset that grows at g strictly dominates the risk-free rate r. Thus, I can assume each arbitrageur invest all of her wealth into the asset before her realization of the overpricing and focus on her problem after the discovery without loss of generality. Thus, I only need to consider arbitrageur t_i's problem at t_i. So, the discount $e^{-r(t-t_i)}$ is also a function of relative time. For convenience, I define the transaction cost as $Ce^{r(t-t_i)}$ to keep the arbitrageurs from trading infinite number of times.

[5]Too see this, I have

$$\Phi((t_0 - \epsilon) - (t_i - \epsilon)|\eta) = \frac{e^{\lambda\eta} - e^{-\lambda((t_0-\epsilon)-(t_i-\epsilon)}}{e^{\lambda\eta} - 1} = \frac{e^{\lambda\eta} - e^{-\lambda(t_0-t_i)}}{e^{\lambda\eta} - 1} = \Phi(t_0 - t_i|\eta),$$

for any $\epsilon > 0$. Moreover, $(t_0 - \epsilon) - (t_i - \epsilon) = t_0 - t_i \in [-\eta, 0]$.

As arbitrageur t_i does not know the absolute time of t_i, the arbitrageur's action cannot be contingent on the absolute time. Thus, the arbitrageur's action must be defined as a function of relative time,

$$\sigma(t-t_i):(-\infty,\infty)\rightarrow[0,1]. \tag{3.1}$$

Here, I define $1-\sigma(t-t_i)$ as proportion of arbitrageur t_i's wealth that is invested in the asset at time t. Thus, wealth times $\sigma(t-t_i)$ is the selling pressure of arbitrageur t_i. Notice the range of σ is positive and thus rules out short-selling. For notation simplicity, relative time will be denoted as

$$\tau_0=t-t_0, \qquad \tau_i=t-t_i, \qquad \tau_i^0=t_0-t_i.$$

3.3.3. The Small Arbitrageurs' Problem

I first define some useful notations and quantities before writing out the small arbitrageurs' problem. I define the cumulative selling pressure as

$$s(\tau_0)=\int_{(\tau_0-\eta)\wedge 0}^{\tau_0}\sigma(\tau_i)d\tau_i. \tag{3.2}$$

The burst time of the bubble given the selling pressure can then be defined as

$$T^*(t_0)=\inf\{\tau_0+t_0|s(\tau_0)\geq\eta\kappa\}=t_0+\inf\{\tau_0|s(\tau_0)\geq\eta\kappa\}=t_0+\bar{T}. \tag{3.3}$$

where $\bar{T}$ is independent of t_0. $\bar{T}$ is also treated as a constant by all arbitrageurs, since each small arbitrageur has infinitesimal weight and cannot affect $s(\tau_0)$. I define $\inf\{\emptyset\}=\bar{\tau}$ as the exogenous burst time. The cumulative probability function of bursting can then be defined as[6]

$$\Pi(\tau_i|\eta) = \Phi(\tau_i - \bar{T}|\eta).$$

I define $\pi(\tau_i|\eta) \equiv \frac{d\Pi(\tau_i|\eta)}{d\tau_i} = \frac{d\Phi(\tau_i-\bar{T}|\eta)}{d\tau_i} \equiv \phi(\tau_i - \bar{T}|\eta)$ to be the probability density functions of bursting.

Small arbitrageur t_i's problem at time t_i is as the following.

$$\max_{\sigma'} \int_0^{\bar{T}} \left[\int_0^{\tau_i'} e^{-r\tau_i''} P\left(\tau_i''\right) \sigma'\left(\tau_i''\right) d\tau_i'' \right.$$
$$\left. + e^{-r\tau_i'} \left(1 - \sigma\left(\tau_i'\right)\right) \left[1 - \beta\left(t_i + \tau_i' - T^{*-1}\left(t_i + \tau_i'\right)\right)\right] P\left(\tau_i'\right) \right] d\Pi(\tau_i'|\eta) - C \int_0^{\tau_i} |\sigma'(\tau_i')| d\tau_i'$$

Notice that since $T^*(t) = t + \bar{T}$, the inputs of the β function above can be simplified to $\beta(\bar{T})$. Also, note that since before small arbitrageur t_i's realization of the overpricing, the asset price strictly dominates the risk-free rate. It is optimal for the risk-neutral small arbitrageur t_i to be fully invested in the stock. The above can be stated more formally as

Lemma 3.1. *Since $P_t > e^{rt}$, $\sigma(\tau_i) = 0$ for all $\tau_i < 0$.*

The proof is trivial as stated above. Therefore, it is not necessary to include $\tau_i < 0$ in the maximization problem. Moreover, since t_i is arbitrary and appears nowhere in the objective function, all small arbitrageurs have the same maximization problem. Thus, if the solution to the small arbitrageur's problem is unique, all the small arbitrageurs should have the same solution in relative time.

[6] $\Pi(\tau_i|\eta) = \int_{T^*(t_0)<t} d\Phi(t_0 - t_i|\eta) = \int_{t_0+\bar{T}<t} d\Phi(t_0 - t_i|\eta) = \int_{t_i+\tau_i^0+\bar{T}<t} d\Phi(\tau_i^0|\eta) = \int_{\tau_i^0<\tau_i-\bar{T}} d\Phi(\tau_i^0|\eta) = \Phi(\tau_i - \bar{T}|\eta)$

3.3.4. The Large Arbitrageur's Problem

In this section, I present the general problem for the large arbitrageur who has wealth $w \leq \eta\kappa$ and realizes the bubble at time g uniformly distributed in $[t_0, t_0+\nu]$, where $\nu \leq \bar{T}$. I call ν the inverse information quality of the large arbitrageur's information, since the large arbitrageur's information becomes more accurate the smaller ν becomes. But before proceeding, I make one simplifying assumption.

Assumption 3.2 (Almost-Complete Information)**.** *Even though the large arbitrageur does not know the realization of absolute time t_ℓ, the small arbitrageurs do. Nevertheless, the small arbitrageurs only know absolute time t_ℓ, not time relative to the realizations of their own t_i's.*

Under this assumption, the small arbitrageurs do not have to be concerned with the uncertainty of the large arbitrageur. Because of this, the selling pressure function $s(\tau_0)$ keeps the same form when including the large arbitrageur, greatly simplifying the analysis. This is comparable to the assumption on the information of small arbitrageurs, since each small arbitrageur knows the absolute time for the realization of all the other small arbitrageurs.

Now, I present the large arbitrageur's problem. I will later show that the large arbitrageur will not reenter the market after she first exists. Thus, the problem simplifies to optimal exit time problem as the following.

(3.4)

$$\max_{\gamma} w \times \int_0^{\gamma} e^{-r\gamma'}(1-\beta(\bar{T}))P(\gamma')\phi(\gamma'-\bar{T}|\nu)d\gamma'$$

$$+\int_{\gamma}^{\bar{T}} e^{-r\gamma} P(\gamma)[(\eta\kappa - \bar{s}(\gamma - \gamma' + \bar{T})) \vee w]\phi(\gamma' - \bar{T}|\nu)d\gamma'$$

$$+\int_{\gamma}^{\bar{T}} e^{-r\gamma}(1 - \beta(\gamma - \gamma' + \bar{T}))P(\gamma)[(\bar{s}(\gamma - \gamma' + \bar{T}) - (\eta\kappa - w)) \wedge 0]\phi(\gamma' - \bar{T}|\nu)d\gamma'$$

$$-C$$

where $\bar{s}(\tau_0) = s(\tau_0|\bar{T} = \tau_0)$, i.e. the selling pressure of all small arbitrageurs at $t_0 + \tau_0$ if they expect the bubble to burst at $t_0 + \tau_0$.

3.4. Equilibrium with only Small Arbitrageurs (Benchmark Case)

In this section, I present the benchmark equilibrium in the economy with only small arbitrageurs. The equilibrium in the is section provides a good benchmark against the equilibrium in the economy with both large and small arbitrageurs. First, I define the equilibrium below.

Definition 3.1 (Equilibrium)**.** An equilibrium is a defined as a Perfect Bayesian Nash Equilibrium, where agents optimizes their actions based on their (correct) beliefs of other agents' optimal actions.

3.4.1. Equilibrium: No updating

Before stating the equilibrium result, I specify a technical assumption.

Assumption 3.3. $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} < \frac{g-r}{\beta(\eta\kappa)}$.

This assumption is very useful as it helps rule out the no-bubble equilibrium and helps compute the equilibrium bubble bursting time.[7]

[7]See section C.1.1

Proposition 3.1. *The bubble bursts exogenously at* $t_0+\bar{\tau}$ *if and only if* $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} \leq \frac{g-r}{\beta}$. *Moreover, the bubble bursts endogenously at* $t_0+\tau_0^* \equiv t_0+\beta^{-1}\left(\frac{(g-r)(1-e^{-\lambda\eta\kappa})}{\lambda}\right)$ *if and only if* $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} > \frac{g-r}{\beta}$.

If the bubble bursts endogenously at or after $\bar{\tau}$, then $\bar{s}(\tau_0) < \eta\kappa$ for all $\tau_0 < \bar{\tau}$. In other words, the small arbitrageurs will not be able to coordinate to reach a cumulative selling pressure of $\eta\kappa$ before $\bar{\tau}$. Therefore, in this equilibrium, the bubble will burst at $\bar{\tau}$ for sure. Thus, one can consider endogenous bursting of the bubble at and after time $\bar{\tau}$ as exogenous bursting. For example, if I were to add an arbitrageur who can reach a selling pressure of $\eta\kappa$ by herself (i.e. the large arbitrageur) to this setting, the rest of the arbitrageurs would simply treat the action of the new arbitrageur, e.g. bursting the bubble by herself, as exogenous and would modify their own strategies accordingly.

3.4.2. Equilibrium with Updating

The only new source of information after t_i for each small arbitrageur t_i is whether her selling of the asset bursts the bubble. On one hand, if the bubble didn't burst after small arbitrageur t_i sells, small arbitrageur t_i would immediately realize that she is one of the first κ small arbitrageurs to realize the overpricing. That is, $t_0 \in [t_i-\eta\kappa, t_i]$. This information would change the small arbitrageur's posterior cumulative distribution function of t_0 to $\Phi(\tau_i^0|\eta\kappa)$. On the other hand, if the bubble bursts before the small arbitrageur sells, the small arbitrageur would realize that she is one of the latter $1-\kappa$ small arbitrageurs to realize the overpricing. However, since the bubble already burst, there is nothing the small arbitrageur can do except exiting the market (or staying in the market if the post-crash price/fundamental value grows at same rate as the risk-free interest rate). Note that the small arbitrageur still has to sell at τ_i^* so that she can get this new information. Thus, the small arbitrageur problem with and without updating only differs after τ_i^*, i.e. small arbitrageur's decision to reenter the market if her sale does not burst the bubble. I write the small arbitrageurs' new problem of reentry as the following, from the perspective of τ_i (if the bubble didn't burst after τ_i^*).

$$\max_{\sigma'} \int_{\tau_i^*}^{\bar{T}} \sigma'\left(\tau_i'\right) \underbrace{\left[e^{-r\tau_i'} P\left(\tau_i'\right)\left(1-\Pi\left(\tau_i'|\eta\kappa\right)\right) - \int_{\tau_i'}^{\bar{T}} e^{-r\tau_i''}\left[1-\beta\left(\bar{T}\right)\right] P\left(\tau_i''\right) d\Pi(\tau_i''|\eta\kappa) - C \times sign(\sigma'(\tau_i'))\right]}_{A(\tau_i')} d\tau_i'$$

With the new problem for the small arbitrageurs, I have the following result.

Proposition 3.2. *The small arbitrageurs do not reenter even with updating. Thus, the equilibrium with updating is equivalent to the equilibrium without updating.*

Proposition 3.2 establishes the result that small arbitrageurs only use symmetric triggering strategy and thus proves the claim of Abreu and Brunnermeier (2003) that when each small arbitrageur's asset holding is less than maximum, she correctly believes that the asset holding of all small arbitrageurs who became aware of the bubble prior to her are also at less than maximum. Since the equilibrium with updating is equivalent to the equilibrium without updating, I will proceed to use the results from Proposition C.1 and 3.1 in the following analysis.

3.5. Equilibrium with Large and Small Arbitrageurs

The equilibrium notion here is same as before except I augment the set of arbitrageurs with the large arbitrageur. First, let us rearrange (3.4) to what follows, so that it's easier to work with.[8]

$$\max_{\gamma} \; w \times \int_0^{\bar{T}} e^{-r\gamma'}(1-\beta(\bar{T}))P(\gamma')\phi(\gamma'-\bar{T}|\nu)d\gamma'$$
$$+\int_{\gamma}^{\bar{T}} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})[(\eta\kappa-\bar{s}(\gamma-\gamma'+\bar{T}))\vee w]\phi(\gamma'-\bar{T}|\nu)d\gamma' - C \tag{3.5}$$

First, I will consider the case in which the large arbitrageur has complete information, i.e. $\nu = 0$, and enough wealth to burst the bubble by herself, i.e. $w = \eta\kappa$. Then, I will consider the other extreme case where the large arbitrageur has minimum information, i.e. $\nu = \bar{T}$, and maximum wealth $w = \eta\kappa$. Though I call $\nu = \bar{T}$ the minimum possible information, I realize that ν can be even larger than $\bar{T}$. However, I will show in section 3.5.3 that the minimum ν, denoted $\underline{\nu}$, is actually less than $\bar{T}$ and that the equilibrium is identical for all $\nu \geq \underline{\nu}$. Finally, I examine the more general case with $\nu \in (0, \bar{T})$ and $w = \eta\kappa$. For tractability, I henceforth define $\beta(\tau_0) = 1 - e^{-(g-r)\tau_0}$.[9] And, I state another assumption.

Assumption 3.4. $\frac{\lambda}{1-e^{-\lambda\eta}} \leq g - r$ ($\Leftrightarrow \frac{1}{\lambda}[\log(g-r) - \log(g-r-\lambda)] \leq \eta$)

Assumption 3.4 is fairly innocuous. It holds automatically in the case of exogenous bursting of the bubble. In the endogenous case, the burst time depends only on $\eta\kappa$. Thus, if the above condition doesn't hold, I can simply increase η (to η') until the above condition holds and decrease κ (to κ') so that $\eta'\kappa' = \eta\kappa$.

[8]For completeness see section C.1.2
[9]See section C.1.3 for detail.

3.5.1. Case 1: Complete information and Maximum Wealth

Proposition 3.3. *There exists a unique equilibrium in which* $0 < \gamma^* \le \bar{T}$ *solves* (C.1) *and the large arbitrageur does not reenter after she sells at* γ^**. Moreover,* $\gamma^* = \bar{T}$ *if and only if* $\bar{T} = \bar{\tau}$*, i.e. the bubble bursts exogenously.*

Given the unique equilibrium, I now examine whether the overpricing lasts long enough to be called a bubble. Since the large arbitrageur has wealth $w = \eta\kappa$ and becomes aware of the overpricing at t_0, the overpricing is a bubble if the price doesn't collapse at t_0. Since the large arbitrageur has enough wealth to burst the bubble anytime, the bubble bursts exactly at γ^* when the large arbitrageur sells. By Proposition 3.3, $\gamma^* > 0$. That is, the large arbitrageur sells after $t_0 + \gamma^* > t_0$. So, there is indeed a bubble equilibrium.

In the case that $\bar{T}$ is endogenous, i.e the bubble bursts endogenously by the small arbitrageurs at $\bar{T}$ in absence of the large arbitrageur, Proposition 3.3 shows that $\gamma^* < \bar{T}$. In other words, the large arbitrageur accelerates the bursting of the bubble. Moreover, the small arbitrageurs attack the bubble earlier when compared to the benchmark case. Formally,

Corollary 3.1. *When there is a the large arbitrageur in the market and* $\bar{T}$ *is endogenous, the time of sale for each small arbitrageur is* $\tau_i^{**} = \gamma^* - \frac{1}{\lambda}(\log(g-r) - \log(g - r - \lambda\beta(\gamma^*)) < \tau_i^*$.

The result from Corollary 3.1 is in line with the findings in Corsetti et al. (2004). The presence of a large arbitrageur makes the small arbitrageur more aggressive in attacking both the bubble in this chapter and the currency peg in Corsetti et al. (2004). However, I must exercise care when comparing the model/results here to that in Corsetti et al.

(2004), since the main uncertainty in this model is time (i.e. synchronization risk a la Abreu and Brunnermeier (2002)) whereas the main uncertainty in Corsetti et al. (2004) is the fundamental value (i.e. fundamental risk). What time is to this model is equivalent to what probability is to Corsetti et al. (2004). Nevertheless, the intuition transcends the differences between this chapter and Corsetti et al. (2004). The reason for the similarity in the results is that the presence of a large arbitrageur in both models improves the coordination of the small arbitrageurs. Intuitively, since the large arbitrageur exerts non-negative selling pressure, the presence of a large arbitrageur decreases the threshold number of small arbitrageurs required to burst the bubble and thus makes it easier for the small arbitrageurs to coordinate. However, the similarity between the results here and those in Corsetti et al. (2004) is only limited to the endogenous case. According to Proposition 3.3, when the small arbitrageurs cannot coordinate to burst the bubble endogenously, it is optimal for the large arbitrageur hold the bubble asset until time $t_0+\bar{\tau}$. In other words, if the bubble bursts exogenously in absence of the large arbitrageur, the large arbitrageur does not accelerate the bubble bursting, nor does she make the small arbitrageurs more aggressive. Formally,

Corollary 3.2. *When there is a the large arbitrageur in the market and $\bar{T}$ is exogenous, the time of sale for each small arbitrageur is* $\tau_i^{**} = \gamma^* - \frac{1}{\lambda}(\log(g-r) - \log(g-r-\lambda\beta(\gamma^*)) = \tau_i^*$.

Thus, while the presence of the large arbitrageur accelerates the endogenous bursting, the presence of the large arbitrageur cannot covert the endogenous bursting to exogenous bursting or vice versa. Here is another way to interpret the result so that it's readily comparable to that of Corsetti et al. (2004). Suppose exogenous bursting means the bubble never bursts, i.e. $\bar{\tau} = \infty$. Then, the large arbitrageur would not burst the bubble if the small arbitrageurs could not do so by themselves. In the words of Corsetti et al. (2004), the presence of the large arbitrageur does not affect the probability of the bubble bursting. This result is in contrast with another result from Corsetti et al. (2004), i.e. the presence of a large arbitrageur unambiguously increases the probability of attack. The difference between Corollary 3.2 and the result from Corsetti et al. (2004) is driven by the difference between an attack on pegged currency and an attack on growing bubble. Since the price of the bubble asset is growing exponentially, the large arbitrageur stands to gain if the small arbitrageurs cannot coordinate. In Corsetti et al. (2004), however, the peg is constant. So, the large arbitrageur gains nothing by not attacking the peg regardless of the small arbitrageurs' coordination ability.

It is important to note, however, that the large arbitrageur's increase probability of the bubble bursting as her information becomes incomplete. Intuitively, as the large arbitrageur loses track of absolute time, she becomes more cautious and would start selling (or attacking the bubble) earlier, even when the bubble would otherwise burst exogenously. I shall first examine the case when the large arbitrageur has worst possible information, i.e. $\nu = \bar{T}$.

3.5.2. Case 2: Minimum Information and Maximum Wealth

In this case, I examine the large arbitrageur's optimal exit time when she has minimum information, i.e. $\nu = \bar{T}$. Since the large arbitrageur should be more cautious, the optimal exit time relative to the time when the large arbitrageur becomes aware of the overpricing should be smaller than γ^* from last subsection.

Proposition 3.4. *There exists a unique solution* $0 < \gamma^{**} < \bar{T}$ *to the large arbitrageur's problem*[10].

Recall t_ℓ is the absolute time when the large arbitrageur becomes aware of the bubble. Fixing $t_\ell = t_0$, i.e. that the large arbitrageur is aware of the bubble since t_0 but thinks that $t_\ell \in [t_0, t_0 + \bar{T}]$, I can compare this case directly to Case 1. In this case, the large arbitrageur sells earlier than $\bar{\tau}$ since $\gamma^{**} < \bar{T} \leq \bar{\tau}$ by Proposition 3.4. Therefore, when the bubble bursts exogenously, the large arbitrageur sells earlier in this case than in the previous case with complete information. In other words she bursts the bubble endogenously. The large arbitrageur also sells earlier when the bubble bursts endogenously.

Corollary 3.3. $\gamma^{**} < \gamma^*$

The presence of the large arbitrageur with incomplete information unambiguously increases the probability of the bubble bursting. Following the footsteps of Corsetti et al. (2004), now I analyze the effect of the large arbitrageur's information on the small arbitrageurs. With Assumption 3.2, I can rule out any direct effect of the large arbitrageur's information on the small arbitrageurs. Thus, the large arbitrageur only has indirect effect on the small arbitrageurs through her time of sale. According to Corollary 3.1, the small arbitrageurs' optimal selling time increases with the large arbitrageur's optimal selling time. Therefore, Corollary 3.3 also implies that small arbitrageurs sell earlier in this case when compared to Case 1 and the Benchmark Case. Now the result is more akin to

[10] See C.2

that of Corsetti et al. (2004). That is, the presence of the large arbitrageur makes the small arbitrageurs more aggressive regardless of whether the bubble bursts endogenously in absence of the large arbitrageur. However, this result is also in direct contrast with yet another result from Corsetti et al. (2004). While the small arbitrageurs here are more aggressive in Case 2 than in Case 1, the small arbitrageurs in Corsetti et al. (2004) are less aggressive in their equivalent of my Case 2 and in their equivalent of my Case 1.[11]

The intuition of the result in Corsetti et al. (2004) is the following. If the large arbitrageur has better information (relative to the smaller arbitrageurs), she can identify the true fundamental value better and attack the peg with higher accuracy and probability. Since the probability of the large arbitrageur directly enters into the small arbitrageurs' payoff function, increased probability would cause the small arbitrageurs to be more aggressive. Knowing this, the large arbitrageur would become more aggressive herself. If, however, each small arbitrageur has better information (relative to the large arbitrageur), the small arbitrageurs may fail to coordinate since each one has independent signals. Thus, to the small arbitrageurs, the coordination-free large arbitrageur's information is more valuable than their own information. In a sense, the information of the small arbitrageurs and the large arbitrageur are substitutes.

In this model, however, the information quality of the large arbitrageur does not affect the information quality of the small arbitrageurs (as ruled out by Assumption 3.2). The information quality of the large arbitrageur only affects the actions of the small arbitrageurs indirectly through the action of the large arbitrageur. Thus, Assumption

[11] If I hold the small arbitrageurs' information constant in Corsetti et al. (2004), I can map Case 1 and 2 to two of the limiting cases in Corsetti et al. (2004).

3.2 essentially eliminates the information effect and allows me to focus on the large-arbitrageur effect, i.e. the effect of changing the large trader's optimal exit time on the small arbitrageurs. With only the large-arbitrageur effect, the information of the large arbitrageur and small arbitrageurs are more like complements.

Moreover, the large arbitrageur attacks the bubble less aggressively the more information she has, since there are gains by holding the exponentially growing bubble asset. In the absence of Assumption 3.2, the better information of the large arbitrageur also gives better information to the small arbitrageurs and would thus make the small arbitrageurs attack more aggressively. Thus, if I were to remove Assumption 3.2, I would get conflicting results with the information effect increasing the small arbitrageurs' aggressiveness and the large-trader effect would decrease the small arbitrageurs' aggressiveness. I want to focus on the large-arbitrageur effect and thus my decision to have Assumption 3.2.

In the next subsection, I solve the general case with $\nu \in (0, \bar{T})$ and $w = \eta\kappa$ and show that the results holds in the intermediate cases as well.

3.5.3. Case 3: Incomplete Information and Maximum Wealth

In this case, I have the following result.[12]

Proposition 3.5. *There exists a unique equilibrium $\gamma^{***} = \arg\max_\gamma H(\gamma)$. Moreover, $\gamma^{***} = \gamma^{**}$ if $\nu \geq \bar{T} - \gamma^{**}$ and $\gamma^{**} < \gamma^{***} \leq \bar{T} - \nu$ if $\nu < \bar{T} - \gamma^{**}$. Also, if $\nu < \bar{T} - \gamma^{**}$, $\gamma^{***} = \bar{T} - \nu$ only if $\bar{T} = \bar{\tau}$.*

There exists a unique equilibrium for the general $\nu < \bar{T}$. I also have $\gamma^{***} \geq \gamma^{**}$. This implies that the optimal exit time for the large arbitrageur (or the burst time of the

[12]See section C.1.5 for additional analysis

bubble) is weakly later than the optimal exit time in Case 2. Thus, the small arbitrageurs attack less aggressively than in Case 2. Moreover, as stated in Proposition 3.5, $\gamma^{***} = \gamma^{**}$ if and only if $\nu \geq \bar{T} - \gamma^{**}$. This mean I overstated the worst-case scenario at the beginning of this section. the large arbitrageur's strategy is not affected by the deterioration of her information if her (inverse) information quality is already $\nu = \bar{T} - \gamma^{**}$. Thus, the worse information quality for the large arbitrageur is $\underline{\nu} = \bar{T} - \gamma^{**}$. Now, two questions remain: 1. Can γ^{***} be larger than γ^{*}? 2. What's the relationship between γ^{***} and ν? The answer to the first question in the exogenous case is trivial. By Proposition 3.5, $\gamma^{***} \leq \bar{T} - \nu \leq \bar{T} = \bar{\tau} = \gamma^{*}$. For the answer to the first question in the endogenous case and the answer to the second question, I have the following lemma.

Lemma 3.2. *$\gamma^{***} \leq \gamma^{*}$. Moreover, γ^{***} is strongly monotonic in $-\nu$ for all $\nu < \underline{\nu}$.*

This Lemma states that the large arbitrageur attacks the bubble less aggressively the more information he has. This would also mean the small arbitrageurs attack the bubble less aggressively when the large arbitrageur has more information. Again note that this result follows from the large-arbitrageur effect.

Thus far, I have only used the case where the large arbitrageur has $\eta\kappa$. In the next section, I will evaluate the comparative statics with respect to wealth.

3.6. Extensions

3.6.1. Comparative Statics on Wealth of the large arbitrageur

Going back to the large arbitrageur's problem in the beginning of Section 3.5 with general w, I have

$$\max_{\gamma}\ w \times \int_0^{\bar{T}} e^{-r\gamma'}(1-\beta(\bar{T}))P(\gamma')\phi(\gamma'-\bar{T}|\nu)d\gamma' \\ + \int_{\gamma}^{\bar{T}} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})[(\eta\kappa-\bar{s}(\gamma-\gamma'+\bar{T}))\vee w]\phi(\gamma'-\bar{T}|\nu)d\gamma' - C$$

For any fixed w, the first and last terms are constants with respect to the problem, so I can simplify it to the following

$$\max_{\gamma} \int_{\gamma}^{\bar{T}} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})[(\eta\kappa-\bar{s}(\gamma-\gamma'+\bar{T}))\vee w]\phi(\gamma'-\bar{T}|\nu)d\gamma'$$

With the min function in the heart of the objective function. I can't very well take the derivative to get the first order condition as it is. However, I can modify split the objective function into two parts, one part unrestricted by w and another part with only w.

$$\max_{\gamma} \int_{\gamma}^{\gamma+\bar{T}-\bar{s}^{-1}(\eta\kappa-w)} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})[\eta\kappa-\bar{s}(\gamma-\gamma'+\bar{T})]\phi(\gamma'-\bar{T}|\nu)d\gamma' \\ + \int_{\gamma+\bar{T}-\bar{s}^{-1}(\eta\kappa-w)}^{\bar{T}} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})w\phi(\gamma'-\bar{T}|\nu)d\gamma' \tag{3.6}$$

For any fixed w, let $\hat{\gamma}(w) \equiv \bar{s}^{-1}(\eta\kappa - w)$.[13] Then, it is clear that for all $\gamma \geq \hat{\gamma}$, the above objective function is exactly the same as the one in Section C.1.5. Thus, if $\hat{\gamma}(w) \leq \gamma^{***}$,

[13] $\bar{s}^{-1}$ exists in the endogenous case and in the exogenous case if $\eta\kappa - w$ is not in the range of $\bar{s}$, I set $\hat{\gamma} = \bar{\tau}$.

the equilibrium with w would be exactly the same as the equilibrium with $w = \eta\kappa$ in section 3.5.3. However, if $\hat{\gamma}(w) > \gamma^{***}$, the optimal selling time for the large arbitrageur is $\gamma^{(4)} \in (\gamma^{***}, \hat{\gamma}(w))$. To see this, first note that for all $\gamma \geq \hat{\gamma}(w)$, objective function in (3.6) is same as H. By properties of H, $H'(\hat{\gamma}(w)) < 0$. Thus, the first order condition for (3.6) is also negative at $\hat{\gamma}(w)$. Let us denote the objective function in (3.6) as L. Then,

$$\begin{aligned} L(\gamma) =& H(\gamma) \\ &+ \int_{\gamma+\bar{T}-\hat{\gamma}(w)}^{\bar{T}} e^{-r\gamma} P(\gamma)\beta(\gamma-\gamma'+\bar{T})(w-[\eta\kappa-\bar{s}(\gamma-\gamma'+\bar{T})])\phi(\gamma'-\bar{T}|\nu)d\gamma'. \end{aligned}$$

This implies

$$\begin{aligned} L'(\gamma^{***}) =& H'(\gamma^{***}) \\ &+ \int_{\gamma^{***}+\bar{T}-\hat{\gamma}(w)}^{\bar{T}} \left\{(g-r)e^{(g-r)\gamma^{***}}\beta(\gamma^{***}-\gamma'+\bar{T})(w-[\eta\kappa-\bar{s}(\gamma^{***}-\gamma'+\bar{T})])\right. \\ &+ e^{(g-r)\gamma^{***}}\beta'(\gamma^{***}-\gamma'+\bar{T})(w-[\eta\kappa-\bar{s}(\gamma^{***}-\gamma'+\bar{T})]) \\ &\left.+ e^{(g-r)\gamma^{***}}\beta'(\gamma^{***}-\gamma'+\bar{T})\bar{s}'(\gamma^{***}-\gamma'+\bar{T})\right\}\phi(\gamma'-\bar{T}|\nu)d\gamma' \\ >& H'(\gamma^{***}) = 0. \end{aligned}$$

where the last inequality is from $\hat{\gamma}(w) > \gamma^{***}$ and $w > \eta\kappa - \bar{s}$ in the range of the integral. Thus, $\gamma^{(4)}(w) > \gamma^{***}$. The exogenous case $\hat{\gamma}(w) = \bar{\tau}$ is trivial with $\gamma^{(4)} = \bar{\tau}$. In this general case, it's difficult to say whether $\gamma^{(4)}$ is monotonic in $-w$ without further analysis. Nevertheless, in Case 1, since $\gamma^{(4)}(w) = \gamma^{*} \wedge \hat{\gamma}(w)$, $\gamma^{(4)}$ is weakly monotonic in $-w$. More specifically, $\gamma^{(4)}$ is (strongly) monotonic in $-w$ if and only if $w \leq \eta\kappa - \bar{s}(\gamma^{*})$.

3.7. Conclusion

In this chapter, I first solve the Abreu and Brunnermeier (2003) model with a different approach. Specifically, I analyze the information that is available to each arbitrageur and conditioned the arbitrageurs' actions on the information available to them. In this case, since all the small arbitrageurs have common prior and the distribution of the main uncertainty is updating-proof, all the small essentially perceive the same information in the time frame that is relative to their own time. Thus, the arbitrageurs' actions in relative time must be identical. This greatly simplifies the problem since now the arbitrageurs no longer need to worry about absolute time. With this novel method, I prove that arbitrageurs use trigger strategies and would not reenter the market even when they update on new information.

Next, I examine the implications introducing a large arbitrageur. Specifically, I study the effect of the presence of the large arbitrageur on the small arbitrageurs. I found that the presence of the large arbitrageur induces the small arbitrageurs to sell earlier by partially alleviating the synchronization/coordination problem between the small arbitrageurs. This result is in line with Corsetti et al. (2004), who also suggest that the presence of a large arbitrageur reduces the coordination problem between small arbitrageurs in a global-games setting. Nevertheless, my results do have some differences with those of Corsetti et al. (2004). There is one result from this model that is absent from Corsetti et al. (2004). If the large arbitrageur has perfect information and the small arbitrageur cannot coordinate, then the large arbitrageur will not attack the bubble until the last moment when the bubble bursts exogenously. This model also suggests that the small arbitrageurs are less aggressive when the large arbitrageur's information improves. This is in direct contrast with one result from Corsetti et al. (2004). In Corsetti et al. (2004), the small arbitrageurs become more aggressive when the large arbitrageur's information improves. The difference between the results here and in Corsetti et al. (2004) lies in the difference between the attack on currency and the attack on an asset bubble. The price of the bubble asset here is growing exponentially while the currency peg in Corsetti et al. (2004) is constant. On one hand, mis-coordination in this model gives birth to a bubble and thus gains from riding the bubble. On the other hand, coordination in Corsetti et al. (2004) allows successful attack on the pegged currency, yielding profit.

Lastly, I examine the effect of wealth of the large arbitrageur on her strategy. I find that in the general case, having less wealth than $\eta\kappa$ induces the large arbitrageur to sell later than the case when she has exactly $\eta\kappa$. In the case when the large arbitrageur has complete information, if the large arbitrageur's wealth is below a certain threshold, her optimal time of sale is strongly monotonically decreasing in her wealth.

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APPENDIX A

Appendix for Chapter 1

A.1. Additional Analysis

A.1.1. Simplifying the problems with Assumption 1.2

Lemma A.1. *Given Assumption 1.2, O would never choose $\tau_{O,P}$ s.t.* (IR-P) *is slack.*

Lemma A.1 is an immediate consequence of Assumption 1.2. By Lemma A.1, I can restrict the attention to $\tau_{O,P}$ such that (IR-P) binds with equality. I shall refer to the equality version of (IR-P) as (IR'-P). Lemma A.1 means P will always accept O's offer and this reduces P's value function to

$$\hat{U}_P\left(\tau_{H,P}\right) \equiv U_P\left(\tau_{O,P}(\tau_{H,P}), \tau_{H,P}\right) = \mathbb{E}_P[(w_P - \tau_{H,P})^+]. \tag{A.1}$$

At $t = 2$, P's wealth, including the trades, is still $(w_P - \tau_{O,P} - \tau_{H,P})^+$. This may be different from $(w_P - \tau_{H,P})^+$. However, ex-ante at $t = 1$, P can be sure that O will offer $\tau_{O,P}$ such that P is indifferent between the two. Thus, at $t = 0$, P is only concerned about $\mathbb{E}_P\left[(w_P - \tau_{H,P})^+\right]$. This decouples P's decision of accepting H's offer from other contracts. This simplifies H's problem.

A.1.2. Simplifying the problems with Assumption 1.3

In section 1.3.2, I make no restriction on $\tau_{O,P}$. Assumption 1.3 helps put some structure on $\tau_{O,P}$ for the baseline result. As defined in (1.2), there are two possibilities, $\tau_{O,P} \in \mathbb{R}_+ \times \mathbb{R}$ and $\tau_{O,P} \in \mathbb{R}_- \times \mathbb{R}$. The first possibility corresponds to O buying insurance from P whereas the second possibility corresponds to O selling insurance to P. Given $\tau_{H,O} = \tau_{H,P} = (0,0)$, O would always choose to sell insurance to P since O is more optimistic about state 1 and O has all the bargaining power. When $\tau_{H,O}$ and $\tau_{H,P}$ are both non-zero, O may choose to buy insurance from P so that P would default on $\tau_{H,P}$. In that case, O would compare the expected revenue from buying insurance from and selling insurance to P and choose the better of the two. The only benefit for O to buy insurance from P is that O can get $\min(\tau_{H,P}(0), w_P)$ for free, since H already paid the price. The cost of doing so is that O is trading against her own interest as O values state 1 more. It's helpful to first study the result when I restrict $\tau_{O,P} \in \mathbb{R}_- \times \mathbb{R}$. Thus, I impose an Assumption 1.3 on w_P to rule out $\tau_{O,P} \in \mathbb{R}_+ \times \mathbb{R}$. In section 1.6, I relax the assumption.

Assumption 1.3 also helps reduce the pessimist's problem to a simple participation constraint. To see this, I have the following Lemma, which follows immediately from Assumption 1.3.

Lemma A.2. *Given Assumption 1.3, O would only consider $\tau_{O,P} \in \mathbb{R}_- \times \mathbb{R}_+$.*

This is very useful as I only need to consider O selling insurance to P. Thus, Lemma A.2 states that O always has commitment problem. Assumption 1.3 also has another implication. The second inequality in Assumption 1.3, which follows from the definition of $h(\cdot)$, implies that P always has enough wealth to insure H. With Assumptions 1.1 and

1.3, I am essentially studying the case when both O and P are wealthy relative to H. Together with P's value function in (A.1), Assumption 1.3 also implies P would not accept any offer from H with $\tau_{H,P}(0) > w_P$, since H cannot afford $\tau_{H,P}(1) = w_P h^{-1}(\pi) > 1$, which is required by (IR-HP) defined in the next section. Thus, H can never offer a contract in which H asks P to pay more than P's wealth. P's value function can be further reduced to

$$\bar{U}_P\left(\tau_{H,P}\right) \equiv \mathbb{E}_P[w_P - \tau_{H,P}].$$

A.1.3. Analysis for Benchmark Case

In this case, $\tau^*_{O,P} = (0,0)$ and H would choose $\tau_{H,P} = (0,0)$. This provides a useful benchmark since O does not have commitment problem. O's value function becomes

$$\mathbb{E}_O[(w_O - \tau_{H,O})^+].$$

In words, O chooses whether to accept $\tau_{H,O}$. With w_O being common knowledge, H would only offer contract $\tau_{H,O} \leq w_O$. This reduces O's objective function to $\mathbb{E}_O[w_O - \tau_{H,O}]$. Thus, O would only accept the contract if $\mathbb{E}_O[w_O - \tau_{H,O}] \geq \mathbb{E}_O[w_O]$ or $\mathbb{E}_O[\tau_{H,O}] \leq 0$. In other words, O only accepts contract $\tau_{H,O}$ if O at least breaks even. Knowing this, H maximizes $\mathbb{E}_H[u(R + \tau_{H,O})]$ subject to O's budget constraint, $\tau_{H,O} \leq w_O$, and individual rationality constraint, $\mathbb{E}_O[\tau_{H,O}] \leq 0$.

A.1.4. Preliminary Analysis

As discussed previously, given Assumption 1.3, (IR-HP) can be reduced to

$$\text{(IR'-HP)} \qquad \mathbb{E}_P[w_P - \tau_{H,P}] \geq \mathbb{E}_P[w_P],$$

as P has enough wealth to insure H. Then, (IR'-HP) implies $\tau_{H,P} \propto \tau_{h^{-1}(\pi)}$. In other words, H only trades with P at the price of $h^{-1}(\pi)$. By Lemma A.2, $w_P - \tau_{H,P}(0) > 0$ implies $w_P - \tau_{O,P}(0) - \tau_{H,P}(0) > 0$. This implies (BC-HP) can be removed and $\tau'_{H,P}$ can be replaced by $\tau_{H,P}$. In other words, P never defaults on H and so I can replace the recovery contract $\tau'_{H,P}$ in H's objective function with the actual contract $\tau_{H,P}$.

Moreover, since $w_P - \tau_{O,P}(0) - \tau_{H,P}(0) > 0$, P's value function reduces to

$$\mathbb{E}_P\left[w_P - \tau_{O,P} - \tau_{H,P}\right].$$

Thus, O's (IR'-P) simplifies further to a break-even condition for P

$$\text{(IR''-P)} \qquad \mathbb{E}_P\left[\tau_{O,P}\right] = 0.$$

This implies that O can sell insurance to P for price of $h^{-1}(\pi)$. (IR'-P) was the only condition that dependents on both $\tau_{H,P}(0)$ and $\tau_{H,P}(1)$. With (IR"-P), O's problem now only depends on $\tau_{H,P}(1)$ with (BC-HP). By Lemma A.2, (BC-O) can be modified to

$$\text{(BC'-O)} \qquad -\tau_{O,P} \leq w_O.$$

Since $\tau_{O,P}$ is in the same direction as $\tau_{H,O}$, O pays out to P exactly when H asks O for payment. Thus, contract $\tau_{H,O}$ does not increase O's budget constraint anymore. Given Lemma A.2 and the above, I have the following Proposition.

Proposition A.1. *Given any $\tau_{H,P}(1)$, there is a unique solution $\tau^*_{O,P}[\tau_{H,P}(1)] \propto -\tau_{h^{-1}(\pi)}$ to O's problem. $\tau^*_{O,P}$ is independent of $\tau_{H,O}$ and $\tau_{H,P}(0)$. Moreover, $\tau^*_{O,P}$ is a corner solution determined by either* (BC'-O) *or* (BC-P)*. $\tau^*_{O,P}$ is only dependent on $\tau_{H,P}(1)$ when* (BC-P) *binds.*

O can sell insurance to P for a price of $h^{-1}(\pi)$, which is higher than O's break-even price of $h^{-1}(\pi')$. Thus, O is willing to sell insurance to P until either O or P runs out of money. This leads to commitment problem as O sells insurance to P to the limit, regardless of O's existing contract with H.

Given Proposition A.1, as long as the recovery contract $\tau'_{H,O}$ breaks even for O, O will happily accept $\tau_{H,O}$. Given independence of $\tau^*_{O,P}$ from $\tau_{H,O}$, I can replace $\tau'_{H,O}$ in H's problem with $\tau_{H,O}$ and modify (IR-HO) and (BC-HO) to

$$\text{(IR'-HO)} \qquad \mathbb{E}_O[\tau_{H,O}] \leq 0,$$

$$\text{(BC'-HO)} \qquad \tau_{H,O} \leq w_O + \tau^*_{O,P}[\tau_{H,P}(1)].$$

Thus, H can offer contract with a price as low as $h^{-1}(\pi')$. H can do this as long as O has enough resources. Moreover, the independence statements in Proposition A.1 imply

Proposition A.2. *Either $\tau^*_{H,O} = (0,0)$ or $\tau^*_{H,P} = (0,0)$.*

In equilibrium, H would only consider trading with O or P. There are two cases. First, H would only trade with O when (BC-P) binds, i.e. P runs out of wealth first while trading with O. Intuitively, when H trades with both O and P, P would take promises from H and use it to trade with O. This diverts state 0 wealth of O away from H's contract with O. Thus, H is essentially competing against herself for O's state 0 wealth. Second, H would only trade with P when (BC'-O) binds, i.e. O runs out of wealth first while trading with P. In that case, O gives all of her state 0 wealth to P. Thus, H cannot expect O to pay her anything; H would only trade with P.

A.1.5. Additional Analysis for Hedging

Recall $\tau_{O,P}$ can be either $\mathbb{R}_- \times \mathbb{R}$ or $\mathbb{R}_+ \times \mathbb{R}$. I can immediately rule out $\tau_{O,P} \in \mathbb{R}_- \times \mathbb{R}_{--}$ and $\tau_{O,P} \in \mathbb{R}_+ \times \mathbb{R}_{++}$. Contracts in the first space means non-negative transfers from O to P in both states while contracts in the second one means non-negative transfers from P to O in both states. O would prefer offering $(0,0)$ to offering any contract $\tau_{O,P} \in \mathbb{R}_- \times \mathbb{R}_{--}$. P would never accept any contract $\tau_{O,P} \in \mathbb{R}_+ \times \mathbb{R}_{++}$ since P is better off with $(0,0)$. Thus, it is only necessary to consider contracts $\tau_{O,P}$ that in either $\mathbb{R}_- \times \mathbb{R}_+$ or $\mathbb{R}_+ \times \mathbb{R}_-$. In other words, I only need to consider O buying insurance from P or selling insurance to P.

First, let us define useful notations. I shall denote

$$\tau_{O,P,+} \equiv \tau_{O,P} \in \mathbb{R}_+ \times \mathbb{R}_- \quad \text{and} \quad \tau_{O,P,-} \equiv \tau_{O,P} \in \mathbb{R}_- \times \mathbb{R}_+ \tag{A.2}$$

In words, $\tau_{O,P,+}$ represents O buying insurance from P while $\tau_{O,P,-}$ represents O selling insurance to P. Moreover, I define

$$U_O(\tau_{H,O}, \tau_{H,P}, i) \equiv \max_{\tau_{O,P,i}} \hat{U}_O(\tau_{O,P,i}|\tau_{H,O}, \tau_{H,P}) \tag{A.3}$$

$$\tau^*_{O,P,i} \equiv \arg\max_{\tau_{O,P,i}} \hat{U}_O(\tau_{O,P,i}|\tau_{H,O}, \tau_{H,P}) \tag{A.4}$$

for $i \in \{+, -\}$. $U_O(\tau_{H,O}, \tau_{H,P}, +)$ is O's problem with the restriction that O can only buy insurance for P. $U_O(\tau_{H,O}, \tau_{H,P}, -)$ is defined analogously. I have the following Lemma.

Lemma A.3. *Given $\tau_{H,P}$ and $\tau_{H,O}$, $\tau^*_{O,P,+}$ is either $(0,0)$ or $h^{-1}(\pi)(w_P, -(w_P - \tau_{H,P}(0))^+$. $\tau^*_{O,P,-}$ is either*

(1) $w_O(-\tau_{h^{-1}(\pi)}) - (0, h^{-1}(\pi)(\tau_{H,P}(0) - w_P)^+)$,
if $h^{-1}(\pi)(w_O - (\tau_{H,P}(0) - w_P)^+) \leq w_P - \tau_{H,P}(1)$,

(2) $h(\pi)(w_P - \tau_{H,P}(1))(-\tau_{h^{-1}(\pi)}) - ((\tau_{H,P}(0) - w_P)^+, 0)$,
if $h^{-1}(\pi)(w_O - (\tau_{H,P}(0) - w_P)^+) > w_P - \tau_{H,P}(1)$,

(3) *or* $(0,0)$ *if 1 and 2 gives lower expected utility.*

Thus, given $\tau_{H,P}$ and $\tau_{H,O}$, I only need to compare $\tau^*_{O,P,+}$ and $\tau^*_{O,P,-}$ to find out whether O prefers to buy insurance from or to sell insurance to P. Now I can define the incentive compatibility constraints of O as the following:

$$\hat{U}_O(\tau^*_{O,P,+}|\tau_{H,O}, \tau_{H,P}) \geq \hat{U}_O(\tau^*_{O,P,-}|\tau_{H,O}, \tau_{H,P}), \tag{IC-O-B}$$

$$\hat{U}_O(\tau^*_{O,P,+}|\tau_{H,O}, \tau_{H,P}) \leq \hat{U}_O(\tau^*_{O,P,-}|\tau_{H,O}, \tau_{H,P}). \tag{IC-O-S}$$

The first IC constraint states O prefers buying insurance from P. The second IC constraint states O prefers selling insurance to P. I augment H's problem with the incentive compatibility constraints. Whether H prefers O to buy insurance from P or otherwise, it must be incentive compatible for O to do so.

For H's problem, (IR-HO) requires comparison between $\hat{U}_O(\tau_{O,P}|\tau_{H,O}, \tau_{H,P})$ and $\hat{U}_O(\tau_{O,P}|(0,0), \tau_{H,P})$ to determine the price of the contract. To aid the comparison in (IR-HO), I have the following Lemma.

Lemma A.4. *Given $\tau_{H,P}$ and $\tau_{H,O}$, $\hat{U}_O(\tau^*_{O,P,+}|\tau_{H,O}, \tau_{H,P}) \geq \hat{U}_O(\tau^*_{O,P,-}|\tau_{H,O}, \tau_{H,P})$ only if $\hat{U}_O(\tau^*_{O,P,+}|(0,0), \tau_{H,P}) \geq \hat{U}_O(\tau^*_{O,P,-}|(0,0), \tau_{H,P})$ for all $\tau^*_{O,P,+}$ and $\tau^*_{O,P,-}$.*

In words, O prefers to buy insurance from P given $\tau_{H,O}$ and $\tau_{H,P}$, only if O also prefers to buy insurance from P when O rejects contract $\tau_{H,O}$ from H. Intuitively, when O sells insurance to P, O can default on $\tau_{H,O}$. Thus, $\tau_{H,O}$ increases O's expected utility more when O sells insurance to P than when O buys insurance from P. So, if O doesn't want to buy insurance from P even when O rejects contract $\tau_{H,O}$, O would not want to buy insurance from P no matter what contract H offers. As a result, the direction of trade between O and P relies heavily on $\tau_{H,P}$.

Lemma A.4 simplifies H's problem. If H wants O to buy insurance from P in equilibrium, H only need to consider $\tau_{H,P}$ such that (IC-O-B) binds for $\tau_{H,O} = (0, 0)$. Given such a $\tau_{H,P}$, H can solve for $\tau_{H,O}$ using (IR-HO). Thus, I have Proposition 1.5.

A.2. Proofs

A.2.1. Proof of Lemmas

A.2.1.1. Proof of Lemma A.1.

Proof. Consider $\tau_{O,P}$ such that (IR-P) is slack and $\tau_{O,P}$ satisfies all other constraints. Let $e_0 = (1, 0)$ and $e_1 = (0, 1)$.

I can always find $\delta_0, \delta_1 > 0$ such that (IR-P) binds with equality for $\tau(\delta_0, \delta_1) = \tau_{O,P} + \delta_0 e_0 + \delta_1 e_2$, since $\tau > (0, 0)$ for sufficiently high δ_i and (IR-P) is violated in that case. Since U_P is a continuous function with respect to $\tau(\delta_0, \delta_1)$, which is continuous with respect to δ_i's, there must be some pair of δ_i's such that (IR-P) binds with equality for $\tau(\delta_0, \delta_1)$. Let's denote the resulting contract as τ'.

Since $\delta_0, \delta_1 > 0$, (BC-O) is satisfied and $\hat{U}_O(\tau'|\cdot) \geq \hat{U}_O(\tau_{O,P}|\cdot)$. By construction, all constraints are satisfied for τ'. Thus, O prefers τ' by Assumption 1.2. Thus, O would never choose $\tau_{O,P}$ such that (IR-P) is slack. $\square$

A.2.1.2. Proof of Lemma A.2.

Proof. Since $\tau_{H,O} \in \mathbb{R}_- \times \mathbb{R}_-$ is weakly dominated by $(0, 0)$ and $\tau_{H,O} \in \mathbb{R}_+ \times \mathbb{R}_+$ violates (IR-P), I only need to consider 2 cases.

Given $\tau_{H,P}$, $\tau_{H,O}$ and Assumption 1.3. Since H receives at most 1 in state 1, Assumption 1.3 together with (IR-HP) implies $\tau_{H,P}(0) \leq h(\pi)$. Consider $\tau_{O,P} = (w_P, -(w_P - \tau_{H,P}(0))h^{-1}(\pi))$. If this is not feasible according to (BC-O), the only feasible $\tau_{O,P} \in \mathbb{R}_+ \times \mathbb{R}_-$ is $\tau_{O,P} \propto (1, -h^{-1}(\pi))$, which is strictly dominated by $(0, 0)$. Then, I am done. So now suppose $\tau_{O,P} = (w_P, -(w_P - \tau_{H,P}(0))h^{-1}(\pi))$ is feasible. Then $\mathbb{E}_O[(w_O - \tau_{H,O} + \tau_{O,P})^+] \leq \mathbb{E}_O[(w_O - \tau_{H,O})^+ + \tau_{O,P}]$ since $\tau_{H,O} \in \mathbb{R}_+ \times [-1, 0]$ and (BC-O). Moreover,

$\mathbb{E}_O[(w_O-\tau_{H,O})^+ + \tau_{O,P}] = \mathbb{E}_O[(w_O-\tau_{H,O})^+] + \mathbb{E}_O[\tau_{O,P}]$. And $\mathbb{E}_O[\tau_{O,P}] = (1-\pi')(-(w_P - \tau_{H,P}(0))h^{-1}(\pi)) + \pi'(w_P) = w_P(1-\pi')(h^{-1}(\pi) - h^{-1}(\pi')) + (1-\pi')h^{-1}(\pi)\tau_{H,P}(0) \leq w_P(1-\pi')(h^{-1}(\pi) - h^{-1}(\pi')) + (1-\pi')h^{-1}(\pi)h(\pi) < 0$, where the last inequality follows from Assumption 1.3. Thus, $\mathbb{E}_O[(w_O - \tau_{H,O} + \tau_{O,P})^+] \leq \mathbb{E}_O[(w_O-\tau_{H,O})^+] + \mathbb{E}_O[\tau_{O,P}] < \mathbb{E}_O[(w_O-\tau_{H,O})^+]$ so again $\tau_{O,P}$ is dominated by $(0,0)$. Thus, given Assumption 1.3, all $\tau_{O,P} \in \mathbb{R}_+ \times \mathbb{R}_-$ are dominated by $(0,0)$. Thus, O will not consider any $\tau_{O,P} \in \mathbb{R}_+ \times \mathbb{R}_-$. □

A.2.1.3. Proof of Lemma A.3.

Proof. First, $\tau^*_{O,P,+}$. O prefers $(0,0)$ to $\tau_{O,P} \propto \tau_{h^{-1}(\pi)}$. (IR'-P) implies $\tau_{O,P} \propto \tau_{h^{-1}(\pi)}$ or $\tau_{O,P} = (x, -(w_P - \tau_{H,P}(0))^+ h^{-1}(\pi))$ for all $x \in [0, w_P]$. O prefers $\tau_{O,P} = (w_P, -(w_P - \tau_{H,P}(0))^+ h^{-1}(\pi))$ over all others. Thus, if $\tau_{O,P} = (w_P, -(w_P - \tau_{H,P}(0))^+ h^{-1}(\pi))$ is feasible, that is $\tau^*_{O,P,+}$. If not, $\tau^*_{O,P,+} = (0,0)$.

Now, $\tau^*_{O,P,-}$. Solution is always corner by same argument as in proof of Proposition A.1. Suppose $(w_O - (\tau_{H,P}(0) - w_P)^+)h^{-1}(\pi) \leq w_P - \tau_{H,P}(1)$. (BC-O) binds first. (IR'-P) implies $\hat{\tau}_{O,P,-} = w_O(-\tau_{h^{-1}(\pi)}) - (0, (\tau_{H,P}(0) - w_P)^+ h^{-1}(\pi))$. Suppose the opposite, (BC-P) binds first. (IR'-P) implies $\hat{\tau}_{O,P,-} = h(\pi)(w_P - \tau_{H,P}(1))(-\tau_{h^{-1}(\pi)}) - ((\tau_{H,P}(0) - w_P)^+, 0)$. $\tau^*_{O,P,-}$ is either $\hat{\tau}_{O,P,-}$ or $(0,0)$ if $\hat{U}_O(\hat{\tau}_{O,P,-}|\cdot) \leq \hat{U}_O((0,0)|\cdot)$. □

A.2.1.4. Proof of Lemma A.4.

Proof. I prove using contrapositive. Suppose

$$\hat{U}_O(\tau_{O,P,+}|(0,0), \tau_{H,P}) < \hat{U}_O(\tau_{O,P,-}|(0,0), \tau_{H,P}).$$

I have

$$\hat{U}_O(\tau_{O,P,+}|\tau_{H,O},\tau_{H,P}) = \hat{U}_O(\tau_{O,P,+}|(0,0),\tau_{H,P}) - \pi'\tau_{H,O}(1)$$
$$- (1-\pi')\min(\tau_{H,O}(0), w_O + w_P)$$
$$\hat{U}_O(\tau_{O,P,-}|\tau_{H,O},\tau_{H,P}) = \hat{U}_O(\tau_{O,P,+}|(0,0),\tau_{H,P}) - \pi'\tau_{H,O}(1)$$
$$- (1-\pi')\min(\tau_{H,O}(0), w_O - h(\pi)(w_P - \tau_{H,P}(1)) - (\tau_{H,P}(0) - w_P)^+)$$

Since

$$w_O + w_P > w_O > w_O - h(\pi)(w_P - \tau_{H,P}(1)) - (\tau_{H,P}(0) - w_P)^+,$$

I have $\hat{U}_O(\tau_{O,P,+}|\tau_{H,O},\tau_{H,P}) < \hat{U}_O(\tau_{O,P,-}|\tau_{H,O},\tau_{H,P})$ as desired. □

A.2.2. Proof of Propositions

A.2.2.1. Proof of Proposition 1.1.

Proof. It is strictly optimal to have $\tau_O^{FB} \propto h(\pi')\tau_{h^{-1}(\pi')}$. So plugging this into objective function, I get FOC: $(1-\pi)u'(h(\pi')x) - \pi u'(1-x) \leq 0$, where $x \in [0,1]$. Since u is strictly concave and FOC is positive at $x = 0$. If there is an interior solution, I am done. If not, $x = 1$. Assumption 1.1 ensures $\tau_O^{FB} \leq w_O$. □

A.2.2.2. Proof of Proposition A.1.

Proof. (IR"-P) implies the price. I first prove that the solution is corner. Suppose $\tau^*_{O,P}[\tau_{H,O}]$ is not a corner solution, I can always multiply $\tau^*_{O,P}[\tau_{H,O}]$ by $1+\epsilon$ for $\epsilon > 0$ small enough so that the constraints are satisfied. However, I can increase the objective function of O by doing so. This contradicts the optimality of $\tau^*_{O,P}[\tau_{H,O}]$. There are only

2 corner solutions. When both are feasible, they are the same. Thus there is a unique solution. Now I prove the independence. None of the constraints depend on $\tau_{H,O}$ and $\tau_{H,P}(0)$. Specifically, the corner constraints that determine the solution do not depend on $\tau_{H,O}$ and $\tau_{H,P}(0)$. Thus, the solution is independent of $\tau_{H,O}$ and $\tau_{H,P}(0)$. Moreover, only (BC-P) depends on $\tau_{H,P}(0)$. Thus, the solution is only dependent on $\tau_{H,P}(1)$ when (BC-P) binds. $\square$

A.2.2.3. Proof of Proposition A.2.

Proof. For any $\tau_{H,P}$, (BC-P) binding implies (BC-P) binds for all $\hat{\tau}_{H,P}$ with $\hat{\tau}_{H,P}(1) \leq \tau_{H,P}(1)$. (BC-P) binding for $\tau_{H,P}$ implies $\tau^*_{O,P} = -h(\pi)(w_P - \hat{\tau}_{H,P}(1))\tau_{h^{-1}(\pi)}$ for all $\hat{\tau}_{H,P}$ with $\hat{\tau}_{H,P}(1) \leq \tau_{H,P}(1)$. Thus, H's problem becomes

$$\begin{aligned} &\max_{\tau_{H,O},\tau_{H,P}} \mathbb{E}_H[u(R + \tau_{H,O} + \tau_{H,P})] \\ s.t.\quad & \mathbb{E}_P[\tau_{H,P}] \leq 0 \\ & \mathbb{E}_O[\tau_{H,O}] \leq 0 \\ & \tau_{H,O}(0) \leq w_O - h(\pi)(w_P - \tau_{H,P}(1)) \end{aligned}$$

Since it's strictly better to have first and second constraint binding, I can replace them with equalities. I can then replace $\tau_{H,P}(1)$ with $-h^{-1}(\pi)\tau_{H,P}(0)$ as implied by first constraint with equality. Thus, H's problem becomes

$$\begin{aligned} &\max_{\tau_{H,O},\tau_{H,P}} \mathbb{E}_H[u(R + \tau_{H,O}(0)\tau_{h^{-1}(\pi')} + \tau_{H,P}(0)\tau_{h^{-1}(\pi)})] \\ s.t.\quad & \tau_{H,O}(0) + \tau_{H,P}(0) \leq w_O - h(\pi)w_P \end{aligned}$$

In this case, since $\tau_{H,O}$ is cheaper, H strictly prefers $\tau_{H,O}$. Since this holds for all $\hat{\tau}_{H,P}$ with $\hat{\tau}_{H,P}(1) \leq \tau_{H,P}(1)$, H prefers $\tau_{H,O}$ to all such $\hat{\tau}_{H,P}$ and picks $\tau_{H,P} \neq (0,0)$. Now if (BC-P) doesn't bind $\tau_{H,P}$, (BC'-O) binds. In that case, $\tau_{H,O} = (0,0)$. Thus, the equilibrium contracts can only be either $\tau^*_{H,P} = (0,0)$ or $\tau^*_{H,O} = (0,0)$. □

A.2.2.4. Proof of Proposition 1.2.

Proof. Given $\tau^{(0)}_{H,O}$ and $\tau^{(0)}_{H,P}$, constraint (BC-P) binds first. Thus, I have $\tau^{(0)}_{O,P}$ by Proposition A.1. Since H can buy insurance from P for price of $h^{-1}(\pi)$ and H can buy insurance from O for price of $h^{-1}(\pi') < h^{-1}(\pi)$, H would choose O for cheaper price. Moreover, since $\tau^{(0)}_{H,O} = \tau^{FB}_{H,O}$, H does not want to purchase any more insurance even at the lower price, much less the higher price.

By Proposition A.2, there are only two possibilities. Since H strictly prefers $\tau^{FB}_{H,O}$, there is a unique equilibrium. □

A.2.2.5. Proof of Proposition 1.3.

Proof. $\tau^{(1)}_{H,O}$ is given by (IR'-HO) and (BC'-HO). $\tau^{(2)}_{H,P}$ is the solution to H's problem given price $h(\pi)$. H will choose whichever gives her the higher utility. By Assumption 1.5, (BC-P) binds for O. Thus, any increase in $\tau_{H,O}(1)$ increases $\tau_{O,P}$, which decreases $\tau_{H,O}$. Thus $\tau_{H,O}$ competes with $\tau_{H,P}$ for w_O and H would never choose both to be non-zero simultaneously due to Proposition A.2.

Suppose $\tau^{(1)}_{H,O}$ gives H higher expected utility. Since $\tau^{(1)}_{H,O}$ is preferred to $\tau^{(2)}_{H,P}$, $\tau^{(1)}_{H,O}$ is preferred to any $\tau_{H,P} \propto \tau_{h^{-1}(\pi)}$. Thus $\tau^{(1)}_{H,O}$ is the unique solution in this case. $\tau^{(1)}_{O,P}$ is implied by $\tau^{(1)}_{H,P} = (0,0)$. Thus is case 1. Suppose $\tau^{(2)}_{H,P}$ gives higher utility. Given any other $\tau_{H,P}$ and $\tau_{H,O}$, H would deviate. So $\tau^{(2)}_{H,P}$ is the unique solution. $\tau^{(2)}_{O,P}$ and $\tau^{(3)}_{O,P}$ are determined by whether (BC-P) or (BC'-O) binds first. Thus, all equilibrium is unique in all 3 cases. Moreover, there is a unique threshold w_O^{**} such that (BC-P) and (BC'-O) binds at the same time. Equilibrium is in case 2 when it's not case 1 and w_O is above the threshold and in case 3 otherwise.

In the upper bound for w_O in Assumption 1.5, $\tau_{H,O}^{(1)} = \tau_{H,O}^{FB}$ which is strictly preferred to $\tau_{H,P}^{(2)}$. In the lower bound for w_O in Assumption 1.5, $\tau_{H,O}^{(1)} = 0$ as $\tau_{O,P}^{(1)}(0) = w_O$. Thus, $\tau_{H,P}^{(2)}$ is strictly preferred to $\tau_{H,O}^{(1)} = 0$. Since $\tau_{H,O}^{(1)}$ is a linear decreasing function of w_O while $\tau_{H,P}^{(2)}$ is constant with respect to w_O, there must a unique w_O^* such that the two are indifferent. $w_O^* \geq h(\pi)w_P$ since $\tau_{H,O}^{(1)} = (0,0)$ at $h(\pi)w_P$. $\square$

A.2.2.6. Proof of Proposition 1.4.

Proof. Given case 1 in section 1.4.2. If O rejects $\tau_{H,O}$, O would choose $\tau_{O,P} = \tau_{O,P}^*$. Thus, for O to accept $\tau_{H,O}$, it must be $\tau_{H,O} = w_O - \tau_{O,P}^* + a \times (1, -h^{-1}(\pi))$. H would solve for $a \in [0, \tau_{H,P}^{**}(0)]$. Since u is concave, there is a unique maximum. $\tau_{H,O}^{CCP}$ is weakly better than corresponding contracts in section 1.4.2. Moreover, $\tau_{H,O}^{CCP}$ weakly dominates $\tau_{H,P}$ since price of $\tau_{H,O}^{CCP}$ is weakly higher. $\square$

A.2.2.7. Proof of Proposition 1.5.

Proof. Consider any $\tau_{H,P}$ such that (IC-O-B) binds for $\tau_{H,O} = (0,0)$. Since (IC-O-B) binds for $\tau_{H,O} = (0,0)$, $\tau'_{H,P} = (0, \tau_{H,P}(1))$ when $\tau_{H,O} = (0,0)$. (IR-HO) implies $\tau_{H,O} \propto \tau_{h^{-1}(\pi')}$ as long as $\tau_{H,O}(0) \leq w_O + w_P$. Since LHS of (IC-O-B) is constant while RHS increases when $\tau_{H,O}(0) \geq w_O - h(\pi)(w_P - \tau_{H,P}(1)) - (\tau_{H,P}(0) - w_P)^+$. The may exist a unique threshold $\hat{\tau}_{H,O}(0)$ such that both (IC-O-B) and (IC-O-S) binds for $\hat{\tau}_{H,O} \equiv \hat{\tau}_{H,O}(0)\tau_{h^{-1}(\pi')}$. If it doesn't exist, let $\hat{\tau}_{H,O}(0) = w_O + w_P$. Then for $\tau_{H,O}(0) \in [0, \hat{\tau}_{H,O}(0)]$, (IC-O-B) binds, $\tau'_{H,O} = \tau_{H,O}$ and $\tau'_{H,P} = (0, \tau_{H,P}(1))$. H's objective function is concave over closed and bounded feasible set. Thus, there is a unique solution $\tilde{\tau}_{H,O}$. Now, if $\hat{\tau}_{H,O}(0) = w_O + w_P$, $\tau^{(4)}_{H,O} = \tilde{\tau}_{H,O}$. Else, when (IC-O-S) binds at $\hat{\tau}_{H,O}$, $\tau'_{H,O} = ((w_O - (w_P - \tau_{H,P}(1))h^{-1}(\pi))^+, \hat{\tau}_{H,O}(1))$ and $\tau'_{H,P} = \tau_{H,P}$. $\hat{\tau}_{H,O}$ is strictly preferred to $\tau_{H,O} = (\hat{\tau}_{H,O}(0), \tau_{H,O}(1))$ with $\tau_{H,O}(1) > \hat{\tau}_{H,O}(1)$. Thus, $\hat{\tau}_{H,O}$ is the argmax when (IC-O-S) binds. In this case, $\tau^{(4)}_{H,O}$ equals to $t\hat{a}u_{H,O}$ or $\tilde{\tau}_{H,O}$, whichever gives H higher utility with corresponding $\tau'_{H,O}$. □

A.2.2.8. Proof of Proposition 1.6.

Proof. By Lemma A.4, I only need to be concerned with $\tau_{H,P}$ such that (IC-O-B) holds for $\tau_{H,O} = (0,0)$. H can always make (IC-O-B) hold by increasing $\tau_{H,P}(0)$ above w_P. Thus, the upper bound of $\tau_{H,P}(1)$ is $h^{-1}(\pi)w_P$, which is implied by (IR-HP) for $\tau_{H,P}(0) > w_P$. The lower bound is determined by the lowest $\tau_{H,P}$ such that (IC-O-B) holds for $\tau_{H,O}$. I only need to consider $\tau_{H,P}(0) \le w_P$. If there is no such $\tau_{H,O}(0)$, the lower bound is equal to upper bound. If (BC-O) binds, (IC-O-B) becomes

$$\pi'[w_O - (w_P - \tau_{H,P}(0))h^{-1}(\pi)] + (1-\pi')(w_O + w_P) \ge \pi'[w_O - w_O h^{-1}(\pi)].$$

I can rearrange term to get hedging cost $\tau_{H,P}(1) \ge (w_O + w_P)(h^{-1}(\pi) - h^{-1}(\pi'))$. If (BC-P) binds, (IC-O-B) becomes

$$\begin{aligned}\pi'[w_O - (w_P - \tau_{H,P}(0))h^{-1}(\pi)] + (1-\pi')(w_O + w_P)\\ \ge \pi'[w_O - (w_P - \tau_{H,P}(1))] + (1-\pi')[w_O - (w_P - \tau_{H,P}(1))h(\pi)].\end{aligned}$$

I can rearrange term to get hedging cost $\tau_{H,P}(1) \geq h^{-1}(\pi)[1+h^{-1}(\pi)][h(\pi') - h(\pi)]w_P$. Thus are the upper and lower bounds.

For lower bound of hedging cost of $\tau_{H,P}$, it's 0 since (IC-O-S) also holds when (IC-O-B) binds with equality at $\tau_{H,O} = (0,0)$. H needs to increase RHS of (IC-O-B) to hedge $\tau_{H,P}$. Thus, the hedging cost increases with the difference between LHS and RHS of (IC-O-B) at $\tau_{H,O} = (0,0)$. The maximum difference between LHS and RHS is when RHS is equal to $\mathbb{E}_O[w_O]$. When that happens, the difference between LHS and RHS is

$$\pi' \min[w_O, h^{-1}(\pi)w_P][h^{-1}(\pi) - h^{-1}(\pi')] - (1-\pi')w_P.$$

The hedging cost needs to compensate O for this amount in state 1. Thus, the hedging cost in this case is $\min[w_O, h^{-1}(\pi)w_P][h^{-1}(\pi) - h^{-1}(\pi')] - h^{-1}(\pi')w_P$. Thus are the upper and lower bounds. $\square$

APPENDIX B

Appendix for Chapter 2

B.1. Additional Analysis

B.1.1. Preliminary Analysis

In the general problem, agents can choose to hold any amount in an infinite number of simple debt contracts. However, the agents would only ever want to choose one simple debt contract or are indifferent between multiple simple debt contracts. To see this, I use Fubini's theorem. First,

$$\mathbb{E}_i\left[\int_{[\underline{s},\bar{s}]}|\min(s,D)\mu_i(D)|\,dD\right] < \mathbb{E}_i\left[\int_{[\underline{s},\bar{s}]}\max\left(|\underline{s}|,|\bar{s}|\right)|\mu_i(D)|\,dD\right] \leq \max\left(|\underline{s}|,|\bar{s}|\right) < \infty.$$

The first inequality follows from $|\min(s,D)| \leq \max\left(|\underline{s}|,|\bar{s}|\right)$. All simple debt contracts must be collateralized by A regardless of the seller. Since there is only one unit of A in the economy, $\int_{[0,\bar{s}]}|\mu_i(D)|dD \leq 1$. Thus, the second inequality follows. Since $\underline{s}$ and $\bar{s}$ are finite, the third inequality follows. Now the condition is satisfied for Fubini's theorem. By Fubini's theorem, I can replace (2.6) with

$$\max_{\alpha_i,c_i,\mu_i} U(\cdot|n_p) \equiv \mathbb{E}_i\left[\alpha_i s\right] + \int_{D\in[\underline{s},\bar{s}]}\mathbb{E}_i\left[\min(s,D)\right]\mu_i(D)dD. \tag{B.1}$$

All the constraints still apply. Written this way, it's easy to see that the benefit of choosing $\mu_i(D)$ is independent from that of $\mu_i(D')$ for any $D' \neq D$.

B.1.2. Solving Pessimists' Problem

By Assumption 2.3, cash constraint (2.3) does not bind for the pessimists and can be removed from the pessimist's problem. So first, I solve the pessimists' maximization problem in (B.1). After plugging the budget constraint (2.5) into (B.1), I get

$$\max_{\alpha_i, c_i, \mu_i(D)} n_p + \alpha_p \mathbb{E}_p [s - q] + \int \mathbb{E}_p [\min(s, D) - \pi(D)] \mu_p(D) dD.$$

The first order condition with respect to α_p is

$$\mathbb{E}_p [s - q] + \lambda_s + \lambda_{col} = 0, \tag{B.2}$$

where $\lambda_s \geq 0$ is the Lagrange multiplier for the short-sell constraint (2.2) and $\lambda_{col} \geq 0$ is the Lagrange multiplier for the collateral constraint (2.4). Since $\lambda_s, \lambda_{col} \geq 0$, this implies

$$\mathbb{E}_p [s - q] \leq 0 \Rightarrow \mathbb{E}_p [s] \leq q$$

Immediately, I have the following corollary.

Corollary B.1. *If an equilibrium exists, it must be that the equilibrium asset price* $q \geq \mathbb{E}_p [s]$.

For each D, taking the first order condition with respect to $\mu_p(D)^-$ gives

$$\mathbb{E}_p [\min(s, D) - \pi(D)] + \lambda_{col} \geq 0. \tag{B.3}$$

If the inequality is strict, it means $\mu_p(D)^- = 0$ is a corner solution. Equation (B.3) implies $\mathbb{E}_p[\min(s, D)] + \lambda_{col} \geq \pi(D)$. Taking first order condition with respect to $\mu_p(D)^+$ yields

$$\mathbb{E}_p[\min(s, D) - \pi(D)] \leq 0 \tag{B.4}$$

Similarly, if the inequality is strict, it means $\mu_p(D)^+ = 0$ is a corner solution. Together, (B.3) and (B.4) imply the next Corollary.

Corollary B.2. *If an equilibrium exists, it must be that the equilibrium debt price*

$$\pi(D) \in [\mathbb{E}_p[\min(s, D)], \mathbb{E}_p[\min(s, D)] + \lambda_{col}] \tag{B.5}$$

for all $D \in [0, \bar{s}]$ *and some* $\lambda_{col} \geq 0$.

With complementary slackness, it is easy to check that the indirect utility $V_p(n_p) = n_p$. In words, it doesn't matter what the pessimists do, their expected utility cannot exceed their endowment. Thus, the pessimists weakly prefer holding cash to holding any position in A or in simple debt contracts.

If a portfolio (α_p, c_p, μ_p) gives the pessimists the expected utility of n_p, it solves the pessimists' problem. Therefore, the pessimists' problem can be reduced to a participation constraint,

$$U_p(\alpha_p, c_p, \mu_p | n_p) \geq n_p, \tag{B.6}$$

along with the other constraints. In any equilibrium, the optimists' allocation will solve the optimists' problem. The pessimists' allocation, determined by the optimists' allocation and market clearing conditions (2.7) and (2.8), must satisfy (B.6).

B.1.3. Solving Optimists' Problem

The optimists' problem is similar. But since the cash constraint may bind, I do not want to plug the budget constraint into the problem first. So the optimists' problem is the same as (B.1). First, the first order condition with respect to α_o is

$$\mathbb{E}_o[s] + \delta_s + \delta_{col} - \delta_b q = 0, \tag{B.7}$$

where δ_s, δ_{col}, and $\delta_b \geq 0$ are the Lagrange multipliers for constraints (2.2), (2.4), and (2.5), respectively. Next, the first order condition with respect to $\mu_o(D)^-$ is

$$\mathbb{E}_o[\min(s, D)] + \delta_{col} - \delta_b \pi(D) \geq 0 \tag{B.8}$$

and the first order condition with respect to $\mu_o(D)^+$ is

$$\mathbb{E}_o[\min(s, D)] - \delta_b \pi(D) \leq 0. \tag{B.9}$$

Note that Finally, the first order condition with respect to c_o is

$$1 - \delta_b + \delta_{cash} = 0, \tag{B.10}$$

where $\delta_{cash} \geq 0$ is the Lagrange multiplier on the cash constraint (2.3). The multiplier δ_b is essentially the return on the optimists' asset and debt investments. If $\delta_b = 1$, the optimists' investments will have a return of one, meaning the optimists are indifferent between investing and holding cash. If $\delta_b > 1$, the optimists' investments will have a return greater than 1, dominating cash holding.

Given δ_b, if $\mathbb{E}_o\left[\min(s,D)\right]-\delta_b\pi(D)<0$ for some D, it must be that $\delta_{col}>0$. Otherwise, (B.8) will be violated. Moreover, for (B.8) to hold for all D, it must be that

$$\delta_{col}=0\vee-\left[\min_D\mathbb{E}_o\left[\min(s,D)\right]-\delta_b\pi(D)\right]. \tag{B.11}$$

And, for any $D'\in\arg\min_D\mathbb{E}_o\left[\min(s,D)\right]-\delta_b\pi(D)$, $\mu_i(D)=\alpha_i$ maximizes the optimists' problem given α_i and λ_b. So, first order condition (B.8) can be replaced by the minimization problem (B.11). To solve (B.11), $\pi(D)$ is needed. Thus, I'll continue in the next subsection where $\pi(D)$ can be determined in equilibrium.

B.1.4. Equilibrium Analysis

Equilibrium is defined in Definition 2.1. This subsection includes analysis and Lemmas that help prove Proposition 2.1. When analyzing the equilibrium, one can connect the pessimists' problem with the optimists' problem by using the market clearing conditions. This way, it is possible to further simplify the problems with 2 observations.

Observation 1: The optimists would only sell simple debt contract for a unique D^* in equilibrium and the equilibrium price of the asset can be determined by D^*. Next, I go through the reasoning. If the optimists sell any simple debt contract in equilibrium, the pessimists must buy those contracts due to the market clearing conditions. When pessimists buy simple debt contract D in equilibrium, their first order condition (B.4) must hold with equality for D. This implies that if the optimists sell simple debt contract D in equilibrium, it must be that

$$\pi(D)=\mathbb{E}_p[\min(s,D)]. \tag{B.12}$$

Note that this is due to Assumption 2.3. Thus, when the optimists borrow or sells debt contracts, it is as if the optimists hold all the bargaining power. Debt price (B.12) can be plugged into (B.11) to get

$$\delta_{col} = 0 \vee -\left[\min_{D} \mathbb{E}_o\left[\min(s, D)\right] - \delta_b \mathbb{E}_p[\min(s, D)]\right]. \tag{B.13}$$

It is helpful to state the following corollary first.

Corollary B.3. *For $i \in \{o, p\}$, $\mathbb{E}_i\left[s\right] = \int_{\underline{s}}^{\bar{s}} 1 - F_i(s)ds + \underline{s}$; $\mathbb{E}_i\left[\min(s, D)\right] = \int_{\underline{s}}^{D} 1 - F_i(s)ds + \underline{s}$; and $\mathbb{E}_i\left[s - \min(s, D)\right] = \int_{D}^{\bar{s}} 1 - F_i(s)ds$.*

To solve this minimization problem, I take the derivative of the objective with respect to D to get

$$\frac{1 - F_o(D)}{1 - F_p(D)} = \delta_b(\geq 1), \tag{B.14}$$

where the optimists' cash constraint (B.10) implies $\delta_b \geq 1$. Since the objective is non-linear, it is necessary to check the second order condition. The second order condition is $-f_o(D) + \delta_b f_p(D) > 0$. Plugging (B.14) into the second order condition gives

$$\frac{f_p(D)}{1 - F_p(D)} > \frac{f_o(D)}{1 - F_o(D)}. \tag{B.15}$$

For D such that (B.14) and (B.15) holds, D maximizes the return from selling debt. Note that (B.15) is implied by Simsek (2013)'s assumption (2.1). Under (2.1), $\frac{1-F_o(D)}{1-F_p(D)}$ is increasing and thus unique for each D. Under the weaker Assumption 2.4 in this model, for D such that (B.15) holds, $\frac{1-F_o(D)}{1-F_p(D)}$ is increasing and thus unique for each D. In words, given the weaker Assumption 2.4, there is still a unique solution D to the minimization problem (B.13). Formally,

Lemma B.1. *Given Assumption 2.4, in any equilibrium,* $\delta_b \in \left[1, \max_D \frac{1-F_o(D)}{1-F_p(D)}\right)$ *and for each* δ_b*, there is a unique interior solution* $D^*(\delta_b)$ *to the minimization problem* (B.13).

This is a very useful result as it will also help us pin down asset price. Given $\delta_b = \frac{1-F_o(D^*)}{1-F_p(D^*)}$, for the asset to be worth investing for the optimists, it must be that

$$q = q_o(D^*) \equiv \mathbb{E}_p\left[\min\left(s, D^*\right)\right] + \frac{1-F_p(D^*)}{1-F_o(D^*)}\mathbb{E}_o\left[s - \min\left(s, D^*\right)\right], \tag{B.16}$$

where I replaced $1/\delta_b$ with $\frac{1-F_p(D^*)}{1-F_o(D^*)}$ to remove δ_b from the equation. One can simply check the derivative to see that the asset price $q_o(D^*)$ is decreasing and unique for each D^*. If the optimists hold the asset in equilibrium, the equilibrium asset price $q_o(D^*)$ will be uniquely determined by D^*, which will be uniquely determined by market clearing conditions and the optimists' budget constraint.

Observation 2: If the pessimists were to hold the asset and borrow, the pessimists' optimal debt contract D is also unique such that $\delta_b = \frac{1-F_o(D)}{1-F_p(D)}$. If the pessimists were to have $\mu_p(D)^- < 0$ for some D, market clearing conditions imply $\mu_o(D)^+ > 0$. Thus, the optimists' first order condition (B.9) gives the debt price,

$$\pi(D) = \frac{1}{\delta_b}\mathbb{E}_o\left[\min\left(s, D\right)\right]. \tag{B.17}$$

Given this debt price, the pessimists must find λ_{col} such that their first order condition (B.3) must hold for all D. Similar to the optimists, the pessimists must solve

$$\lambda_{col} = 0 \vee - \left[\min_{D} \mathbb{E}_p \left[\min(s, D)\right] - \frac{1}{\delta_b}\mathbb{E}_o[\min(s, D)]\right]. \tag{B.18}$$

Upon closer inspection, one may realize that the objective function in the pessimists' minimization problem (B.18) is simply the negative of the objective function in the optimists' minimization problem (B.13). Thus, this problem is equivalent to maximizing the objective function in the optimists' problem (B.13). The first order condition for the maximization problem is exactly the same as the first order condition in the minimization problem. It is the second order condition that differentiates maximization from minimization. Thus, the first order condition to (B.18) is again (B.14). The second order condition, however, is the opposite of (B.15),

$$\frac{f_p(D)}{1 - F_p(D)} < \frac{f_o(D)}{1 - F_o(D)}. \tag{B.19}$$

Notice that $\delta_b \geq 1$ still have to hold. So, under some belief structure, an interior solution may not exist for (B.18). When that happens, it is easy to see that $D = \bar{s}$ is the solution, since $\mathbb{E}_p\left[s\right] - \delta_b \mathbb{E}_o\left[s\right] < (1 - \delta_b)\underline{s} \leq 0$. Per Assumption 2.1, if an interior solution does not exist, the pessimists hold no asset. Analogous to observation 1, given δ_b, for the optimists to hold the asset, it must be that

$$q = q_p(D) \equiv \frac{1 - F_p(D)}{1 - F_o(D)}\mathbb{E}_o\left[\min\left(s, D\right)\right] + \mathbb{E}_p\left[s - \min\left(s, D\right)\right]. \tag{B.20}$$

This follows immediately from the pessimists' first order conditions (B.2) and (B.3).

B.2. Proofs

B.2.1. Proof of Corollaries

B.2.1.1. Proof of Corollary 2.1.

Proof. First, $1 - F_o(\underline{s}) = 1 = 1 - F_p(\underline{s})$. Let $G(s) = \frac{1-F_o(s)}{1-F_p(s)}$. So, $G(\underline{s}) = 1$. Moreover, $G'(s) \propto \left(\frac{f_p(s)}{1-F_p(s)} - \frac{f_o(s)}{1-F_o(s)}\right) > 0$ for all $s \in (\underline{s}, \bar{s})$ by Assumption A2 from Simsek (2013). Since f_o and f_p are both continuous, $\frac{f_p(s)}{1-F_p(s)} \geq \frac{f_o(s)}{1-F_o(s)}$ for $s = \underline{s}$. Thus, $G(s)$ is an increasing function that starts at 1. So, $G(s) > 1$ for all $s \in (\underline{s}, \bar{s})$. This implies $1 - F_o(s) > 1 - F_p(s)$ for all $s \in (\underline{s}, \bar{s})$. □

B.2.1.2. Proof of Corollary 2.2.

Proof. By Assumption 2.2 and Corollary B.3, there must be an non-empty interval $[s', s''] \subset [\underline{s}, \bar{s}]$ such that $1 - F_o(s) > 1 - F_p(s)$ for all s in the interval. By Assumption 2.4, it must be that $s' = \underline{s}$ or $s'' = \bar{s}$ and the interval is unique. In either case, $s_{sc} = s'$. If slope of $\frac{1-F_o(s)}{1-F_p(s)}$ crosses zero from above, the crossing point is s_m. Otherwise, $s_m = \bar{s}$. It's unique because of Assumption 2.4. In either case, $s_m > s_{sc}$ since the maximum of $\frac{1-F_o(s)}{1-F_p(s)}$ is greater than 1 (by Assumption 2.2) and the slope of $\frac{1-F_o(s)}{1-F_p(s)}$ is positive at s_{sc}. □

B.2.1.3. Proof of Corollary B.3.

Proof. This can be done with integration by parts. $\mathbb{E}_i\left[\min(s, D)\right] = \int_{\underline{s}}^{D} s dF_i(s) + D(1 - F_i(D)) = D(F_i(s)) - \int_{\underline{s}}^{D} F_i(s) ds + D(1 - F_i(D)) = \int_{\underline{s}}^{D} 1 - F_i(s) ds + \underline{s}$. Similarly, $\mathbb{E}_i\left[s\right] = \int_{\underline{s}}^{\bar{s}} s dF_i(s) = \bar{s} - \int_{\underline{s}}^{\bar{s}} F_i(s) ds = \int_{\underline{s}}^{\bar{s}} 1 - F_i(s) ds + \underline{s}$. So, taking the difference gives $\mathbb{E}_i\left[s - \min(s, D)\right] = \int_{D}^{\bar{s}} 1 - F_i(s) ds$. □

B.2.2. Proof of Lemmas

B.2.2.1. Proof of Lemma B.1.

Proof. I will prove the existence and uniqueness of D^* first. By Assumption 2.2 and Corollary B.3, the interval $\left[1, \max_D \frac{1-F_o(D)}{1-F_p(D)}\right)$ is non-empty. Given Assumption 2.4, there are 3 cases. First, the slope does not cross 0. The slope must be positive, otherwise Assumption 2.2 will be violated. This is the same as Simsek (2013)'s (2.1). Uniqueness is established per Simsek (2013). Second, the slope crosses 0 from below. This means there is a D' such that slope is negative before D' and positive after D'. For all $D > D'$, slope is positive. So $\frac{1-F_o(D)}{1-F_p(D)}$ is increasing and unique for each $D > D'$. Since $\frac{1-F_o(\underline{s})}{1-F_p(\underline{s}))} = 1$ and the slope is negative for all $D < D'$, $\frac{1-F_o(D)}{1-F_p(D)} < 1$ for all $D < D'$. Since $\max_D \frac{1-F_o(D)}{1-F_p(D)} > 1$, there must be $D'' > D'$ such that $\frac{1-F_o(D)}{1-F_p(D)} \geq 1$ and is increasing for all $D \geq D''$ Third, the slope crosses 0 from above. This means there is a D' such that slope is positive before D' and negative after D'. For all $D < D'$, slope is positive. So $\frac{1-F_o(D)}{1-F_p(D)}$ is increasing and unique for each $D < D'$. Since $\frac{1-F_o(\underline{s})}{1-F_p(\underline{s}))} = 1$ and the slope is positive for all $D < D'$, $\frac{1-F_o(D)}{1-F_p(D)} > 1$ for all $D < D'$.

Now, I prove the restriction on δ_b. (B.10) gives the lower bound. I will prove the upper bound by contradiction. First, I will make an observation.

$$\max_D \frac{1-F_o(D)}{1-F_p(D)} > \frac{\mathbb{E}_o[s]}{\mathbb{E}_p[s]}. \tag{B.21}$$

Let's simplify the notation a bit. Let's denote $G(D) = \frac{1-F_o(D)}{1-F_p(D)}$ and $\hat{D} = \arg\max_D G(D)$. Note $\hat{D}$ is unique. To see (B.21), consider the following. $\mathbb{E}_p[s]\, G(\hat{D}) = \int 1 - F_p(s) ds G(\hat{D}) + \underline{s} G(\hat{D}) > \int 1 - F_p(s) ds G(\hat{D}) + \underline{s} > \int 1 - F_o(s) ds + \underline{s} = \mathbb{E}_o[s]$. The first and last equalities follow from Corollary B.3. The first inequality follows from $G(\hat{D}) > 1$ and the second inequality follows from the maximality of $G(\hat{D})$. Now, suppose $\delta_b > \max_D \frac{1-F_o(D)}{1-F_p(D)} > \frac{\mathbb{E}_o[s]}{\mathbb{E}_p[s]}$, $D^* = \bar{s}$ is the unique solution and

$$\delta_{col} = -\left[\mathbb{E}_o\left[s\right] - \delta_b \mathbb{E}_p[s]\right] > 0. \tag{B.22}$$

Plugging (B.22) into the optimists' first order condition w.r.t. α_o (B.7) gives

$$\mathbb{E}_o\left[s\right] + \delta_s - \mathbb{E}_o\left[s\right] + \delta_b \mathbb{E}_p[s] - \delta_b q = 0 \Longrightarrow \delta_s = \delta_b(q - \mathbb{E}_p[s]) \leq 0.$$

where the last inequality follows from Corollary B.1. Since $\delta_s \geq 0$ by definition, $q = \mathbb{E}_p[s]$. Since the collateral constraint binds, the asset position and debt positions cancel out. Since $\delta_b > 1$, the budge constraint reduces to

$$n_o = \int \pi(D)\mu_o(D)^+ dD.$$

There must be at least one D' such that $\mu_o(D')^+ > 0$, meaning $\mu_p(D')^- < 0$ by market clearing condition. By (B.9) and maximality of $G(\hat{D})$, $\mu_o(D')^+ > 0$ implies $\pi(D') = \frac{\mathbb{E}_o[\min(s,D')]}{\delta_b} < \mathbb{E}_p\left[\min(s, D')\right]$. This implies $\lambda_{col} < 0$ for the pessimists. Contradiction! Thus, δ_b is bounded by the interval $\left[1, \max_D \frac{1-F_o(D)}{1-F_p(D)}\right)$. Thus, a unique interior solution always exists. $\square$

B.2.3. Proof of Propositions

B.2.3.1. Proof of Proposition 2.1.

Proof. First, the problem can be simplified as shown in Appendix B.1. I now prove existence. I claim there always exists an equilibrium in which only the optimists hold the asset. I prove this by construction. When only the optimists hold the asset, $\alpha_o = 1$ and $\alpha_p = 0$ by market clearing. Given the equilibrium asset price $q_o(D)$ for some D, the optimists will also have $\mu_o(D) = -1$. The optimists' budget constraint becomes

(B.23)

$$n_o = q_o(D) - E_p[\min(s, D)] = \frac{1 - F_p(D)}{1 - F_o(D)} E_o[s - \min(s, D)] = \frac{1 - F_p(D)}{1 - F_o(D)} \int_D^{\bar{s}} 1 - F_o(s) ds,$$

where the second equality follows from definition of $q_o(D)$ and the last equality follows from Corollary B.3. By Assumption 2.5 and the fact that $\frac{1-F_p(D)}{1-F_o(D)} \int_D^{\bar{s}} 1-F_o(s)ds$ is decreasing in D, there exists a unique $D^* \in [s_{sc}, s_m]$ such that the above equation holds. One can check the following prices and allocations solve both the pessimists and the optimists' problems and the market clears. $q_o(D^*)$ is the asset price. For all D, simple debt contract D has the price $\pi(D) = \max\left[\mathbb{E}_p[\min(s, D)], \frac{1-F_p(D^*)}{1-F_o(D^*)}\mathbb{E}_o[\min(s, D)]\right]$. The allocations are $(\alpha_o = 1, \alpha_p = 0, \mu_o(D^*) = -1, \mu_p(D^*) = 1, \mu_o(D) = \mu_p(D) = 0 \ \forall \ D \neq D^*)$. Thus, equilibrium exists.

I now prove the above equilibrium is also unique. First, note that under Assumption 2.6, there is no equilibrium in which the pessimists hold asset A. For the pessimists to hold asset A and borrow, (B.14) and (B.19) must both hold. However, under Assumption 2.6, there is no D such that both (B.14) and (B.19) hold. This is because Assumption 2.6 implies either the slope of $\frac{1-F_o(D)}{1-F_p(D)}$ is always positive or it is first negative and then positive. Thus, I only need consider equilibria in which only the optimists hold asset A. When only the optimists hold asset A, equilibrium D^* is determined by (B.23). By

(B.23) and Assumption 2.5, D^* is strictly decreasing in n_o. Thus, there is a one-to-one mapping from n_o to D^*. Moreover, there is a one-to-one mapping from D^* to $q_o(D^*)$ by (B.16). Thus, for each n_o, there is a unique equilibrium price $q_o(D^*)$ that clears the market. Given $q_o(D^*)$, there is a unique set of allocations that solves the optimists and pessimists' problems. The price of the traded simple debt contract D^* is given by (B.4). Thus, the allocations and equilibrium prices of the asset and simple debt contract D^* are unique. The price $\pi(D)$ of any non-traded simple debt contract D has to satisfy

$$\pi(D) \geq \max \left[\mathbb{E}_p[\min(s, D)], \frac{1 - F_p(D^*)}{1 - F_o(D^*)} \mathbb{E}_o[\min(s, D)] \right] \tag{B.24}$$

$$\begin{aligned} \pi(D) \leq \min \Bigg[& q_o(D^*) - \mathbb{E}_p[s - \min(s, D)], \\ & \mathbb{E}_p[\min(s, D^*)] + \frac{1 - F_p(D^*)}{1 - F_o(D^*)} \mathbb{E}_o[\min(s, D) - \min(s, D')] \Bigg]. \end{aligned} \tag{B.25}$$

In the first inequality, the first term inside min function is from (B.3) and the second term is from(B.9). In the second inequality, the first term in the min function is from (B.8) and the second term is from (B.2) and (B.3). Thus, the equilibrium is unique up to the allocations and the prices of traded contracts. □

B.2.3.2. Proof of Proposition 2.2.

Proof. Given Proposition 2.1, the equilibrium price in unique for each n_o. When the slope of $\frac{1-F_o(s)}{1-F_p(s)}$ crosses zero exactly once from below, the CCDFs' cross each other for the first time at some interior $s_{sc} > \underline{s}$. Then the price

$$q = q_o(s_{sc}) \equiv \mathbb{E}_p \left[\min \left(s, s_{sc} \right) \right] + \frac{1 - F_p(s_{sc})}{1 - F_o(s_{sc})} \mathbb{E}_o \left[s - \min \left(s, s_{sc} \right) \right]$$

is the equilibrium price for n_o such that $n_o \geq \frac{1-F_p(s_{sc})}{1-F_o(s_{sc})}\mathbb{E}_o[s - \min(s, s_{sc})]$. Since the slope of $\frac{1-F_o(s)}{1-F_p(s)}$ is negative for $s < s_{sc}$, $\frac{1-F_o(s)}{1-F_p(s)} < 1$ for all $s \in (\underline{s}, s_{sc})$. This means $1 - F_o(s) < 1 - F_p(s)$ for all $s \in (\underline{s}, s_{sc})$. So,

$$\mathbb{E}_p\left[\min\left(s, s_{sc}\right)\right] = \int_{\underline{s}}^{s_{sc}} 1 - F_p(s)ds + \underline{s} > \int_{\underline{s}}^{s_{sc}} 1 - F_o(s)ds + \underline{s} = \mathbb{E}_o\left[\min\left(s, s_{sc}\right)\right].$$

The equality follows from Corollary B.3 and the inequality follows from $1 - F_o(s) < 1 - F_p(s)$ for all $s \in (\underline{s}, s_{sc})$. Putting this back into the price gives

$$q_o(s_{sc}) > \mathbb{E}_o\left[\min\left(s, s_{sc}\right)\right] + \frac{1 - F_p(s_{sc})}{1 - F_o(s_{sc})}\mathbb{E}_o\left[s - \min\left(s, s_{sc}\right)\right] = \mathbb{E}_o[s].$$

The equality follows from the fact that $\frac{1-F_p(s_{sc})}{1-F_o(s_{sc})} = 1$. $\square$

APPENDIX C

Appendix for Chapter 3

C.1. Additional Analysis

C.1.1. Additional Analysis for section 3.4.1

The small arbitrageurs' maximization problem defined above does not condition on any new information. Thus, here I present the equilibrium where each small arbitrageur t_i does not use any information from after time t_i. Since Π and Φ does not condition on new information, it's always conditioned on the constant η. Thus, I can drop the conditioning on η for notational convenience. To derive the equilibrium where the small arbitrageurs condition their actions on new information, I would have to modify Π and Φ. First I establish some useful conditions to help us solve for the equilibrium.

Lemma C.1. *Since the small arbitrageurs are risk-neutral,* $\sigma(\tau_i) \in \{0, 1\}$ *for all* τ_i*.*

Corollary C.1. *For any* τ_i*,* $\sigma'(\tau_i) \in \{0, 2\sigma(\tau_i) - 1\}$

The proofs are immediate. Note $\sigma(\tau_i)$ is the result of adding $\sigma'(\tau_i)$ to the selling pressure from the previous instant, i.e. $\sigma(\tau_i) = \lim_{\tau_i' \to \tau_i^-} \sigma(\tau_i') + \sigma'(\tau_i)$. Corollary C.1 implies any two consecutive non-zero σ' has to have opposite signs.

Corollary C.2. *For any* τ_i *such that* $\sigma'(\tau_i) \neq 0$*, let* $\tau_i' = \inf\{\tau_i'' | \sigma'(\tau_i'') \neq 0$ *and* $\tau_i'' > \tau_i\}$*. Then,* $\sigma'(\tau_i) = -\sigma'(\tau_i'')$*.*

Intuitively, during any two consecutive changes of action, the changes must be in opposite directions so that the action stays in the set $\{0,1\}$. Now, I present the following Proposition.

Proposition C.1. *Let*

$$\tau_i^* \equiv \inf\left\{\tau_i \,\middle|\, \frac{\lambda}{1-e^{\lambda(\tau_i-\bar{T})}} \geq \frac{g-r}{\beta(\bar{T})}\right\} < \bar{T}.$$

The small arbitrageurs optimally exit the market at τ_i^*. Moreover, once the small arbitrageurs exit the market, they will not reenter, i.e. $\sigma(\tau_i) = 1$ for all $\tau_i \geq \tau_i^*$.

With the exit condition in Proposition C.1, I immediately have the following Corollary.

Corollary C.3. *If $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} < \frac{g-r}{\beta(\eta\kappa)}$, I can rule out no-bubble equilibrium.*

Proof by contrapositive. In a no-bubble equilibrium, all small arbitrageurs sell at $\tau_i = 0$ and the price collapse exactly at $t_0 + \bar{T} = t_0 + \eta\kappa$. In order for the small arbitrageurs to sell at $\tau_i = 0$, I must have (by Proposition C.1)

$$\frac{\lambda}{1-e^{\lambda(\tau_i-\bar{T})}} \geq \frac{g-r}{\beta(\bar{T})} \Leftrightarrow \frac{\lambda}{1-e^{-\lambda\eta\kappa}} \geq \frac{g-r}{\beta(\eta\kappa)},$$

since $\tau_i - \bar{T} = 0 - \eta\kappa = -\eta\kappa$ and $\bar{T} = \eta\kappa$. □

Now that I can rule out no-bubble equilibrium, I can simplify the exit condition in Proposition C.1 in the following Corollary.

Corollary C.4. *If $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} < \frac{g-r}{\beta(\eta\kappa)}$, $\tau_i^* = \bar{T} - \frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\bar{T})\right)$.*

Proof. $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} < \frac{g-r}{\beta(\eta\kappa)}$ implies

$$0 < \tau_i^* \equiv \inf\left\{\tau_i \,\middle|\, \frac{\lambda}{1-e^{\lambda(\tau_i-\bar{T})}} \geq \frac{g-r}{\beta(\bar{T})}\right\} < \bar{T}.$$

Since the solution is interior, the first order condition must hold with equality, i.e.

$$\frac{\lambda}{1-e^{\lambda(\tau_i^*-\bar{T})}} = \frac{g-r}{\beta(\bar{T})}.$$

I can then rearrange and get the desired time τ_i^* as a function of $\bar{T}$. □

Thus, I add Assumption 3.3 as it has more uses beyond the last two Corollaries. With Assumption 3.3, I can plug the expression in Corollary C.4 into equation (3.2) and compute the equilibrium burst time of the bubble.

C.1.2. Rearranging (3.4)

$$\begin{aligned}
&w \times \int_0^{\bar{T}} e^{-r\gamma'}(1-\beta(\bar{T}))P(\gamma')\phi(\gamma'-\bar{T}|\nu)d\gamma' \\
&\quad + \int_\gamma^{\bar{T}} e^{-r\gamma}P(\gamma)[(\eta\kappa - \bar{s}(\gamma-\gamma'+\bar{T}))\vee w]\phi(\gamma'-\bar{T}|\nu)d\gamma' \\
&\quad + \int_\gamma^{\bar{T}} e^{-r\gamma}(1-\beta(\gamma-\gamma'+\bar{T}))P(\gamma)[(\bar{s}(\gamma-\gamma'+\bar{T})-\eta\kappa)\wedge -w]\phi(\gamma'-\bar{T}|\nu)d\gamma' - C \\
=&w \times \int_0^{\bar{T}} e^{-r\gamma'}(1-\beta(\bar{T}))P(\gamma')\phi(\gamma'-\bar{T}|\nu)d\gamma' \\
&\quad + \int_\gamma^{\bar{T}} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})[(\eta\kappa - \bar{s}(\gamma-\gamma'+\bar{T}))\vee w]\phi(\gamma'-\bar{T}|\nu)d\gamma' - C
\end{aligned}$$

C.1.3. Additional Analysis for section 3.5.1

In the base case, the problem for the large arbitrageur reduces to the following

$$\max_{\gamma}\ \eta\kappa \times \int_0^{\bar{T}} e^{-r\gamma'}(1-\beta(\bar{T}))P(\gamma')\phi(\gamma'-\bar{T}|0)d\gamma' \\ + \int_{\gamma}^{\bar{T}} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})[\eta\kappa - \bar{s}(\gamma-\gamma'+\bar{T})]\phi(\gamma'-\bar{T}|0)d\gamma' - C$$

Since $\phi(\cdot|0)$ is a Dirac delta function, the above problem can be further simplified to

$$\max_{\gamma} \eta\kappa \left[e^{-r\bar{T}}(1-\beta(\bar{T}))P(\bar{T})\right] + e^{-r\gamma}P(\gamma)\beta(\gamma)[\eta\kappa - \bar{s}(\gamma)]\mathbb{1}_{\{\gamma \le \bar{T}\}}$$

Lemma C.2. *If* $\gamma^* = \arg\max_{\gamma} \eta\kappa \left[e^{-r\bar{T}}(1-\beta(\bar{T}))P(\bar{T})\right] + e^{-r\gamma}P(\gamma)\beta(\gamma)[\eta\kappa - \bar{s}(\gamma)] \times \mathbb{1}_{\{\gamma \le \bar{T}\}}$ *exists, it must be that* $\gamma^* \le \bar{T}$*.*

Proof. Since $e^{-r\gamma}P(\gamma)\beta(\gamma)[\eta\kappa - \bar{s}(\gamma)] \ge 0$ for $\gamma \le \bar{T}$, it is $\gamma \le \bar{T}$ (weakly) dominates $\gamma' > \bar{T}$. □

At this point, the analysis cannot proceed without specifying a β function or at least more properties of the β function since the β function is also embedded in $\bar{s}$. However, instead of deriving properties of β function that guarantees existence of unique solution, I restrict my attention to a special case where $\beta(\tau_0) = 1 - e^{-(g-r)\tau_0}$. In this case, the fundamental value after t_0 becomes

$$f_t = (1-\beta(\tau_0))e^{gt} = e^{gt_0 + r\tau_0}.$$

In this case, Assumption 3.1 binds with equality. As a result, the arbitrageurs trapped in the asset after the crash will keep holding the asset as there is cost of selling the asset and the arbitrageurs are indifferent between the asset and the risk-free interest rate. I call this a special case because there are other β functions that fit the criteria, i.e. strictly increasing and continuous with range between 0 and 1. However, this choice of β function is almost natural, because the fundamental value of the stock is essentially risk-free after t_0 and, therefore, should grow at the risk-free rate. Since the small arbitrageurs' problem does not depend on any additional properties of β function, the special case β function does not affect the small arbitrageurs' problem. Therefore, I can simply plug the special case β function into $\bar{s}$ without additional qualifications. With the β function, the large arbitrageur's problem simplifies even further to

$$\max_{\gamma} e^{(g-r)\gamma}[\eta\kappa - \bar{s}(\gamma)] + \bar{s}(\gamma) \tag{C.1}$$

Moreover, the choice of β function and Assumption 3.4 also simplifies $\bar{s}$ so that it's differentiable everywhere. One immediate result from this Assumption 3.4 is

Corollary C.5. $g - r - \lambda \geq (g-r)e^{-\lambda\eta} > 0$.

Up until now, I have been using log without qualifications. Corollary C.5 ensures that all of the natural logs have real values since $g - r - \lambda\beta(\tau_0) > g - r - \lambda > 0$. Another result immediately follows.

Corollary C.6. *Let* $B(\tau_0) = \frac{1}{\lambda}[\log(g-r) - \log(g - r - \lambda + \lambda e^{-(g-r)\tau_0})]$,

$$\frac{dB(\tau_0)}{d\tau_0} = \frac{(g-r)e^{-(g-r)\tau_0}}{g - r - \lambda + \lambda e^{-(g-r)\tau_0}} = \frac{(g-r)}{\lambda + (g-r-\lambda)e^{(g-r)\tau_0}} > 0,$$
$$\frac{d^2B(\tau_0)}{d\tau_0^2} = \frac{-(g-r)^2(g-r-\lambda)e^{(g-r)\tau_0}}{(\lambda + (g-r-\lambda)e^{(g-r)\tau_0})^2} < 0.$$

Corollary C.6 shows that $B(\tau_0)$ is concave. This fact along with Assumption 3.4 establishes following Lemma.

Lemma C.3. $\bar{s}(\tau_0) = \frac{1}{\lambda}[\log(g-r) - \log(g-r-\lambda+\lambda e^{-(g-r)\tau_0})]$.

C.1.4. Additional Analysis for section 3.5.2

I will provide a more formal proof after the next Proposition. Now I derive the large arbitrageur's problem for this special case. Plugging in $\nu = \bar{T}$ and $w = \eta\kappa$, into (3.5), I get

$$\max_{\gamma} \; \eta\kappa \times \int_0^{\bar{T}} e^{-r\gamma'}(1-\beta(\bar{T}))P(\gamma')\phi(\gamma'-\bar{T}|\bar{T})d\gamma' \\ + \int_{\gamma}^{\bar{T}} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})[\eta\kappa - \bar{s}(\gamma-\gamma'+\bar{T})]\phi(\gamma'-\bar{T}|\bar{T})d\gamma' - C$$

Since the first term and last term are simply constants with respect to γ, the above maximization problem is equivalent to

$$\text{(C.2)} \qquad \max_{\gamma} \int_{\gamma}^{\bar{T}} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})[\eta\kappa - \bar{s}(\gamma-\gamma'+\bar{T})]\phi(\gamma'-\bar{T}|\bar{T})d\gamma'$$

Let's denote the objective function as $G(\gamma) \equiv \int_{\gamma}^{\bar{T}} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})[\eta\kappa - \bar{s}(\gamma-\gamma'+\bar{T})]\phi(\gamma'-\bar{T}|\bar{T})d\gamma'$. The equilibrium result is as the following.

C.1.5. Additional Analysis for section 3.5.3

The only complication involved with incomplete information is that $\phi(\tau_i^0|\nu)$ is not continuous and is 0 for all $\tau_i^0 \in (-\infty, -\nu) \cup (0, \infty)$. Thus, I can rewrite (3.5) as the following

$$\max_{\gamma}\ \eta\kappa \times \int_0^{\bar{T}} e^{-r\gamma'}(1-\beta(\bar{T}))P(\gamma')\phi(\gamma'-\bar{T}|\nu)d\gamma'$$
$$+\int_{\gamma\wedge(\bar{T}-\nu)}^{\bar{T}} e^{-r\gamma}P(\gamma)\beta(\gamma-\gamma'+\bar{T})[\eta\kappa-\bar{s}(\gamma-\gamma'+\bar{T})]\phi(\gamma'-\bar{T}|\nu)d\gamma'-C$$

As usual, I can remove the constants in the maximization problem. Let's denote the objection as H, i.e.

$$H(\gamma) \equiv \int_{\gamma\wedge(\bar{T}-\nu)}^{\bar{T}} (e^{(g-r)\gamma}-e^{(g-r)(\gamma'-\bar{T})})[\eta\kappa-\bar{s}(\gamma-\gamma'+\bar{T})]\phi(\gamma'-\bar{T}|\nu)d\gamma'$$

Before I proceed, here are some useful results.

Lemma C.4. *H is continuous at $\bar{T}-\nu$ for endogenous and exogenous $\bar{T}$. H' is continuous at $\bar{T}-\nu$ for endogenous $\bar{T}$. For exogenous $\bar{T}$, H' has a negative jump at $\bar{T}-\nu$.*

Let's define

$$\underline{H}(\gamma) \equiv \int_{\bar{T}-\nu}^{\bar{T}} (e^{(g-r)\gamma}-e^{(g-r)(\gamma'-\bar{T})})[\eta\kappa-\bar{s}(\gamma-\gamma'+\bar{T})]\phi(\gamma'-\bar{T}|\nu)d\gamma',$$
$$\bar{H}(\gamma) \equiv \int_{\gamma}^{\bar{T}} (e^{(g-r)\gamma}-e^{(g-r)(\gamma'-\bar{T})})[\eta\kappa-\bar{s}(\gamma-\gamma'+\bar{T})]\bar{\phi}(\gamma'-\bar{T}|\nu)d\gamma'.$$

so that $H(\gamma)=\bar{H}(\gamma)$ for $\gamma \geq \bar{T}-\nu$ and $H(\gamma)=\underline{H}(\gamma)$ for $\gamma < \bar{T}-\nu$. Then, I have two more useful results.

Lemma C.5. *$\underline{H}'$ crosses 0 at most one time and if it crosses 0, it does so from above.*

Lemma C.6. *$\underline{H}'(\gamma^{**}) > 0$.*

C.2. Proof of Lemmas and Corollaries

C.2.1. Proof of Corollary 3.1

Proof. Let $D(\tau_0) \equiv \tau_0 - B(\tau_0) = \tau_0 - \frac{1}{\lambda}(\log(g-r) - \log(g-r-\lambda\beta(\tau_0)))$. Since the small arbitrageurs treat the bubble bursting caused by the large arbitrageur as if it's exogenous, their optimal selling time, according to Proposition C.1 and Corollary C.4 is $\tau_i^{**} = D(\gamma^*) = \gamma^* - \frac{1}{\lambda}(\log(g-r) - \log(g-r-\lambda\beta(\gamma^*))$. Since $B'(0) = 1$ and $B''(\tau_0) < 0$ for all τ_0 (by Corollary C.6), $B'(\tau_0) < 1$ for all $\tau > 0$. Thus, $D'(\tau_0) = 1 - B'(\tau_0) > 0$ for all $\tau_0 > 0$. Proposition 3.3 shows that $\gamma^* < \bar{T}$ when $\bar{T}$ is endogenous. Thus, $\tau_i^{**} = D(\gamma^*) < D(\bar{T}) = \tau_i^*$. □

C.2.2. Proof of Corollary 3.2

Proof. Since $\gamma^* = \bar{T} = \bar{\tau}$ in the exogenous case, $\tau_i^{**} = D(\gamma^*) = D(\bar{T}) = \tau_i^*$. □

C.2.3. Proof of Corollary 3.3

Proof. In the exogenous case, by Proposition 3.3 and 3.4, I have $\gamma^{**} < \bar{T} = \bar{\tau} = \gamma^*$. Now I prove the endogenous case. Recall that by Proposition 3.3, $F(\gamma^*) = 0$. Moreover, in Proposition 3.4, I have that $G'(\gamma^{**}) = 0$ and $G''(\gamma^{**}) < 0$. I also have that $G''(\gamma) = (g-r-\lambda)G'(\gamma) - \frac{\lambda}{e^{\lambda\bar{T}}-1}F(\gamma)$ for any γ. Therefore, $F(\gamma^{**}) = \frac{e^{\lambda\bar{T}}-1}{\lambda}[(g-r-\lambda)G'(\gamma^{**}) - G''(\gamma^{**})] = -\frac{e^{\lambda\bar{T}}-1}{\lambda}G''(\gamma^{**}) > 0$. Since γ^* is the unique maximum and there is no reentry, $F(\gamma^{**}) > 0$ implies $\gamma^{**} < \gamma$. □

C.2.4. Proof of Lemma C.3

Proof. Recall that $\bar{s}(\tau_0) = \tau_0 \vee \eta \vee B(\tau_0)$. By Assumption 3.4, $B(\tau_0) = \frac{1}{\lambda}[\log(g-r) - \log(g-r-\lambda+\lambda e^{-(g-r)\tau_0})] \leq \frac{1}{\lambda}(\log(g-r) - \log(g-r-\lambda\bar{\beta})) \leq \eta$. Thus, $\bar{s}(\tau_0) = \tau_0 \vee B(\tau_0)$. Moreover, since $B(\tau_0)$ is concave by Corollary C.6, I have

$$B(\tau_0) \leq B(0) + B'(0)(\tau_0)$$

Since $B(0) = 0$ and $B'(0) = 1$, I have $B(\tau_0) \leq \tau_0$. So, $\bar{s}(\tau_0) = B(\tau_0) = \frac{1}{\lambda}[\log(g-r) - \log(g-r-\lambda+\lambda e^{-(g-r)\tau_0})]$. □

C.2.5. Proof of Lemma C.4

Proof. Let

$$\underline{H}(\gamma) \equiv \int_{\bar{T}-\nu}^{\bar{T}} (e^{(g-r)\gamma} - e^{(g-r)(\gamma'-\bar{T})})[\eta\kappa - \bar{s}(\gamma-\gamma'+\bar{T})]\phi(\gamma'-\bar{T}|\nu)d\gamma',$$

$$\bar{H}(\gamma) \equiv \int_{\gamma}^{\bar{T}} (e^{(g-r)\gamma} - e^{(g-r)(\gamma'-\bar{T})})[\eta\kappa - \bar{s}(\gamma-\gamma'+\bar{T})]\bar{\phi}(\gamma'-\bar{T}|\nu)d\gamma'.$$

where $\bar{\phi}(\gamma|\nu) = \frac{\lambda e^{-\lambda\gamma}}{e^{\lambda\nu}-1}$. Clearly, $H(\gamma) = \bar{H}(\gamma)$ for $\gamma \geq \bar{T}-\nu$ and $H(\gamma) = \underline{H}(\gamma)$ for $\gamma \leq \bar{T}-\nu$. Note that $\bar{H}(\bar{T}-\nu) = \underline{H}(\bar{T}-\nu)$. Therefore, H is continuous at $\bar{T}-\nu$. Similarly, $H' = \bar{H}'$ when $\gamma \geq \bar{T}-\nu$ and $H' = \underline{H}'$ when $\gamma < \bar{T}-\nu$. And

$$\begin{aligned}\underline{H}'(\gamma) = \int_{\bar{T}-\nu}^{\bar{T}} \Big[&(g-r)e^{(g-r)\gamma}[\eta\kappa \\ &- \bar{s}(\gamma-\gamma'+\bar{T})] - (e^{(g-r)\gamma} - e^{(g-r)(\gamma'-\bar{T})})\bar{s}'(\gamma-\gamma'+\bar{T})\Big]\phi(\gamma'-\bar{T}|\nu)d\gamma'\end{aligned}$$

$$\bar{H}'(\gamma) = \int_{\gamma}^{\bar{T}} \Big[(g-r)e^{(g-r)\gamma}[\eta\kappa$$

$$- \bar{s}(\gamma - \gamma' + \bar{T})] - (e^{(g-r)\gamma} - e^{(g-r)(\gamma'-\bar{T})})\bar{s}'(\gamma - \gamma' + \bar{T})\Big]\phi(\gamma' - \bar{T}|\nu)d\gamma'$$

$$- (e^{(g-r)\gamma} - e^{(g-r)(\gamma-\bar{T})})[\eta\kappa - \bar{s}(\bar{T})]\phi(\gamma - \bar{T}|\nu)$$

In the endogenous case, $\bar{s}(\bar{T}) = \eta\kappa$. So $\underline{H}'(\bar{T} - \nu) = \bar{H}'(\bar{T} - \nu)$, i.e. H' is continuous at $\bar{T} - \nu$. In the exogenous case, however, $\bar{s}(\bar{T}) < \eta\kappa$. So $\bar{H}'(\bar{T} - \nu) < \underline{H}'(\bar{T} - \nu)$, i.e. H' has a negative jump at $\bar{T} - \nu$. □

C.2.6. Proof of Lemma C.5

Proof. First I need the second derivative.

$$\begin{aligned}
\underline{H}''(\gamma) &= (g-r)\left(\underline{H}'(\gamma) - \int_{\bar{T}-\nu}^{\bar{T}} e^{(g-r)(\gamma'-\bar{T})}\bar{s}'(\gamma - \gamma' + \bar{T})\phi(\gamma' - \bar{T}|\nu)d\gamma'\right) \\
&\quad - \int_{\bar{T}-\nu}^{\bar{T}} (g-r)e^{(g-r)\gamma}\bar{s}'(\gamma - \gamma' + \bar{T})\phi(\gamma' - \bar{T}|\nu)d\gamma' \\
&\quad - \int_{\bar{T}-\nu}^{\bar{T}} (e^{(g-r)\gamma} - e^{(g-r)(\gamma'-\bar{T})})\bar{s}''(\gamma - \gamma' + \bar{T})\phi(\gamma' - \bar{T}|\nu)d\gamma' \\
&= (g-r)\left(\underline{H}'(\gamma) - \int_{\bar{T}-\nu}^{\bar{T}} e^{(g-r)(\gamma'-\bar{T})}\bar{s}'(\gamma - \gamma' + \bar{T})\phi(\gamma' - \bar{T}|\nu)d\gamma'\right) \\
&\quad + \int_{\bar{T}-\nu}^{\bar{T}} \Big[- (g-r)e^{(g-r)\gamma}\bar{s}'(\gamma - \gamma' + \bar{T}) \\
&\qquad - (e^{(g-r)\gamma} - e^{(g-r)(\gamma'-\bar{T})})\bar{s}''(\gamma - \gamma' + \bar{T})\Big]\phi(\gamma' - \bar{T}|\nu)d\gamma'
\end{aligned}$$

Notice the term in square brackets. Using similar method as in Proposition 3.3, I can simplify the second derivative to the following

$$\underline{H}''(\gamma) = (g-r)\left(\underline{H}'(\gamma) - \int_{\bar{T}-\nu}^{\bar{T}} e^{(g-r)(\gamma'-\bar{T})}\bar{s}'(\gamma - \gamma' + \bar{T})\phi(\gamma' - \bar{T}|\nu)d\gamma'\right)$$

$$-\int_{\bar{T}-\nu}^{\bar{T}} \frac{(g-r)^2\lambda e^{(g-r)\gamma} + (g-r)^2(g-r-\lambda)e^{(g-r)\gamma}}{(\lambda+(g-r-\lambda)e^{(g-r)(\gamma-\gamma'+\bar{T})})^2}\phi(\gamma'-\bar{T}|\nu)d\gamma'$$

Now it is easy to see that whenever $\underline{H}'(\gamma) \leq 0$, $\underline{H}''(\gamma) < 0$. Thus, whenever $\underline{H}'(\gamma) = 0$, $\underline{H}'(\gamma') < 0$ for all $\gamma' > \gamma$. □

C.2.7. Proof of Lemma C.6

Proof. Let us denote the integrand of $\underline{H}'(\gamma)$ and $\bar{H}'(\gamma)$ as $E(\gamma, \gamma')$, i.e.

$$\begin{aligned} E(\gamma,\gamma') \equiv \Big[&(g-r)e^{(g-r)\gamma}[\eta\kappa - \bar{s}(\gamma-\gamma'+\bar{T})] \\ &-(e^{(g-r)\gamma} - e^{(g-r)(\gamma'-\bar{T})})\bar{s}'(\gamma-\gamma'+\bar{T})\Big]\phi(\gamma'-\bar{T}|\nu) \end{aligned}$$

Then, let's take a look at the derivative of E with respect to γ'.

$$\begin{aligned} \frac{\partial E(\gamma,\gamma')}{\partial \gamma'} = \Big[&(g-r)e^{(g-r)\gamma}\bar{s}'(\gamma-\gamma'+\bar{T}) \\ &+ (e^{(g-r)\gamma} - e^{(g-r)(\gamma'-\bar{T})})\bar{s}''(\gamma-\gamma'+\bar{T})\Big]\phi(\gamma'-\bar{T}|\nu) \\ &+ \phi'(\gamma'-\bar{T}|\nu)\frac{E(\gamma,\gamma')}{\phi}(\gamma'-\bar{T}|\nu)] \\ = \phi(\gamma'-\bar{T}|\nu)&\frac{(g-r)^2\lambda e^{(g-r)\gamma} + (g-r)^2(g-r-\lambda)e^{(g-r)\gamma}}{(\lambda+(g-r-\lambda)e^{(g-r)(\gamma-\gamma'+\bar{T})})^2} \\ &+ \phi'(\gamma'-\bar{T}|\nu)\frac{E(\gamma,\gamma')}{\phi}(\gamma'-\bar{T}|\nu)] \end{aligned}$$

Since $\phi'(\gamma'-\bar{T}|\nu) = -\lambda^2 e^{-\lambda(\gamma'-\bar{T})}/(e^{\lambda\nu}-1) < 0$, whenever $E(\gamma,\gamma') \leq 0$, I have $\frac{\partial E(\gamma,\gamma')}{\partial\gamma'} > 0$. Therefore, if $E(\gamma,\gamma') > 0$, then $E(\gamma,\gamma'') > 0$ for all $\gamma'' \geq \gamma'$.

Now Recall that G' is the first order condition in Proposition 3.4. First note that $(e^{\gamma\nu}-1)\bar{H} = (e^{\gamma\bar{T}}-1)G$. Thus, $(e^{\gamma\nu}-1)\bar{H}' = (e^{\gamma\bar{T}}-1)G'$. By Proposition 3.4, I know

$\bar{H}'(\gamma^{**}) = \frac{e^{\lambda\bar{T}}}{e^{\lambda\nu}-1}G'(\gamma^{**}) = 0$. Since $0 = \bar{H}'(\gamma^{**}) = \int_{\gamma^{**}}^{\bar{T}} E(\gamma^{**},\gamma')d\gamma'$, it must be that $\int_{\gamma^{**}}^{\bar{T}-\nu} E(\gamma^{**},\gamma')d\gamma' < 0$. (If not, the property of E from above implies $E(\gamma^{**},\bar{T}-\nu) > 0$, which in turn implies $\bar{H}'(\gamma^{**}) > 0$, a contradiction.) By Lemma C.5, $\bar{H}'(\gamma^{**}) - \underline{H}'(\gamma^{**}) \leq \int_{\gamma^{**}}^{\bar{T}-\nu} E(\gamma^{**},\gamma')d\gamma' < 0$. Thus, $\underline{H}'(\gamma^{**}) > \bar{H}'(\gamma^{**}) = 0$. $\square$

C.2.8. Proof of Lemma 3.2

Proof. The statement is true for the exogenous case as explained above. The case is also trivial for $\nu \geq \bar{T} - \gamma^{**}$, since $\gamma^{***} = \gamma^{**} < \gamma^*$ (by Corollary 3.3). Thus, I only need to prove for the endogenous case with $\nu > \bar{T} - \gamma^{**}$. By Lemma C.4 and property of G', I have $\underline{H}'(\bar{T}-\nu) = \bar{H}'(\bar{T}-\nu) = \frac{e^{\lambda\bar{T}}}{e^{\lambda\nu}-1}G'(\bar{T}-\nu) < 0$. Moreover, by Lemma C.5, $\underline{H}'(\gamma) < 0$ for all $\gamma > \bar{T}-\nu$. Thus, if $\gamma^* > \bar{T}-\nu$, $\underline{H}'(\gamma^*) < 0$. Since ν in ϕ is only causes monotonic transformation in $\underline{H}'$ and $\bar{H}'$, I can disregard the effect from ν in ϕ. Thus,

$$\begin{aligned}
\underline{H}'_\nu(\gamma,\nu) \equiv \frac{d\underline{H}'(\gamma)}{d\nu} &= -\frac{\lambda e^{\lambda\nu}}{e^{\lambda\nu}-1}\underline{H}'(\gamma) + E(\gamma,\bar{T}-\nu) \\
&= -\frac{\lambda e^{\lambda\nu}}{e^{\lambda\nu}-1}\underline{H}'(\gamma) + e^{-(g-r)\nu}\phi(-\nu|\nu)F(\gamma+\nu)
\end{aligned}$$

Since $F(\gamma^*+\nu) < 0$, $\underline{H}'_\nu(\gamma^*,\nu) < -\frac{\lambda e^{\lambda\nu}}{e^{\lambda\nu}-1}\underline{H}'(\gamma^*)$, for all $\nu > 0$. I already know, by Proposition 3.5 that when $\nu = \bar{T}-\gamma^{**} < \gamma^*$, $\underline{H}'(\gamma^*) < 0$. Moreover, whenever $\underline{H}'(\gamma^*) = 0$ for some $\nu < \underline{\nu}$, $\underline{H}'_\nu(\gamma,\nu) < 0$. Thus, $\underline{H}'(\gamma^*) \leq 0$ for all $\nu < \underline{\nu}$. By Lemma C.5, $\underline{H}'(\gamma) < 0$ for all $\gamma > \gamma^*$. Thus, $\gamma^{***} \leq \gamma^*$.

Now I prove the second result. First I look at the cases of $\nu > \bar{\nu}$ where $\gamma^{***} < \bar{T}-\nu$. Since $\underline{H}'(\gamma^{***}) = 0$, $\underline{H}'_\nu(\gamma^{***},\nu) = e^{-(g-r)\nu}\phi(-\nu|\nu)F(\gamma^{***}+\nu)$. It's more useful to think of comparative statics with respect to $-\nu$, so I relabel $\underline{H}'_{-\nu}(\gamma^{***},\nu) =$

$-e^{-(g-r)\nu}\phi(-\nu|\nu)F(\gamma^{***}+\nu)$. Thus, for $\gamma^{***}+\nu>\gamma^*$, $\underline{H}'_{-\nu}(\gamma^{***},\nu)>0$, which implies γ^{***} is strongly monotonic in $-\nu$. If, however, $\gamma^{***}+\nu\leq\gamma^*$, $\underline{H}'_{-\nu}(\gamma^{***},\nu)\leq 0$, which would imply γ^{***} is weakly decreasing in $-\nu$. Thus, whenever $\gamma^{***}+\nu\leq\gamma^*$, and $\gamma^{***}(\nu')+\nu'<\gamma^*$ for all $\nu'<\nu$. Thus, $\gamma^{***}(\nu')<\gamma^{***}(\nu)$ for all $\nu<\nu'$. Therefore, if $\gamma^{***}(\nu)\leq\gamma^*$ for some $\nu>0$, $\lim_{\nu\to 0}\gamma^{***}(\nu)<\gamma^*$. This would contradict the fact that $\lim_{\nu\to 0}\underline{H}'=F$. Thus, I must have either $\gamma^{***}(\nu)>\gamma^*$ or $\gamma^{***}(\nu)+\nu>\gamma^*$ for all $0<\nu<\underline{\nu}$. Since the first results rules out $\gamma^{***}(\nu)>\gamma^*$, I must have $\gamma^{***}(\nu)+\nu>\gamma^*$ for all $0<\nu<\underline{\nu}$. Thus, γ^{***} is strongly monotonic in $-\nu$.

In some exogenous cases, there may be $\nu<\underline{\nu}$ such that $\underline{H}'(\bar{T}-\nu)\geq 0$. Since $\bar{T}-\nu>\gamma^*-\nu$, $\underline{H}'_{-\nu}(\gamma^*,\nu)>0$. Thus, γ^{***} is strongly monotonic in $-\nu$. □

C.3. Proof of Propositions

C.3.1. Proof of Proposition C.1

Proof. I can change the order of the integral of the objective function to get

$$\max_{\sigma'}\int_0^{\bar{T}} e^{-r\tau_i'}P\left(\tau_i'\right)\sigma'\left(\tau_i'\right)\left(1-\Pi\left(\tau_i'\right)\right)d\tau_i'$$
$$+\int_0^{\tau_i} e^{-r\tau_i'}\left(1-\sigma\left(\tau_i'\right)\right)\left[1-\beta\left(\bar{T}\right)\right]P\left(\tau_i'\right)d\Pi(\tau_i')-C\int_0^{\tau_i}|\sigma'(\tau_i')|d\tau_i',$$

Note that by Lemma 3.1, $\sigma(\tau_i')=\int_{-\infty}^{\tau_i'}\sigma'(\tau_i)d\tau_i=\int_0^{\tau_i'}\sigma'(\tau_i)d\tau_i$. Thus, I can rewrite and reorder the objective function as the following

$$\max_{\sigma'}\int_0^{\bar{T}} e^{-r\tau_i'}P\left(\tau_i'\right)\sigma'\left(\tau_i'\right)\left(1-\Pi\left(\tau_i'\right)\right)d\tau_i'-C\int_0^{\bar{T}}|\sigma'(\tau_i')|d\tau_i'$$
$$+\int_0^{\bar{T}} e^{-r\tau_i'}\left(1-\int_0^{\tau_i'}\sigma'\left(\tau_i''\right)d\tau_i''\right)\left[1-\beta\left(\bar{T}\right)\right]P\left(\tau_i'\right)d\Pi(\tau_i')$$

$$
\begin{aligned}
&= \max_{\sigma'} \int_0^{\bar{T}} e^{-r\tau_i'} P\left(\tau_i'\right) \sigma'\left(\tau_i'\right)\left(1-\Pi\left(\tau_i'\right)\right) d\tau_i' - C \int_0^{\bar{T}} |\sigma'(\tau_i')| d\tau_i' \\
&\qquad - \int_0^{\bar{T}} e^{-r\tau_i'} \int_0^{\tau_i'} \sigma'\left(\tau_i''\right)\left[1-\beta\left(\bar{T}\right)\right] P\left(\tau_i'\right) d\tau_i'' d\Pi(\tau_i') \\
&= \max_{\sigma'} \int_0^{\bar{T}} e^{-r\tau_i'} P\left(\tau_i'\right) \sigma'\left(\tau_i'\right)\left(1-\Pi\left(\tau_i'\right)\right) d\tau_i' - C \int_0^{\bar{T}} |\sigma'(\tau_i')| d\tau_i' \\
&\qquad - \int_0^{\bar{T}} \sigma'\left(\tau_i'\right) \int_{\tau_i'}^{\bar{T}} e^{-r\tau_i''}\left[1-\beta\left(\bar{T}\right)\right] P\left(\tau_i''\right) d\Pi(\tau_i'') d\tau_i' \\
&= \max_{\sigma'} \int_0^{\bar{T}} \sigma'\left(\tau_i'\right) \\
&\quad \underbrace{\left[e^{-r\tau_i'} P\left(\tau_i'\right)\left(1-\Pi\left(\tau_i'\right)\right) - \int_{\tau_i'}^{\bar{T}} e^{-r\tau_i''}\left[1-\beta\left(\bar{T}\right)\right] P\left(\tau_i''\right) d\Pi(\tau_i'') - C \times sign(\sigma'(\tau_i'))\right] d\tau_i'}_{A(\tau_i')}
\end{aligned}
$$

There are a few things to note. First, by Lemma 3.1, $\sigma'(\tau_i^*) = 1$, where $\tau_i^* \equiv \inf\{\tau_i | \sigma'(\tau_i) \neq 0 \text{ and } \tau_i \geq 0\}$. This means I want to maximize $A(\tau_i)$. Therefore, $\tau_i^* = \inf\{\tau_i' \,| A'(\tau_i') \leq 0\}$. Moreover,

$$
A'(\tau_i) = e^{(g-r)\tau_i}\left[(g-r)(1-\Pi(\tau_i)) - \beta(\bar{T})\pi(\tau_i)\right] \leq 0 \Leftrightarrow \frac{\pi(\tau_i)}{1-\Pi(\tau_i)} \geq \frac{g-r}{\beta(\bar{T})}
$$

which I can further simplify as

$$
\frac{\lambda}{1-e^{\lambda(\tau_i-\bar{T})}} \geq \frac{g-r}{\beta(\bar{T})}. \tag{C.3}
$$

Note that LHS is strictly increasing in $\tau_i < \bar{T}$, whereas RHS is a constant. Therefore, the above condition is true for all $\tau_i > \tau_i^*$. However, by Corollary C.2, two consecutive non-zero σ' cannot have the same signs. Thus, $\sigma'(\tau_i) = 0$ for all $\tau_i > \tau_i^*$, i.e. once the individuals leave the market they will not reenter. Moreover, the LHS of (C.3) is ∞ at $\bar{T}$

and the RHS is only a finite constant. Thus, $\tau_i^* < \bar{T}$. Finally note that inequality (C.3) is exactly the same selling condition as in Abreu and Brunnermeier (2003). $\square$

C.3.2. Proof of Proposition 3.1

Proof. Since $\sigma(\tau_i) = 1$ for all $\tau_i \geq \tau_i^*$, I can plug the exit time expression from Corollary C.4 into (3.2) and get

$$
\begin{aligned}
s(\tau_0, \bar{T}) &= \int_{(\tau_0-\eta)\wedge 0}^{\tau_0} \mathbb{1}_{\tau_i \geq \bar{T} - \frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\bar{T})\right)} d\tau_i \\
&= \tau_0 \vee \eta \vee \left[\tau_0 - \left(\bar{T} - \frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\bar{T}))\right)\right)\right] \qquad \text{(C.4)}
\end{aligned}
$$

(*Endogenous Bursting*) Now I will show that there exists a unique endogenous solution if and only if $\frac{\lambda}{1-e^{\lambda\eta\kappa}} > \frac{g-r}{\beta}$.

($\Leftarrow$) Suppose $\frac{\lambda}{1-e^{\lambda\eta\kappa}} > \frac{g-r}{\beta}$. For the bubble to burst endogenously, I need the following conditions to hold simultaneously: $\bar{T} = \tau_0^*$ and $s(\tau_0^*) = \eta\kappa$. First, I plug $\bar{T} = \tau_0^*$ into (C.4).

$$
\begin{aligned}
\bar{s}(\tau_0^*) \equiv s(\tau_0^*, \tau_0^*) &= \tau_0^* \vee \eta \vee \left[\tau_0^* - \left(\tau_0^* - \frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\tau_0^*)\right)\right)\right] \\
&= \tau_0^* \vee \eta \vee \left[\frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\tau_0^*)\right)\right]
\end{aligned}
$$

If there exists a τ_0^* such that $\bar{s}(\tau_0^*) = \eta\kappa$, then I am done. It's useful to note that $\frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\tau_0^*)\right)$ is a strictly increasing in τ_0^*. Now recall Assumption 3.3 states that $\frac{\lambda}{1-e^{-\lambda\eta\kappa}} < \frac{g-r}{\beta(\eta\kappa)}$, which can be rewritten as

$$
\frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\eta\kappa)\right) < \eta\kappa
$$

Since $\eta\kappa = \eta\kappa$ and $\eta > \eta\kappa$, the above condition implies

$$\begin{aligned}\bar{s}(\eta\kappa) &= \eta\kappa \vee \eta \vee \left[\frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\eta\kappa))\right]\right. \\ &= \frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\eta\kappa)) < \eta\kappa.\right.\end{aligned}$$

So, $\tau_0^* \neq \eta\kappa$. Moreover, for all $\tau_0 < \eta\kappa$, $\bar{s}(\tau_0) \leq \tau_0 < \eta\kappa$. Therefore, if a solution τ_0^* exists, it must be that $\tau_0^* > \eta\kappa$. Furthermore, since $\tau_0^* > \eta\kappa$ and $\eta > \eta\kappa$, if $\bar{s}(\tau_0^*) = \eta\kappa$, it must be that

$$\bar{s}(\tau_0^*) = \frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\tau_0^*)) = \eta\kappa.\right.$$

And here is where the new assumption, $\frac{\lambda}{1-e^{\lambda\eta\kappa}} > \frac{g-r}{\beta}$, comes in. I can rearrange this assumption and replace $\bar{\beta} = \beta(\bar{\tau})$, where $\bar{\tau} > \eta\kappa$ is some arbitrarily large number representing the duration of exogenously bursting bubble.

$$\frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\bar{\tau})) > \eta\kappa.\right.$$

Since $\frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\tau_0^*))\right.$ is also strictly increasing and continuous in τ_0^*, there exists a unique $\tau_0^* \in (\eta\kappa, \bar{\tau})$ such that $\bar{s}(\tau_0^*) = \frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\tau_0^*)) = \right.$ $\eta\kappa$. Existence and uniqueness are established by the continuity (Intermediate Value Theorem) and monotonicity, respectively.

($\Rightarrow$) If the bubble bursts endogenously, there exists a τ_0^* such that $\bar{s}(\tau_0^*) = \eta\kappa$. With same argument as above, Assumption 3.3 implies $\tau_0^* > \eta\kappa$. So,

$$\bar{s}(\tau_0^*) = \frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\tau_0^*)) = \eta\kappa < \tau_0^* \vee \eta\right.$$

Since the bubble bursts endogenously, it burst before it bursts exogenously, i.e. $\tau_0^* < \bar{\tau}$. Thus, by monotonicity, $\frac{1}{\lambda}\left(\log(g-r) - \log(g-r-\lambda\beta(\bar{\tau})\right) > \eta\kappa$ which can be rearranged into $\frac{\lambda}{1-e^{\lambda\eta\kappa}} > \frac{g-r}{\bar{\beta}}$. Notice here I consider endogenous bursting at time $t_0 + \bar{\tau}$ as exogenous bursting. I can also choose an arbitrarily large $\bar{\tau}$ so that the bubble would never burst endogenously at $\bar{\tau}$.

(*Exogenous Bursting*) I proved above that the bubble bursts endogenously if and only if $\frac{\lambda}{1-e^{\lambda\eta\kappa}} > \frac{g-r}{\bar{\beta}}$. Therefore, by contrapositives, the bubble does not burst endogenously if and only if $\frac{\lambda}{1-e^{\lambda\eta\kappa}} \leq \frac{g-r}{\bar{\beta}}$. By definition, if the bubble does not bursts endogenously, i.e. $\{\tau_0 | \bar{s}(\tau_0) \geq \eta\kappa\} = \emptyset$, it bursts exogenously, i.e. $T^* - t_0 = \inf\{\emptyset\} = \bar{\tau}$. Moreover if the bubble bursts exogenously at $\bar{\tau}$, then $\{\tau_0 | \bar{s}(\tau_0) \geq \eta\kappa\} = \emptyset$. Thus, exogenous bursting also implies no endogenous bursting. Therefore, the bubble bursts exogenously if and only if the bubble does not burst endogenously if and only if $\frac{\lambda}{1-e^{\lambda\eta\kappa}} \leq \frac{g-r}{\bar{\beta}}$. $\square$

C.3.3. Proof of Proposition 3.2

Proof. As before, it is immediate that the small arbitrageur only reenters if $A'(\tau_i') > 0$ for some $\tau_i' > \tau_i^*$. I can simplify the condition as the following

$$A'(\tau_i) = e^{(g-r)\tau_i}\left[(g-r)(1-\Pi(\tau_i)) - \beta(\bar{T})\pi(\tau_i)\right] > 0$$

$$\Leftrightarrow \frac{\pi(\tau_i|\eta\kappa)}{1-\Pi(\tau_i|\eta\kappa)} < \frac{g-r}{\beta(\bar{T})} \Leftrightarrow \frac{\lambda}{1-e^{\lambda(\tau_i-\bar{T})}} < \frac{g-r}{\beta(\bar{T})}$$

However since I know that $\frac{\lambda}{1-e^{\lambda(\tau_i^*-\bar{T})}} = \frac{g-r}{\beta(\bar{T})}$ and LHS increases with τ_i', I have

$$\frac{\lambda}{1-e^{\lambda(\tau_i-\bar{T})}} < \frac{\lambda}{1-e^{\lambda(\tau_i^*-\bar{T})}} = \frac{g-r}{\beta(\bar{T})}$$

for all $\tau_i' > \tau_i^*$. Thus, the small arbitrageur would never reenter, even when she updates her beliefs on the information that her sale didn't burst the bubble. (Note the reentering condition here is identical to the one in Proposition C.1. It's the result of a special property of exponential distribution.) $\square$

C.3.4. Proof of Proposition 3.3

Proof. The first order condition to (C.1) is as the following

$$F(\gamma^*) \equiv (g-r)e^{(g-r)\gamma^*}(\eta\kappa - \bar{s}(\gamma^*)) - (e^{(g-r)\gamma^*} - 1)\bar{s}'(\gamma^*) = 0$$

Since $\bar{s}(0) = 0$ and $\bar{s}'(0) = 1$, I have $F(0) = (g-r)\eta\kappa > 0$.

(Endogenous) I first start with the case of endogenous bursting by the small arbitrageurs. In endogenous bursting, $\bar{s}(\bar{T}) = \eta\kappa$. Moreover, by Corollary C.6, I have $\bar{s}'(\tau_0) = B'(\tau_0) > 0$ for all τ_0. So, $F(\bar{T}) = -(e^{(g-r)\bar{T}} - 1)\bar{s}'(\bar{T}) < 0$. Continuity of $F(\gamma)$ establishes existence of $0 < \gamma^* < \bar{T}$. The uniqueness is established as what follows. Whenever $F(\gamma) \leq 0$,

$$\begin{aligned}
F'(\gamma) &= (g-r)\left[(g-r)e^{(g-r)\gamma}(\eta\kappa - \bar{s}(\gamma)) - e^{(g-r)\gamma}\bar{s}'(\gamma)\right] - (g-r)e^{(g-r)\gamma}\bar{s}'(\gamma) - (e^{(g-r)\gamma} - 1)\bar{s}''(\gamma) \\
&= (g-r)\left[F(\gamma) - \bar{s}'(\gamma)\right] - (g-r)e^{(g-r)\gamma}\bar{s}'(\gamma) - (e^{(g-r)\gamma} - 1)\bar{s}''(\gamma) \\
&\leq -(g-r)\bar{s}'(\gamma) - (g-r)e^{(g-r)\gamma}\bar{s}'(\gamma) - (e^{(g-r)\gamma} - 1)\bar{s}''(\gamma) \\
&= -(g-r)(e^{(g-r)\gamma} + 1)\bar{s}'(\gamma) - (e^{(g-r)\gamma} - 1)\bar{s}''(\gamma) \\
&= \frac{-(g-r)^2(e^{(g-r)\gamma} + 1)}{\lambda + (g-r-\lambda)e^{(g-r)\tau_0}} - \frac{-(g-r)^2(g-r-\lambda)e^{(g-r)\tau_0}(e^{(g-r)\gamma} - 1)}{(\lambda + (g-r-\lambda)e^{(g-r)\tau_0})^2} \\
&= \frac{-(g-r)^2(e^{(g-r)\gamma} + 1)\lambda - (g-r)^2(g-r-\lambda)(e^{(g-r)\gamma} + 1)e^{(g-r)\tau_0}}{(\lambda + (g-r-\lambda)e^{(g-r)\tau_0})^2}
\end{aligned}$$

$$+ \frac{(g-r)^2(g-r-\lambda)(e^{(g-r)\gamma}-1)e^{(g-r)\tau_0}}{(\lambda+(g-r-\lambda)e^{(g-r)\tau_0})^2}$$
$$= \frac{-(g-r)^2(e^{(g-r)\gamma}+1)\lambda - 2(g-r)^2(g-r-\lambda)e^{(g-r)\tau_0}}{(\lambda+(g-r-\lambda)e^{(g-r)\tau_0})^2} < 0$$

Thus, whenever the first order condition holds, it is maximum. Moreover, the above implies that $F(\gamma) < 0$ for all $\gamma > \gamma^*$, and thus establishing uniqueness and no-reentry.

(Exogenous) I now move on to the exogenous bursting case. Since the bubble bursts exogenously, $\bar{s}(\gamma) < \eta\kappa$ for all $\gamma \in [0, \infty)$. Since, $\bar{s}'(\gamma) > 0$ for all $\gamma > 0$, by monotone convergence theorem, $\lim_{\gamma\to\infty} \bar{s}(\gamma) < \infty$ exists. So, $\eta\kappa > \lim_{\gamma\to\infty} \bar{s}(\gamma)$. Similarly, since $\bar{s}'(\gamma) > 0$ and $\bar{s}''(\gamma) < 0$ for all $\gamma > 0$, $\lim_{\gamma\to\infty} \bar{s}'(\gamma) < \infty$ exists. Moreover, since $\lim_{\gamma\to\infty} \bar{s}(\gamma) < \infty$ exists, a simple proof by contradiction shows that $\lim_{\gamma\to\infty} \bar{s}'(\gamma) = 0$. Therefore,

$$\begin{aligned}
\lim_{\gamma\to\infty} F(\gamma) &= \lim_{\gamma\to\infty} (g-r)e^{(g-r)\gamma}(\eta\kappa - \bar{s}(\gamma)) - (e^{(g-r)\bar{T}} - 1)\bar{s}'(\gamma) \\
&= \lim_{\gamma\to\infty} (g-r)e^{(g-r)\gamma}(\eta\kappa - \bar{s}(\gamma)) \\
&\geq \lim_{\gamma\to\infty} (g-r)e^{(g-r)\gamma}(\eta\kappa - \lim_{\gamma\to\infty} \bar{s}(\gamma)) = \infty
\end{aligned}$$

The inequality on the third line follows from $\bar{s}'(\gamma) > 0$ and the equality on the third line follows from $\eta\kappa - \lim_{\gamma\to\infty} \bar{s}(\gamma)$ being a constant. In addition, I have shown above (in the endogenous case) that whenever, $F(\gamma) \leq 0$, $F(\gamma') < 0$ for all $\gamma' > \gamma$. Therefore, if $F(\gamma) = 0$ for some γ, it's impossible to have $\lim_{\gamma\to\infty} F(\gamma) = \infty$. So, by contradiction, $F(\gamma) > 0$ for all γ. Therefore, I must have a boundary solution and must be the upper bound. By Lemma C.2 and a version of Lemma 3.1 for the large arbitrageur, $0 \leq \gamma^* \leq \bar{T}$

Hence, the boundary solution is exactly $\gamma^* = \bar{T} = \bar{\tau}$. Now I prove the converse. Suppose the solution is exactly $\gamma^* = \bar{T}$, the bubble must burst exogenously, since $\gamma^* < \bar{T}$ in the endogenous case. Therefore, $\bar{T} = \bar{\tau}$. □

C.3.5. Proof of Proposition 3.4

Proof. Plugging $P(t)$ and β into the objective function G, I get

$$\begin{aligned}
G(\gamma) &= \int_{\gamma}^{\bar{T}} e^{(g-r)\gamma}(1 - e^{-(g-r)(\gamma-\gamma'+\bar{T})})[\eta\kappa - \bar{s}(\gamma - \gamma' + \bar{T})]\phi(\gamma' - \bar{T}|\bar{T})d\gamma' \\
&= \int_{0}^{\bar{T}-\gamma} e^{(g-r)\gamma}(1 - e^{-(g-r)(-\gamma'+\bar{T})})[\eta\kappa - \bar{s}(-\gamma' + \bar{T})]\phi(\gamma + \gamma' - \bar{T}|\bar{T})d\gamma' \\
&= \int_{0}^{\bar{T}-\gamma} e^{(g-r)\gamma}(1 - e^{-(g-r)(-\gamma'+\bar{T})})[\eta\kappa - \bar{s}(-\gamma' + \bar{T})]\phi(\gamma' - \bar{T}|\bar{T})e^{-\lambda\gamma}d\gamma' \\
&= e^{(g-r-\lambda)\gamma} \int_{0}^{\bar{T}-\gamma} (1 - e^{-(g-r)(-\gamma'+\bar{T})})[\eta\kappa - \bar{s}(-\gamma' + \bar{T})]\phi(\gamma' - \bar{T}|\bar{T})d\gamma'
\end{aligned}$$

To get the first order condition, I differentiate the objective function using the Leibniz Rule.

$$G'(\gamma) = (g - r - \lambda)G(\gamma) - e^{(g-r-\lambda)\gamma}(1 - e^{-(g-r)\gamma})[\eta\kappa - \bar{s}(\gamma)]\phi(-\gamma|\bar{T})$$

Recall $\phi(-\gamma|\bar{T}) = 0$ for all $\gamma > \bar{T}$. Thus, $G(\gamma) = G'(\gamma) = 0$ for all $\gamma > \bar{T}$. However, since $\bar{s}(0) < \eta\kappa$, $G(0) > 0$. So $\gamma = 0$ strictly dominates all $\gamma > \bar{T}$. Thus, I only consider $\gamma \leq \bar{T}$. Moreover, a version of Lemma 3.1 for the large arbitrageur establishes that $\gamma \geq 0$. Therefore, I only need to restrict ourselves to $\gamma \in [0, \bar{T}]$, over which interval ϕ is continuous. Thus, I can simplify/expand G' as what follows. (G' is also continuous.)

$$G'(\gamma) = (g-r-\lambda)G(\gamma) - \frac{\lambda}{e^{\lambda\bar{T}}-1}(e^{(g-r)\gamma}-1)[\eta\kappa - \bar{s}(\gamma)]$$

Since $G(0) > 0$, $G'(0) = (g-r-\lambda)G(0) > 0$ (by Corollary C.5). Now I will prove the endogenous case.

(Endogenous) In the endogenous case, I have $\bar{s}(\bar{T}) = \eta\kappa$. In addition, $G(\bar{T}) = 0$. So, $G'(\bar{T}) = (g-r-\lambda)G(\bar{T}) = 0$. To know whether $\bar{T}$ is the maximum that I desired, I must examine the second order condition, i.e.

$$G''(\gamma) = (g-r-\lambda)G'(\gamma) - \frac{\lambda}{e^{\lambda\bar{T}}-1}\left[(g-r)e^{(g-r)\gamma}[\eta\kappa - \bar{s}(\gamma)] - (e^{(g-r)\gamma}-1)\bar{s}'(\gamma)\right]$$

At $\bar{T}$, $G''(\bar{T}) = \frac{\lambda}{e^{\lambda\bar{T}}-1}(e^{(g-r)\bar{T}}-1)\bar{s}'(\bar{T}) > 0$. Thus, $\bar{T}$ is not the argmax but the argmin. However, since $G''(\bar{T}) > 0$ and G'' is continuous (since it's a sum of continuous functions) for $\gamma \in [0, \bar{T}]$, there exists a $\delta > 0$ such that for all $\gamma \in (\bar{T}-\delta, \bar{T})$ such that $|G''(\gamma) - G''(\bar{T})| < \epsilon \equiv \frac{G''(\bar{T})}{2}$. Thus, $G''(\gamma) > 0$ for all $\gamma \in (\bar{T}-\delta, \bar{T})$. Thus, by The First Fundamental Theorem of Calculus,

$$G'(\bar{T}-\delta) = G'(\bar{T}) - \int_{\bar{T}-\delta}^{\bar{T}} G''(\gamma)d\gamma = 0 - \int_{\bar{T}-\delta}^{\bar{T}} G''(\gamma)d\gamma < 0$$

Then, existence of $0 < \gamma^* < \bar{T}-\delta < \bar{T}$ is established by the continuity of G'. The uniqueness can be proved by contradiction. Without loss of generality let $\gamma' < \gamma''$. Suppose $G'(\gamma') = G'(\gamma'') = 0$, $G''(\gamma') < 0$, and $G''(\gamma'') < 0$. (i.e. there are two maxima). However, if $G''(\gamma''') < 0$ for all $\gamma''' \in [\gamma', \gamma'']$, then $G'(\gamma'') < 0$ (by The First Fundamental Theorem of Calculus). Therefore, there must exist some $\gamma''' \in [\gamma', \gamma'']$ such that $G''(\gamma''') \geq 0$.

Moreover, similar to the existence proof, there must exist some $\gamma''' \in [\gamma', \gamma'']$ such that $G''(\gamma''') \geq 0$ and $G'(\gamma''') = 0$, i.e. $G'(\gamma''')$ cross 0 from below.

Recall $F(\gamma) = (g-r)e^{(g-r)\gamma}[\eta\kappa - \bar{s}(\gamma)] - (e^{(g-r)\gamma} - 1)\bar{s}'(\gamma)$. So, $G''(\gamma) = (g - r - \lambda)G'(\gamma) - \frac{\lambda}{e^{\lambda\bar{T}}-1}F(\gamma)$. Then,

$$0 \leq G''(\gamma''') = (g-r-\lambda)G'(\gamma''') - \frac{\lambda}{e^{\lambda\bar{T}}-1}F(\gamma''') = -\frac{\lambda}{e^{\lambda\bar{T}}-1}F(\gamma''') \Leftrightarrow F(\gamma''') \leq 0$$

And recall $F'(\gamma) < 0$ whenever $F(\gamma) \leq 0$. Thus, $F(\gamma) < 0$ for all $\gamma > \gamma'''$. Again, by The First Fundamental Theorem of Calculus,

$$\begin{aligned} G'(\gamma'') &= G'(\gamma''') + \int_{\gamma'''}^{\gamma''} G''(x)dx = \int_{\gamma'''}^{\gamma''} (g-r-\lambda)G'(x) - \frac{\lambda}{e^{\lambda\bar{T}}-1}F(x)dx \\ &= (g-r-\lambda)(G(\gamma'') - G(\gamma''')) - \frac{\lambda}{e^{\lambda\bar{T}}-1}\int_{\gamma'''}^{\gamma''} F(x)dx \\ &> -\frac{\lambda}{e^{\lambda\bar{T}}-1}\int_{\gamma'''}^{\gamma''} F(x)dx > 0 \end{aligned}$$

which contradicts the assumption that $G'(\gamma'') = 0$ and thus establishing uniqueness and no reentry. The equality on the second line follows from another application of The First Fundamental Theorem of Calculus. The first inequality on the second line follows from γ'' being local maximum and γ''' being local minimum.

(Exogenous) In the exogenous case, $\bar{s}(\bar{T}) < \eta\kappa$. So, $G'(\bar{T}) = -\frac{\lambda}{e^{\lambda\bar{T}}-1}(e^{(g-r)\bar{T}} - 1)[\eta\kappa - \bar{s}(\bar{T})] < 0$. Since G' is continuous, existence of $0 < \gamma^{**} < \bar{T}$ is established. Uniqueness is established by the same proof as in the endogenous case. $\square$

C.3.6. Proof of Proposition 3.5

Proof. Recall that G' is the first order condition in Proposition 3.4. First note that $(e^{\lambda\nu}-1)\bar{H} = (e^{\lambda\bar{T}}-1)G$. Thus, $(e^{\lambda\nu}-1)\bar{H}' = (e^{\lambda\bar{T}}-1)G'$. Then, $\bar{H}'(\gamma^{**}) = \frac{e^{\lambda\bar{T}}}{e^{\lambda\nu}-1}G'(\gamma^{**}) = 0$. And since $H' = \bar{H}'$ for $\nu \geq \bar{T}-\gamma$, $H'(\gamma^{**}) = 0$ if $\nu \geq \bar{T}-\gamma^{**}$. Moreover, $H''(\gamma^{**}) = \bar{H}''(\gamma^{**}) = \frac{e^{\lambda\bar{T}}}{e^{\lambda\nu}-1}G''(\gamma^{**}) < 0$. Thus, existence is established for the case $\nu \geq \bar{T}-\gamma^{**}$. Moreover, by Lemma C.4 and uniqueness of solution to G' (and no reentry), if $\nu \geq \bar{T}-\gamma$, then $\underline{H}'(\bar{T}-\nu) \geq \bar{H}'(\bar{T}-\nu) = \frac{e^{\lambda\bar{T}}}{e^{\lambda\nu}-1}G'(\bar{T}-\nu) > 0$. By Lemma C.5, $H(\gamma) = \underline{H}'(\gamma) > 0$ for all $\gamma < \bar{T}-\nu$. And by properties of G', $H'(\gamma) = \bar{H}'(\gamma) = \frac{e^{\lambda\bar{T}}}{e^{\lambda\nu}-1}G'(\gamma) > 0$ for all $\gamma \in [\bar{T}-\nu, \gamma^{**})$ and $H'(\gamma) = \bar{H}'(\gamma) = \frac{e^{\lambda\bar{T}}}{e^{\lambda\nu}-1}G'(\gamma) < 0$ for all $\gamma > \gamma^{**}$. Thus, $\gamma^{***} = \gamma^{**}$ is the unique solution when $\nu \geq \bar{T}-\gamma^{**}$.

[$\nu < \bar{T}-\gamma^{**}$ case] Now I consider the case when $\nu < \bar{T}-\gamma^{**}$. Then, by properties of G', I know $H'(\gamma) = \bar{H}'(\gamma) = \frac{e^{\lambda\bar{T}}}{e^{\lambda\nu}-1}G'(\gamma) < 0$ for all $\gamma \geq \bar{T}-\nu (> \gamma^{**})$. By Lemma C.4, $\underline{H}'(\bar{T}-\nu) \geq \bar{H}'(\bar{T}-\nu)$. Since $\bar{H}'(\bar{T}-\nu) < 0$, there are two possible cases. First (for both endogenous and exogenous $\bar{T}$), $\underline{H}'(\bar{T}-\nu) \geq 0$. Second (only for exogenous $\bar{T}$, $\underline{H}'(\bar{T}-\nu) < 0$. In the first case, by Lemma C.5, $\underline{H}'(\gamma) > 0$ for all $\gamma < \bar{T}-\nu$. Thus, the unique solution is $\gamma^{***} = \inf\{\gamma | H'(\gamma) \leq 0\} = \bar{T}-\nu > \gamma^{**}$. In the second case, I have $\underline{H}'(\gamma^{**}) > 0$ by Lemma C.6. So, existence of $\gamma^{***} \in (\gamma^{**}, \bar{T}-\nu)$ is established by continuity of $\underline{H}'$ and uniqueness is established by Lemma C.5. $\square$