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Zero Sound in Liquid ^{3}He near the Superfluid Transition

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ABSTRACT

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Fluctuations are ubiquitous in physics. For a system close to a phase transition, the order parameter fluctuations are strong. In this thesis we study the pair fluctuations in liquid ^{3}He when it is close to the superfluid phase transition. Liquid ^{3}He is a neutral Fermi liquid, which allows zero sound to propagate at low temperatures. We focus on the zero sound properties in the presence of pair fluctuations. The existing theory is based on a heuristic collision integral. Although the zero sound attenuation generated by this collision integral is consistent with experiment, it lacks a microscopic foundation. The motivation of this work is to derive the collision integral from a more microscopic point of view. We investigate two different approaches. The first approach is based on the conventional kinetic equation, and the second one is novel but closer to the fluctuation theory for superconductors. We obtain a collision integral which is qualitatively different from the heuristic one. Furthermore, results beyond the collision integral are obtained. Our results are compared with the existing experiments, and we obtain close agreement on the attenuation.

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Preface

Almost all substances on Earth become solid when the temperature is low enough. The only exception is helium, which remains liquid even at absolute zero.[1] Although this property seems exotic, the research of liquid helium is actually connected to various branches of physics. Helium has two isotopes, ^{4}He and ^{3}He. Although they are chemically similar, ^{4}He is a boson and ^{3}He is a fermion. At low temperatures, quantum statistics becomes important and the two isotopes behave very differently. At low temperatures, ^{4}He atoms can occupy the same quantum state. The coherence of these condensed atoms makes the liquid ^{4}He a superfluid, which can flow without viscosity.[2] This exotic behavior is a manifestation of the quantum nature of the liquid, and thus superfluidity is considered to be a macroscopic quantum phenomenon and has been studied for many years since its discovery.

The other isotope, ^{3}He, is a fermion and follows the exclusion principle, like the electron. At low temperatures, electrons in some solids can flow without resistivity, which is similar to superfluidity and called superconductivity. Since fermions cannot occupy the same quantum state, superconductivity is generated through a different but related mechanism. If there is attraction between fermions, two fermions can bind into a Cooper pair. Cooper pairs are bosons, and thus they can form a condensate. ^{3}He

[1] This statement is for normal pressures. Helium can become solid at high pressures.

[2] More precisely, the liquid contains the normal part and the superfluid part. It is the superfluid part which flows without viscosity.

atoms are fermions and can also become a superfluid through Cooper pairing. Superfluid ^{3}He thus provides an analogue of superconductivity, in a very different system. Indeed, the ^{3}He atom is neutral instead of charged, and the strongest pairing interaction is in the p-wave channel, which makes the order parameter much richer than that in the conventional superconductors. Therefore, superfluid ^{3}He has both similarities and differences compared to superconductors. Moreover, the normal phase of liquid ^{3}He also shares similarities with electrons in metals, as electrons can move in a metal like a liquid. These systems are called Fermi liquids. In summary, liquid ^{3}He is a unique system, which provides insights and connections to other condensed matter systems, but also has its own special properties.

In this thesis we focus on zero sound near the transition temperature. Zero sound is a collective mode in liquid ^{3}He, which has been studied for a long time (see Chapter 2). When the temperature is above but near the superfluid transition temperature of liquid ^{3}He, pair fluctuations become strong and should have an influence on zero sound (see Chapter 1). We consider two different approaches to this problem. The first approach is based on the conventional kinetic equation. To include the influence of pair fluctuations, we use the Keldysh method to derive the kinetic equation, where the leading-order contribution to the self-energy gives rise to the standard Boltzmann-Landau equation. The pairing interaction gives a subleading contribution, which becomes singular near the Cooper instability. We then apply the kinetic equation with corrections from pair fluctuations to the zero sound problem. This approach can easily be generalized to other quasiparticle transport problems.

In our second approach, we consider a novel way to describe zero sound. The theory is formulated in terms of an action with fermion fields representing the ^{3}He quasiparticles. Using the Hubbard-Stratonovich transformations, we introduce two boson fields for zero sound and the pair fluctuations. With fermion degrees of freedom integrated out, we obtain the collisionless kinetic equation describing zero sound and the second-order Ginzburg-Landau equation describing pair fluctuations from the leading-order expansion. We then consider the higher-order terms in the expansion, which give the interactions between zero sound and pair fluctuations. We obtain diagrams for the self-energy for zero sound phonons that are similar to the diagrams in the theory of paraconductivity. The corrections, however, are too large and have strong frequency dependence. We have not understood the cause of the discrepancy between this result and experiments, but given this approach considers the collective mode and the pair fluctuations on an equal footing and suggests a connection to the paraconductivity theory, we think it might be useful for the future. As we shall see, the pair fluctuations in liquid ^{3}He have not been studied well yet. We hope this work will shed some light on this area.

Table of Contents

List of Figures

CHAPTER 1

Introduction

In our world, there are many different materials and it seems impossible to find any general principle for these materials from a microscopic point of view. The development of condensed matter physics in the last century is thus astonishing. In particular, although there is still no general principle or theory which can be applied to all materials, we have a good theory for metals. The simplest model for metals is the free electron model, which considers a metal to be a box containing independent electrons. Since the electrons are independent, this model is effectively a one-body problem, and can be solved easily. The many-body nature of this model appears only in the Pauli exclusion principle, which implies a Fermi sea as the ground state. Surprisingly, this simple model works well for many metals [1], considering that the Coulomb interaction between electrons is strong.

The effectiveness of the free electron model was explained by Landau's Fermi liquid theory [2]. According to this theory, the strong interaction does not totally destroy key features of the free electron model. For a Fermi liquid, the Fermi surface exists and the properties of the original fermions are changed by the interaction. Around the Fermi surface, Landau argued that the fermions have long lifetime, compared to their energy, and called them *quasiparticles*. Therefore, the low-energy behavior of the Fermi liquid can be described by the "gas" of the quasiparticles around the Fermi surface. By "gas" we mean the lifetime of the quasiparticles is long, similar to

free fermions. In a Fermi liquid, the quasiparticles still interact with each other, and the interaction can be strong and lead to bosonic modes. This picture explains why the free electron model works well for many metals. Landau's Fermi liquid theory and the concept of quasiparticles have become the cornerstone of condensed matter physics. Among others, the most successful case is plausibly liquid ^{3}He.[1] The ^{3}He atom contains two protons, two electrons, and one neutron, and the whole atom is a spin-$\frac{1}{2}$ fermion, due to the unpaired neutron. Like its isotope ^{4}He, ^{3}He becomes a liquid only at low temperatures. The Fermi energy of ^{3}He is about a few Kelvin. For temperatures well below T_F, liquid ^{3}He can be described quantitatively by Fermi liquid theory.

Liquid ^{3}He can be made extremely pure at low temperatures. In contrast, it is difficult to achieve the same level of purity in electronic systems because of the existence of crystal lattices and relative ease for impurity contamination. On the other hand, since liquid ^{3}He is a Fermi system, it shares some similarities with electronic systems. In particular, liquid ^{3}He can become a superfluid by Cooper pairing [3]. The Fermi liquid behavior and the Cooper pairing make liquid ^{3}He a great model system for understanding interacting fermions.

In any system with a symmetry-broken phase, the "boundary" between the disordered and ordered phases has interesting properties because of the order parameter fluctuations. Physical properties show abrupt change at the phase transition temperature. Nonetheless, if we carefully check the neighborhood of the transition temperature, we may see some precursor to the ordered phase. The most well-known

[1] Indeed, this system is what Landau considered in his original paper [2].

example is paraconductivity, in which the resistivity of a material drops below the normal value right before entering the superconducting state. The order parameter fluctuations thus "smear" the phase transition. This might be an obstacle for precise measurement of the transition temperature.[2] Therefore, studying order parameter fluctuations is not just interesting in theory, but it also has practical value. In this thesis we focus on liquid ^{3}He with temperature approaching the superfluid phase transition from above. In particular, we investigate how the fluctuations change the kinetic equation for a Fermi liquid. In the following section, we will give a review of the history on this topic.

1.1. Pair fluctuations in liquid ^{3}He

The superfluidity of liquid ^{3}He was discovered by Osheroff, Richardson, and Lee in 1971 [4]. One of the interesting properties of superfluid ^{3}He is the angular momentum of the Cooper pairs. Before the discovery of the superfluid phase, there were studies about the possible pairing in ^{3}He, including p-wave and d-wave pairings; see Vollhardt and Wölfle [3]. In order to check the angular momentum of the pairing, Emery considered the pair fluctuations due to pairings with different angular momenta, which can give corrections to transport properties like viscosity above the transition temperature [5]. He then argued that by measuring the corrections, we can determine which is the correct pairing. However, the correction due to pair fluctuations is small and can only be observed by checking a narrow range of temperatures near the transition temperature T_c. To the best of the author's knowledge, the only relevant experiment was done by Parpia et al. [6], in which they observed unexpected

[2] I want to thank Man Nguyen, who pointed this out to me.

behavior around T_c. Unfortunately, the experiment suffered from a strange deviation from the Fermi liquid behavior when the temperature was below 4 mK, and thus there was no clear conclusion.

The fluctuations do not only change the viscosity. Indeed, the superfluid phase does not just affect the viscosity, but sound propagation is also influenced [7]. The experiment by Paulson and Wheatley shows that the attenuation of zero sound is enhanced near T_c [8]. As shown in Figure 1.1, the zero sound attenuation increases by a few percent from the normal value when the temperature is within about 10% of T_c. Since the T^2 behavior of the zero sound attenuation is the prediction of Fermi liquid theory, which has been confirmed in liquid ^{3}He experimentally [9], the deviation from the T^2 scaling is an interesting phenomenon. Based on Emery's collision integral, Samalam and Serene proposed a theory to explain the additional attenuation in the vicinity of T_c [10]. Their result is consistent with Paulson and Wheatley's experiment, as shown in Figure 1.2. A similar study by Pal and Bhattacharyya [11] obtained the same result.

Another experiment by Lee et al. [12] focused on the zero sound velocity. The sound frequency used in this experiment is higher than the energy gap at zero temperature ($\omega > 2\Delta_0$), and they observed a velocity drop near T_c (Figure 1.3). The observed velocity drop is about 30 ppm. They pointed out this phenomenon might be related to pair fluctuations, but no conclusion could be made until "more careful and rigorous experimental and theoretical work is performed."

The above discussion summarizes the studies on fluctuation phenomena in liquid ^{3}He up to the point prior to this work. As we can see, there are only a few studies

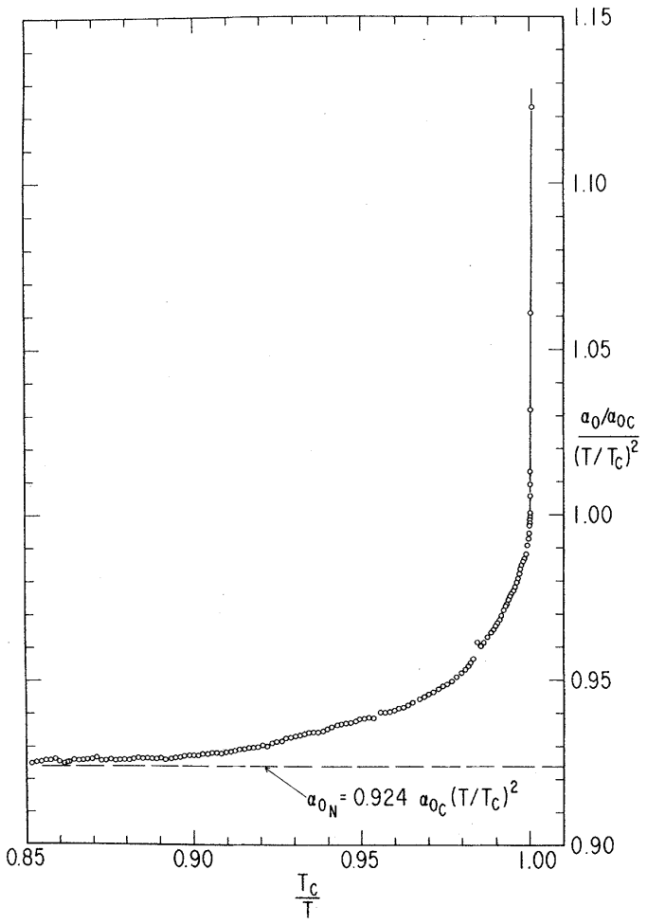


Figure 1.1. This figure is taken from Paulson and Wheatley [8]. The sound frequency used in the experiment was $\omega/2\pi = 25\,\mathrm{MHz}$, and the pressure was 30.87 bar. The zero sound attenuation is normalized by the attenuation at T_c, denoted by α_{0c}. Furthermore, the vertical axis is normalized by the factor T^2, so the Fermi liquid behavior is represented by the horizontal dashed line.

on the pair fluctuations in liquid ^{3}He, and our understanding of them is far from satisfactory. In particular, although Samalam and Serene's result is consistent with the experiment [10], their theory is based on a heuristic collision integral and cannot explain the velocity drop. On the other hand, the velocity and attenuation are given by the real and imaginary part of the dispersion relation, respectively. Considering that

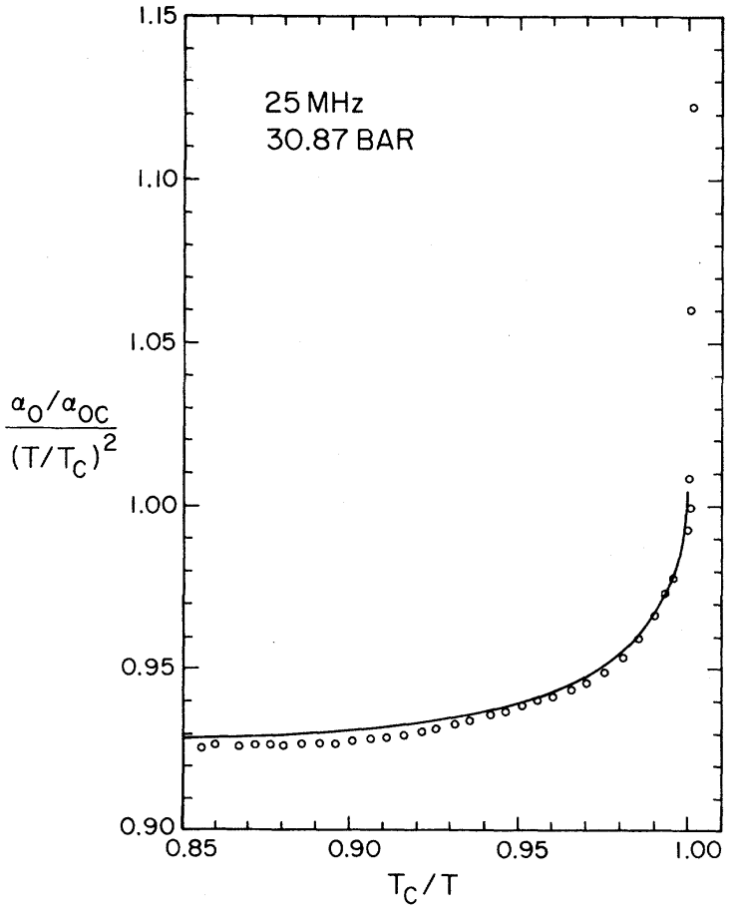


Figure 1.2. This figure is taken from Samalam and Serene [10]. The circles show the zero sound attenuation data from Paulson and Wheatley [8], normalized by T^2 and the attenuation measured at T_c. The solid line is the theoretical result obtained by Samalam and Serene [10].

the pair fluctuations can influence the attenuation, it is almost certain that the velocity is also affected. From this point of view, the theory is not complete, and thus we want to perform a more thorough theoretical study. It should be remarked that in contrast, the fluctuation phenomena in superconductors have been studied extensively [13–20]. As mentioned above, liquid ^{3}He shares similarities and differences with electronic systems, which also motivated us to study the pair fluctuations in liquid ^{3}He.

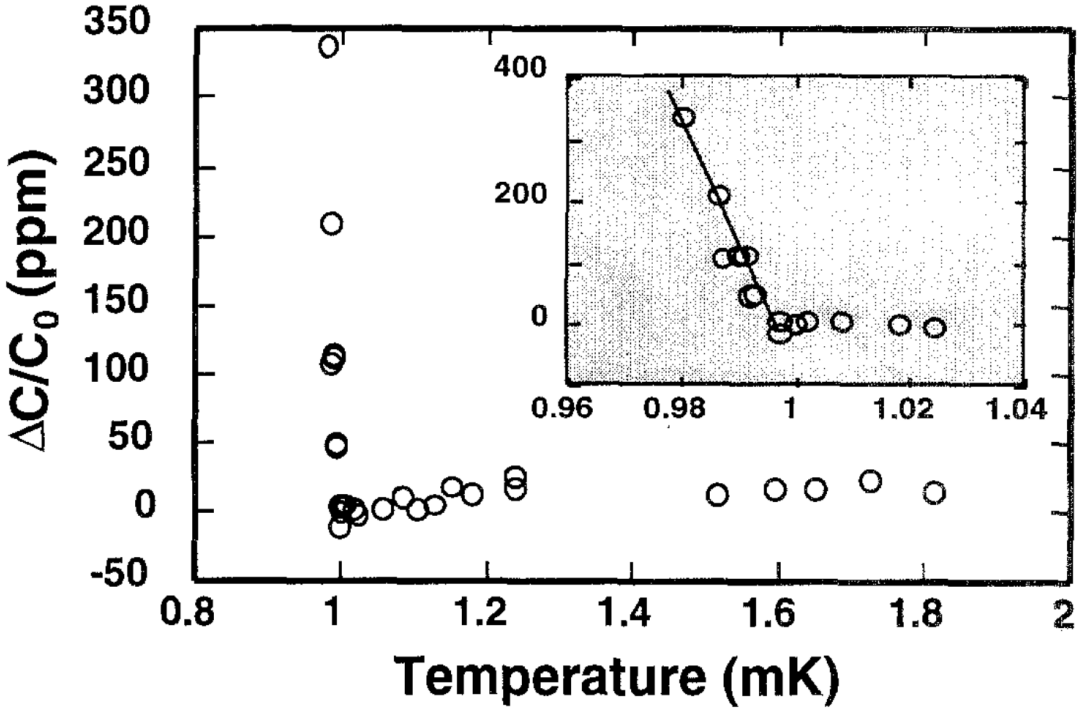


Figure 1.3. This figure is taken from Lee et al. [12]. The pressure used in the experiment was 0.6 bar, and the transition temperature at this pressure is about 1 mK. The sound frequency is 113 MHz. Below 1.2 mK, the velocity has a drop about 30 ppm.

1.2. Thesis outline

In this thesis we focus on zero sound, which is a important feature of neutral Fermi liquids. In the next chapter we briefly review Landau's Fermi liquid theory and the theory of zero sound. In particular, we develop a perturbation theory for zero sound attenuation at the end of this chapter, which will be used later. The current theory for the additional zero sound attenuation observed by Paulson and Wheatley [8] is based on a collision integral with the transition probability given by Emery [5]. In order to understand the collision integral more thoroughly, we derive the kinetic theory from the quantum field theory of interacting fermions, which is described in Chapter 3. In

Chapter 4 we calculate the sum of ladder diagrams in the particle-particle channel using the Keldysh method. The sum gives the effective interaction between two particles with opposite momenta, which diverges at the superfluid phase transition and signifies the Cooper instability. We then calculate the self-energy induced by this effective interaction, which gives an correction to the quasiparticle dispersion relation and the collision integral to the kinetic equation. The collision integral has a different form from the one proposed by Emery [5] using a heuristic approach. In particular, there is no quantum interference in our collision integral, which is more reasonable since no phase coherence exists in the normal state. The zero sound attenuation calculated from our collision integral shows close agreement with the experiment on the temperature dependence. The pressure dependence also follows the tendency observed by Paulson and Wheatley [8]. Moreover, the correction to the quasiparticle energy leads to a velocity drop of zero sound.

We study a totally different theory in Chapter 5, where we introduce two boson fields for the zero sound and the pair fluctuations using the Hubbard-Stratonovich transformation. Integrating out the fermion fields gives the dynamics and interactions for the two boson fields. Thus we can study the influence of pair fluctuations on zero sound using the interactions. We give the conclusion in Chapter 6.

CHAPTER 2

Fermi Liquid Theory

Landau's Fermi liquid theory plays a fundamental role in the theory of interacting fermions. In condensed matter, the interactions between fermions are usually large. For electrons in metals, both the kinetic energy and the Coulomb interaction are of the order of 10 eV (see textbooks on condensed matter physics, for example, [1, 21]). Since the interaction energy is as strong as the kinetic energy, the interaction cannot be considered to be a small perturbation in the free Fermi gas, and thus direct perturbation methods cannot help us understand interacting fermions. For liquid ^{3}He, the interaction between Helium atoms is also not weak, because the interparticle separation is of the atomic scale and there exists strong repulsion between particles. Landau proposed a phenomenological theory to describe interacting fermions [2]. This Fermi liquid theory has been successfully applied to liquid ^{3}He, in which the predictions agree with experiments quantitatively [22, 23].

In this chapter, we review the basic concepts of Fermi liquid theory and the relevant results which will be used later. We begin with the theory of a free Fermi gas, in which the essential concepts like Fermi surface and low-energy excitations are developed. Those essential concepts are used in Fermi liquid theory as well. Landau's phenomenological theory of Fermi liquids is introduced as an extension of the theory of the free Fermi gas, with some important additional ingredients and modifications.

We discuss zero sound as an important result of Landau's theory, which also motivated this work and will be studied further in later chapters.

2.1. Free Fermi gas

In this section we consider the free Fermi gas in a 3D box with volume V. The single-particle states of this system are labeled by momentum $\mathbf{p}$ and spin σ. A nonrelativistic particle in the box has energy $\epsilon_{\mathbf{p}\sigma} = p^2/2m$ in the absence of a magnetic field. To study the Fermi gas, we consider the grand canonical ensemble with temperature T and chemical potential μ. The partition function can be written as

$$\mathcal{Z} = \operatorname{Tr} e^{-\beta(H-\mu N)}\,, \tag{2.1}$$

with

$$H - \mu N = \sum_{\mathbf{p}\sigma} (\epsilon_{\mathbf{p}\sigma} - \mu) \psi^\dagger_{\mathbf{p}\sigma} \psi_{\mathbf{p}\sigma}\,, \tag{2.2}$$

where $\psi^\dagger$ and ψ are the creation and annihilation operators for the fermions.[1] The operator $e^{-\beta(H-\mu N)}$ can be diagonalized using the Fock states $|\{n_{\mathbf{p}\sigma}\}\rangle$, and thus the partition function becomes

$$\begin{aligned} \mathcal{Z} &= \sum_{\{n_{\mathbf{p}\sigma}\}} e^{-\beta \sum_{\mathbf{p}\sigma} (\epsilon_{\mathbf{p}\sigma} - \mu) n_{\mathbf{p}\sigma}} \\ &= \prod_{\text{all } \mathbf{p}\sigma} \sum_{n_{\mathbf{p}\sigma}=0,1} e^{-\beta(\epsilon_{\mathbf{p}\sigma}-\mu) n_{\mathbf{p}\sigma}} = \prod_{\text{all } \mathbf{p}\sigma} \left(1 + e^{-\beta(\epsilon_{\mathbf{p}\sigma}-\mu)}\right). \end{aligned} \tag{2.3}$$

[1] For simplicity, we do not put a hat on a quantum operator, and sometimes the same symbol is used for both the quantum operator and its expectation value. It should be easy to distinguish them from the context.

Note that the Pauli exclusion principle requires that $n_{\mathbf{p}\sigma} = 0, 1$ for each state. The average occupation number in each state can be obtained by taking the variation with respect to the singe-particle energy $\epsilon_{\mathbf{p}\sigma}$,

$$n_{\mathbf{p}\sigma} \equiv -\frac{1}{\beta}\frac{\delta \ln \mathcal{Z}}{\delta \epsilon_{\mathbf{p}\sigma}} = \frac{1}{e^{\beta(\epsilon_{\mathbf{p}\sigma}-\mu)}+1}\,. \tag{2.4}$$

Similarly, taking the derivative with respect to the chemical potential μ gives the average number of total particles

$$N = \sum_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma}\,. \tag{2.5}$$

The average energy can also be expressed in terms of the occupation numbers,

$$E \equiv \langle H \rangle = \sum_{\mathbf{p}\sigma} \epsilon_{\mathbf{p}\sigma} n_{\mathbf{p}\sigma}\,. \tag{2.6}$$

Other thermodynamics quantities can be derived from the grand potential $\Omega = -T \ln \mathcal{Z}$. In particular, we have the entropy

$$S = -\sum_{\mathbf{p}\sigma} \Big[n_{\mathbf{p}\sigma} \ln n_{\mathbf{p}\sigma} + (1 - n_{\mathbf{p}\sigma}) \ln(1 - n_{\mathbf{p}\sigma}) \Big]\,. \tag{2.7}$$

Note that all these quantities can be expressed in terms of the distribution function $n_{\mathbf{p}\sigma}$.

The distinctive feature of fermions is the Pauli exclusion principle, which determines the distribution at low temperatures. Consider the free Fermi gas at $T = 0$. Using the expression (2.4), the occupation number is $n_{\mathbf{p}\sigma} = 0$ for $\epsilon_{\mathbf{p}\sigma} - \mu > 0$, while

$n_{\mathbf{p}\sigma} = 1$ for $\epsilon_{\mathbf{p}\sigma} - \mu < 0$. Thus, the surface determined by $\epsilon_{\mathbf{p}\sigma} = \mu(T = 0)$ in the momentum space separates the states which are fully occupied and the states which are empty. This surface is called **Fermi surface**, and the corresponding energy is called **Fermi energy**, denoted by ϵ_{F}. For isotropic systems like the free Fermi gas, the Fermi surface is a sphere, and the radius is called Fermi momentum, denoted by p_{F}. For the free Fermi gas, we have the relation $\epsilon_{\mathrm{F}} = p_{\mathrm{F}}^2/2m$, and thus $\epsilon_{\mathrm{F}} = \mu(T = 0)$. If the total number of particles N is fixed, then the chemical potential is determined from the relation (2.5), and thus the Fermi energy is related to the particle number density. Indeed, since the occupation number $n_{\mathbf{p}\sigma} = 1$ for $p < p_{\mathrm{F}}$ and vanishes otherwise, we immediately obtain

$$n \equiv \frac{N}{V} = 2 \times \frac{1}{(2\pi)^3} \times \frac{4\pi}{3} p_{\mathrm{F}}^3 = \frac{p_{\mathrm{F}}^3}{3\pi^2}\,, \tag{2.8}$$

where we have approximated the summation by an integral,

$$\sum_{\mathbf{p}} = V \int \frac{d^3p}{(2\pi)^3}\,, \tag{2.9}$$

and the factor of 2 is from the two degenerate spin states. The Fermi energy can then be written as

$$\epsilon_{\mathrm{F}} = \frac{(3\pi^2 n)^{2/3}}{2m}\,. \tag{2.10}$$

Note that pressure, which depends on the number density, can influence the Fermi energy. For metals, the order of magnitude of the Fermi energy is usually about 10^4 K to 10^5 K [1], which is much higher than the temperature for normal applications. For liquid ^{3}He, the Fermi energy is a few Kelvin, but we will focus on temperatures near

the superfluid phase transition, which happens at a few millikelvins. Therefore, we usually can describe the system by its ground state and low-energy excited states.

The ground state of the free Fermi gas is a filled Fermi sphere. Due to the Pauli exclusion principle, excited states can only be created by adding particles to the single-particle states above the Fermi surface, or removing particles from the occupied states below the Fermi surface (another way to describe this process is adding holes below the Fermi surface). At a temperature $T \ll \epsilon_F$, these processes can only happen within a thin shell around the Fermi surface with the width determined by the energy scale T, as shown by the expression (2.4). We remark that although the above argument is made for an equilibrium state with temperature T, it is obvious that the same argument applies to any low-energy state with a characteristic energy $\epsilon \ll \epsilon_F$. Therefore, the low-energy excited states, as well as the physical properties of the gas, are thus *determined by the distribution function with distortions around the Fermi surface*.

As we have seen above, the two main features of a low-energy free Fermi gas are the Fermi surface, associated with an energy scale ϵ_F, and the particles and holes around the Fermi surface, which determines the physical properties of the gas. In the next section, we will see the same features appear in a Fermi liquid.

2.2. Quasiparticles

Fermi liquid theory was proposed by Landau [2] to describe interacting fermions. The original theory was applied to an isotropic liquid with short-range interactions,[2] such as liquid ^{3}He. Landau's original argument is phenomenological, which means

[2]For long-range interactions, such as Coulomb interaction, a modified version of Fermi liquid theory was proposed by Silin [24].

it relies on some assumptions based on general observations, like symmetries and the absence of phase transitions, and no calculation based on microscopic details is involved. The basic idea of Landau's theory is the following assumption: When the interaction is turned on slowly, there is a one-to-one correspondence[3] between the *low-energy states* of the free Fermi gas, and of the interacting fermions. In order to understand this assumption, we consider the simplest state created by adding a particle above the Fermi surface. In the absence of interactions, the particle moves individually. After gradually turning on the interaction, the motion of the particle must influence other particles and induce the motion of a group of particles. We can think of the group of particles as a single entity called **quasiparticle**.[4] According to the one-to-one correspondence, a state near the Fermi surface with one additional particle becomes a state with one additional quasiparticle. If the interaction has translation invariance and spin-rotation invariance, the quasiparticle state has the same momentum $\mathbf{p}$ and spin σ as the noninteracting single-particle state, and thus it can be labeled by the same $\mathbf{p}\sigma$. As a result, the quasiparticles also obey the Pauli exclusion principle. Similarly, the multi-particle low-energy states correspond to multi-quasiparticles states in the presence of the interaction, and they can be described by the distribution function of quasiparticles, $n_{\mathbf{p}\sigma}$. The ground state of the interacting system can be thought of as a sphere in momentum space of occupied states, with the radius given

[3]This idea is termed the principle of **adiabatic continuity** by Anderson [25], who argues that this principle is important for developing new theories.

[4]Similar ideas are used for other collective motions. For example, a phonon involves the motion of many particles on a lattice.

by the same Fermi momentum as the noninteracting gas, as a result of the one-to-one correspondence.[5]

The above argument seems to imply that the interacting state is nothing but a free gas of quasiparticles. This viewpoint is incorrect because *there is interaction between quasiparticles*, resulting from the interaction between particles. We thus need to consider the lifetime of quasiparticles. If the quasiparticles decay fast, it is not reasonable to use them to describe the physical properties of the system. Consider the simplest decay process, in which a quasiparticle collides with another one, and then they are scattered into new states. The lifetime can be estimated by

$$\frac{1}{\tau} \sim \int d^3p_2\, d^3p_{1'}\, W(1,2;1',2') \times \delta(\epsilon_1 + \epsilon_2 - \epsilon_{1'} - \epsilon_{2'}) \times \left[n_1 n_2 (1-n_{1'})(1-n_{2'}) - (1-n_1)(1-n_2) n_{1'} n_{2'}\right], \quad (2.11)$$

where we have used the momentum conservation law, so the momentum $\mathbf{p}_{2'}$ is given by $\mathbf{p}_{2'} \equiv \mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_{1'}$. The function $W(1,2;1',2')$ is the transition probability, which is assumed to be smooth around the Fermi surface, and n_i is the occupation number for the state i. At low temperatures, a quasiparticle near the Fermi surface can only be scattered into a state near the Fermi surface, because of energy conservation and Pauli exclusion principle, which is enforced by the δ function and the occupation numbers in the square brackets in (2.11). These requirements thus restrict the two momentum integrals in (2.11) to a thin shell around the Fermi surface, and we have $1/\tau \sim (\epsilon - \epsilon_{\mathrm{F}})^2/\epsilon_{\mathrm{F}}$ from dimensional analysis. Similar arguments can be made for

[5] If the Fermi momentum was changed, a quasiparticle would have momentum different from the corresponding particle added before the interaction was switched on.

higher-order processes, in which more quasiparticle-quasihole pairs are generated, and the decay rate is proportional to $(\epsilon - \epsilon_F)^n$ with $n > 2$, which can be neglected if the quasiparticle energy ϵ is close to the Fermi energy. Therefore, for a quasiparticle near the Fermi surface, the lifetime is inversely proportional to $(\epsilon - \epsilon_F)^2$ to leading order. Since the quasiparticle energy is $(\epsilon - \epsilon_F)$, the quasiparticle state is a *well-defined excitation*.[6]

The same argument gives the decay rate due to finite temperatures. At temperature $T \ll \epsilon_F$, only the quasiparticles within a thin shell around the Fermi surface, with the width being approximately T, can be excited. Because of the energy conservation law and Pauli exclusion principle, only these thermal quasiparticles can contribute to the scattering processes, and thus the decay rate due to thermal agitation is $1/\tau \sim T^2/\epsilon_F$. In conclusion, we have shown that at low temperatures, the low-energy states described by the quasiparticles have long enough lifetime, compared to their excitation energies. Thus the states composed of quasiparticles are well-defined states and can be used to describe the low-energy properties of the system.

In this section, we discuss the results which can be obtained from the one-to-one correspondence between low-energy states, including the quasiparticles and their lifetime. We call a system of interacting fermions satisfying these results a **Fermi liquid**. In the next section, we describe a Fermi liquid more quantitatively using the quasiparticle distribution function.

[6]This is similar to an underdamped oscillator. If the dissipation rate, $1/\tau$, is much smaller than the frequency of the oscillator, then the oscillation is very close to a perfect oscillator for the time $t \ll \tau$.

2.3. Energy functional and Fermi liquid parameters

We have argued that the low-energy states of a Fermi liquid can be described by the distribution of quasiparticles. To understand the thermodynamic and transport properties of a Fermi liquid, we need the energy of the low-energy states. Since those states are described by the quasiparticle distribution function $n_{\mathbf{p}\sigma}$, the energy is a functional of the distribution. The absolute value of the energy is not important, and thus we can characterize the energy of a Fermi liquid by expanding the energy functional around the ground state distribution $n^0_{\mathbf{p}\sigma} = \Theta(p_{\mathrm{F}} - p)$. The energy functional thus has the form[7]

$$E = E_0 + \sum_1 \epsilon^0_1 \delta n_1 + \frac{1}{2V} \sum_{1,2} f_{1,2} \delta n_1 \delta n_2 \,, \tag{2.12}$$

where E_0 is the ground state energy, and $\delta n = n - n^0$ represents the deviation from the ground state distribution. The linear term represents the energy increase by adding (removing) one quasiparticle to (from) the ground state, where the energy ϵ^0 is the renormalized single-particle energy, including interaction between particles, not simply $p^2/2m$ for the free gas. The coefficient $f_{1,2}$ in the second-order term is called **Landau function**. The second-order term is normalized by the volume V because the energy δE and the number δn are extensive quantities in thermodynamics, and the Landau function $f_{1,2}$, which characterizes the strength of interaction between quasiparticles, should be an intensive quantity. The energy relative to the ground state

[7] Note that we use the shorthand notation $i \equiv (\mathbf{p}_i \sigma_i)$ for the subscripts.

energy is denoted by $\delta E \equiv E - E_0$. The above expression can be written as

$$\delta E = \mu \delta N + \sum_1 (\epsilon_1^0 - \mu)\delta n_1 + \frac{1}{2V}\sum_{1,2} f_{1,2}\delta n_1 \delta n_2 \,, \tag{2.13}$$

where $\delta N \equiv \sum_1 \delta n_1$ is the change of total number of quasiparticles.[8] In the following, we will assume the total number of particles is conserved, and thus we have $\delta N = 0$.

The last term in (2.13) is essential, because its order of magnitude is the same as the first term. This can be seen as follows. Assume the variation $\delta n_{\mathbf{p}}$ is restricted to a thin shell around the Fermi surface of thickness Λ. Thus $(\epsilon_1 - \mu) \sim \Lambda$ and

$$\sum_1 (\epsilon_1^0 - \mu)\delta n_1 \sim \Lambda \times N\frac{\Lambda}{\mu} = N\frac{\Lambda^2}{\mu} \,. \tag{2.14}$$

As for the second term, we have

$$\frac{1}{2V}\sum_{1,2} f_{1,2}\delta n_1 \delta n_2 \sim \frac{F}{N(0)V} N^2 \frac{\Lambda^2}{\mu^2} \,, \tag{2.15}$$

where $N(0)$ is the density of states at Fermi energy and $F \sim N(0)f$ represents the order of magnitude of the dimensionless Landau function. Recall that $N(0) \sim k_{\mathrm{F}}^3/\epsilon_{\mathrm{F}}$, $k_{\mathrm{F}}^3 \sim N/V$ and $\epsilon_{\mathrm{F}} \sim \mu$ at low temperatures. Thus we have

$$\frac{1}{2V}\sum_{1,2} f_{1,2}\delta n_1 \delta n_2 \sim F \times N\frac{\Lambda^2}{\mu}. \tag{2.16}$$

The dimensionless parameter F should be of order N^0, and thus the second term is as important as the first term.[9] On the other hand, for higher-order terms in δn, the

[8] Note that the chemical potential approximately equals the Fermi energy at low temperatures, $\mu \approx \epsilon_{\mathrm{F}}$.
[9] In fact, the parameter F can be much larger than one for liquid ^{3}He, so the second term cannot be neglected in this case.

same argument shows that their magnitude is $N(\Lambda^2/\mu) \times (\Lambda/\mu)^n$ with $n \geq 1$. Since $\Lambda/\mu \ll 1$, the higher-order terms are much smaller than the first two terms and can be neglected in the expansion (2.12).

We now discuss the expansion coefficients ϵ_1^0 and $f_{1,2}$. Since the deviation from equilibrium is restricted to a thin shell around the Fermi surface, i.e., $\delta n_{\mathbf{p}} \approx 0$ as $\mathbf{p}$ is far from the Fermi surface, we only need to consider the angular dependence of the expansion coefficients. For a system with rotational symmetry, like liquid ^{3}He, the coefficient $\epsilon^0_{\mathbf{p}\sigma}$ can be parameterized by

$$\epsilon^0_{\mathbf{p}\sigma} - \epsilon_{\mathrm{F}} = v_{\mathrm{F}}(p - p_{\mathrm{F}}) \tag{2.17}$$

with a single parameter, called **Fermi velocity** v_{F}. Note that in the non-interacting case, the Fermi velocity is given by $v_{\mathrm{F}} = p_{\mathrm{F}}/m$, where m is the particle mass. For a Fermi liquid, we can define the **effective mass** by

$$m^* = \frac{p_{\mathrm{F}}}{v_{\mathrm{F}}}\,. \tag{2.18}$$

The effective mass, as well as the Fermi velocity, contains the influence from the interaction.

The presence of the interaction means that the total energy of the system is not just the sum of the energies of individual particles, $\sum_1(\epsilon_1^0 - \mu)\delta n_1$. The last term in (2.13) also contributes to the energy. We can see this effect by considering

$$\epsilon_1 - \mu \equiv \frac{\delta E}{\delta n_1} = (\epsilon_1^0 - \mu) + \frac{1}{V}\sum_2 f_{1,2}\delta n_2\,, \tag{2.19}$$

which is the energy increase due to an additional quasiparticle. Note that in addition to the energy of the quasiparticle itself (the first term), the interaction with other quasiparticles contributes to the energy. The quasiparticle interaction is given by the Landau function $f_{1,2} \equiv f_{\hat{\mathbf{p}}\sigma,\hat{\mathbf{p}}'\sigma'}$, which can be split into the spin-symmetric and spin-antisymmetric part under the assumption of spin-rotation invariance

$$f = f^{\mathrm{s}} + f^{\mathrm{a}}\sigma\sigma' \,, \tag{2.20}$$

where $\sigma, \sigma' = \pm 1$ represents the spin-up and spin-down components. For a system having rotational symmetry, the Landau function depends only on $\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'$, or the angle between the momenta $\mathbf{p}$ and $\mathbf{p}'$. When the distribution δn is symmetric in σ, the antisymmetric part of the interaction does not contribute. In this case, the interaction term becomes[10]

$$\int \frac{d^3p}{(2\pi)^3} 2 f^{\mathrm{s}}_{\mathbf{p},\mathbf{p}'} \delta n_{\mathbf{p}'} \,, \tag{2.21}$$

where the factor of 2 comes from the two spin species, and the distribution $\delta n_{\mathbf{p}} \equiv \delta n_{\mathbf{p},\sigma}$ for either spin σ. At low temperatures and assuming the external perturbation is weak compared to the Fermi energy, the variation δn is peaked at Fermi surface and we can simplify the integral,

$$\int \frac{d^3p}{(2\pi)^3} 2 f^{\mathrm{s}}_{\mathbf{p},\mathbf{p}'} \delta n_{\mathbf{p}'} = \int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} 2N(0) f^{\mathrm{s}}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') \nu_{\hat{\mathbf{p}}'} \,, \tag{2.22}$$

where we have assumed the variation

$$\delta n_{\mathbf{p}} \approx \delta(\epsilon_{\mathbf{p}}) \nu_{\hat{\mathbf{p}}} \tag{2.23}$$

[10] We have used $\frac{1}{V}\sum_{\mathbf{p}} = \int \frac{d^3p}{(2\pi)^3}$ in the continuum limit.

and $N(0)$ is the single-spin density of states at the Fermi energy. Motivated by the above expression, we define the **Landau parameters** F_l^{s} and F_l^{a} by

$$F^{\mathrm{s}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') \equiv 2N(0)f^{\mathrm{s}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') = \sum_l F_l^{\mathrm{s}} P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\,, \tag{2.24}$$

$$F^{\mathrm{a}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') \equiv 2N(0)f^{\mathrm{a}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}') = \sum_l F_l^{\mathrm{a}} P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\,. \tag{2.25}$$

We can also expand the function

$$\nu_{\hat{\mathbf{p}}'} = \sum_{l'} \nu_{l'} P_{l'}(\hat{\mathbf{p}}'\cdot\hat{\mathbf{k}})\,, \tag{2.26}$$

where $\hat{\mathbf{k}}$ is some given direction, and the interaction energy becomes

$$\int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} 2N(0)f^{\mathrm{s}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\nu_{\hat{\mathbf{p}}'} = \sum_l \frac{F_l^{\mathrm{s}}\nu_l}{2l+1} P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})\,. \tag{2.27}$$

This expression will be used later.

2.4. Basic properties of Fermi liquids

We have introduced the energy functional, which is characterized by Fermi liquid parameters, including the Fermi velocity (or the effective mass) and the Landau parameters. Physical properties of a Fermi liquid are controlled by these parameters. In this section, we discuss some equilibrium properties of a Fermi liquid which will be used later. We will not cover all important equilibrium properties. More details can be found in the references [22, 23].

We first consider the equilibrium distribution at finite temperatures. The entropy of the Fermi liquid, according to the one-to-one correspondence, should have the

expression as the free Fermi gas:

$$S = -\sum_1 [n_1 \ln n_1 + (1 - n_1) \ln(1 - n_1)] \,. \tag{2.28}$$

From the first law of thermodynamics, we have $\delta E = T\delta S + \mu\delta N$. The variance in the total energy is given by the expression (2.19):

$$\delta E = \sum_1 (\epsilon_1 - \mu)\delta n_1 \,. \tag{2.29}$$

Note that the above expression implicitly contains the constraint $\delta N = 0$. With the fixed temperature T and the chemical potential μ, the equilibrium state requires that the free energy is a minimum, and thus we have $0 = \delta\Omega = \delta E - T\delta S$ for any variation in the distribution. Hence we obtain

$$n^{\text{eq}}_{\mathbf{p}\sigma} = \frac{1}{e^{\beta(\epsilon_{\mathbf{p}\sigma} - \mu)} + 1} \,. \tag{2.30}$$

Although the expression is the same as the free Fermi gas, we should remember that the energy $\epsilon_{\mathbf{p}\sigma} = \epsilon_{\mathbf{p}\sigma}[\delta n]$ depends on the distribution itself. The above simple expression is actually a complicated nonlinear equation.

Since the variation of the distribution δn is restricted to a thin shell around the Fermi surface, the calculations in Fermi liquid theory can be simplified by assuming $\delta n \sim \delta(\epsilon)$. The energy integral then gives the density of states at the Fermi energy, $N(0)$, as seen at the end of the last section. The single-spin density of states can be

derived as follows.

$$\begin{aligned} N(0) &= \int \frac{d^3p}{(2\pi)^3}\,\delta(\epsilon-\epsilon^0_{\mathbf{p}})\Big|_{\epsilon=\epsilon_{\mathrm{F}}} \\ &= \int \frac{d^3p}{(2\pi)^3}\,\frac{\delta(p-p(\epsilon))}{|\nabla_{\mathbf{p}}\epsilon^0_{\mathbf{p}}|}\Big|_{\epsilon=\epsilon_{\mathrm{F}}} \\ &= \frac{4\pi p_{\mathrm{F}}^2}{(2\pi)^3}\times\frac{1}{v_{\mathrm{F}}} \\ &= \frac{m^* p_{\mathrm{F}}}{2\pi^2}\,. \end{aligned} \tag{2.31}$$

This expression is again similar to the non-interacting case, with the bare mass replaced with the effective mass.

Since the basic properties of a Fermi liquid are different from the free Fermi gas only by some renormalized parameters, we might think that a Fermi liquid contains no new physics compared to its non-interacting counterpart. This statement is not true, and we will see that the kinetic equation for quasiparticles shows that Fermi liquids support *a unique collective mode, called zero sound, which does not exist in the free Fermi gas*.

2.5. Kinetic equation

The above discussion assumes that the whole liquid is uniform in both space and time. In other words, the distribution function and other physical properties do not have spatial variance. As a macroscopic system, the distribution of the liquid can vary over space and time. Indeed, we can imagine that the liquid is partitioned into many subsystems in space, and the size of each partition is still much larger than the Fermi wavelength. Thus each partition can be described by the Fermi liquid theory

developed before and has its own distribution function. The distribution function then has spatial dependence, with the length scale of variation much larger than the Fermi wavelength. In the same way, the distribution can change over time, so long as the variation is slow compared to the Fermi frequency. In summary, we can consider the distribution to be a function of macroscopic position and time, $n_{\mathbf{p}\sigma}(\mathbf{r}, t)$, when the variation is slow compared to the microscopic scales.

Since the quasiparticle energy depends on the distribution function, we also have[11]

$$\epsilon_{\mathbf{p}\sigma}(\mathbf{r}, t) = \epsilon_{\mathrm{F}} + v_{\mathrm{F}}(p - p_{\mathrm{F}}) + \int_{\mathbf{p}} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} \delta n_{\mathbf{p}'\sigma'}(\mathrm{r}, t)\,. \tag{2.32}$$

Note that the Landau f function is the same as before, which is true for short-range interactions.[12] The quasiparticle velocity is given by $\nabla_{\mathbf{p}}\epsilon$, and in the presence of the spatial dependence of the quasiparticle distribution, the energy gradient gives rise to a force. Therefore, the nonequilibrium distribution follows the kinetic equation[13]

$$\frac{\partial n}{\partial t} + \nabla_{\mathbf{r}} n \cdot \nabla_{\mathbf{p}} \epsilon + \nabla_{\mathbf{p}} n \cdot (-\nabla_{\mathbf{r}} \epsilon) = I[n]\,, \tag{2.33}$$

where $I[n]$ is the collision integral.

The kinetic equation is nonlinear, because the quasiparticle energy depends on the distribution and the collision integral $I[n]$ is also nonlinear. For distributions near equilibrium, we can linearize the kinetic equation around the equilibrium distribution.

[11] For simplicity, we define

$$\int_{\mathbf{p}} \equiv \int \frac{d^3 p}{(2\pi)^3}\,,$$

and the summation convention over spin indices is used.

[12] The long-range case has to be treated carefully. See the paper by Silin [24].

[13] In the following, we will drop the momentum variables and the spin variables when they are not necessary. The dependence on these variables should be seen easily from the context.

Let $n = n^{\mathrm{eq}} + \delta n$, where n^{eq} is the global equilibrium distribution given in (2.30). The linearization gives

$$\frac{\partial \delta n}{\partial t} + \nabla_{\mathbf{r}} \delta n \cdot v_{\mathrm{F}} \hat{\mathbf{p}} + \nabla_{\mathbf{p}} n^{\mathrm{eq}} \cdot (-\nabla_{\mathbf{r}} \delta \epsilon) = I[\delta n] \,. \tag{2.34}$$

We make two remarks here. First, the quasiparticle energy in equilibrium is given by

$$\epsilon^{\mathrm{eq}}_{\mathbf{p}\sigma} = v_{\mathrm{F}}(p - p_{\mathrm{F}}) + \frac{1}{V} \sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} (n^{\mathrm{eq}}_{\mathbf{p}'\sigma'} - n^{0}_{\mathbf{p}'\sigma'}) \,, \tag{2.35}$$

where the equilibrium distribution follows the Fermi-Dirac distribution, $n^{\mathrm{eq}}_{\mathbf{p}\sigma} = n_{\mathrm{F}}(\epsilon^{\mathrm{eq}}_{\mathbf{p}\sigma})$. The Landau f function has weak dependence on the radial component of $\mathbf{p}$ and $\mathbf{p}'$ around the Fermi surface, and thus the derivative of the second term with respect to p can be assumed to vanish. Thus we have $\nabla_{\mathbf{p}} \epsilon^{\mathrm{eq}} \approx \nabla_{\mathbf{p}} \epsilon^{0} = v_{\mathrm{F}} \hat{\mathbf{p}}$. Second, the collision integral vanishes in equilibrium, namely $I[n^{\mathrm{eq}}] = 0$, so it is tempting to expand the collision integral around the global equilibrium. However, the best way to make the expansion is actually *around the local equilibrium*, instead of the global equilibrium. We will discuss this point in more details in the last section.

We can use the kinetic equation to study transport phenomena in Fermi liquids. For example, conservation laws can be derived from the kinetic equation. For more details, see Baym and Pethick [22]. In the next section, we focus on zero sound, which is the central topic of this thesis.

2.6. Zero sound

The linearized kinetic equation (2.34) specifies how the disturbance of the distribution propagates in space. In the seminal paper by Landau from 1957 [26], he showed

that the disturbance propagates as a wave, when the quasiparticle relaxation time is much longer than the period of the oscillation. Landau called this wave **zero sound**, because it is a sound wave which can propagate at absolute zero temperature. The normal sound exists in the long-wavelength and low-frequency limit, which means that the quasiparticle relaxation rate is short compared to the frequency of the sound. In this case the local equilibrium can be established within each cycle of the oscillation. Since normal sound is a wave in the hydrodynamic limit, all of its physical properties can be described by macroscopic parameters in hydrodynamics. For example, the velocity of normal sound is determined by the compressibility of the liquid. In particular, the attenuation of normal sound is proportional to the viscosity, and thus proportional to the mean free path of quasiparticles. We have shown that the lifetime of quasiparticles is inversely proportional to T^2 in Section 2.2 Thus the mean free path of quasiparticles and the normal sound attenuation are proportional to $1/T^2$. Therefore, normal sound is hard to propagate at low temperatures. Another way to understand this is comparing the sound frequency and the quasiparticle relaxation rate. When the sound frequency is higher than the relaxation rate, there is *not enough time to establish local equilibrium*. Hence the nature of the sound wave, if it exists, must be changed.

Consider the limit where the quasiparticle relaxation rate is much smaller than the frequency. To leading order we can neglect the collision integral and have the *collisionless kinetic equation*

$$\frac{\partial \delta n_{\mathbf{p}\sigma}}{\partial t} + \nabla_{\mathbf{r}} \delta n_{\mathbf{p}\sigma} \cdot v_{\mathrm{F}} \hat{\mathbf{p}} + \nabla_{\mathbf{p}} n^{\mathrm{eq}}_{\mathbf{p}\sigma} \cdot (-\nabla_{\mathbf{r}} \delta \epsilon_{\mathbf{p}\sigma}) = 0 \,. \tag{2.36}$$

We can understand the propagation of sound waves in the collisionless regime as follows. If we disturb the liquid locally in this case, the nonequilibrium quasiparticles will give rise to a nonzero $\nabla_{\mathbf{r}}\delta\epsilon_{\mathbf{p}\sigma}$, through the Landau interaction. This term is the force due to the quasiparticles in the neighborhood. The force then drives other quasiparticles out of equilibrium, and the disturbance is able to propagate in the liquid. The Landau interaction term in the energy functional (2.12) plays the role of the potential energy in an elastic wave, which generates the restoring force and makes the wave propagate.

The linearized equation (2.36) can be considered to be the wave equation of zero sound, which has infinitely many components specified by $\mathbf{p}$. In accordance with the assumption that the nonequilibrium disturbance is restricted around the Fermi surface, we consider the ansatz[14]

$$\delta n_{\mathbf{p}\sigma} = -\frac{\partial n^{\mathrm{eq}}_{\mathbf{p}}}{\partial \epsilon^{\mathrm{eq}}_{\mathbf{p}\sigma}} \nu_{\hat{\mathbf{p}}\sigma}\,. \tag{2.37}$$

At low temperatures, the function $-\partial n^{\mathrm{eq}}/\partial\epsilon^{\mathrm{eq}}_{\mathbf{p}\sigma}$ is approximately a delta function peaked at the Fermi energy. Hence the disturbance is nonzero only near the Fermi surface. The function $\nu_{\mathbf{p}}$ represents the variation of the distribution at different positions on the Fermi surface. Schematically, it can be thought of as a *distortion of the Fermi surface*. Using the Fourier component $\delta n(\mathbf{r},t) = \delta n(\mathbf{k},\omega)e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ and the ansatz (2.37), we can reduce the linearized kinetic equation to

$$(\omega - \mathbf{v}_{\mathrm{F}}\cdot\mathbf{k})\nu_{\hat{\mathbf{p}}\sigma} + \mathbf{v}_{\mathrm{F}}\cdot\mathbf{k}\frac{1}{V}\sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}\frac{\partial n^{\mathrm{eq}}}{\partial\epsilon^{\mathrm{eq}}_{\mathbf{p}'\sigma'}}\nu_{\hat{\mathbf{p}}'\sigma'} = 0\,. \tag{2.38}$$

[14]Note the similarity between this ansatz and the expression (2.23).

Note that we have used the notation $\mathbf{v}_{\mathrm{F}} \equiv v_{\mathrm{F}}\hat{\mathbf{p}}$. In the following, we assume that there is no spin dependence in the problem. Thus we have $\delta n_{\mathbf{p}\uparrow} = \delta n_{\mathbf{p}\downarrow} \equiv \delta n_{\mathbf{p}}$ and the above equation can be reduced to the

$$\nu_{\hat{\mathbf{p}}} - \frac{\mathbf{v}_{\mathrm{F}} \cdot \mathbf{k}}{\omega - \mathbf{v}_{\mathrm{F}} \cdot \mathbf{k}} \sum_{l'} \frac{F^{\mathrm{s}}_{l'} \nu_{l'}}{2l'+1} P_{l'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) = 0\,, \tag{2.39}$$

where we have used the calculation at the end of Section 2.3. Define $s \equiv \omega / v_{\mathrm{F}} k$ and

$$\Omega_{ll'}(s) \equiv \int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi} P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) \frac{\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}}{\hat{\mathbf{p}} \cdot \hat{\mathbf{k}} - s} P_{l'}(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}})\,. \tag{2.40}$$

Equation (2.39) can be expanded in terms of spherical harmonics:

$$\frac{\nu_l}{2l+1} + \sum_{l'} \Omega_{ll'}(s) F^{\mathrm{s}}_{l'} \frac{\nu_{l'}}{2l'+1} = 0\,. \tag{2.41}$$

The equation (2.41) gives rise to a set of equations for each mode, which are coupled together through the Landau parameters. Recall that the function $\nu_{\hat{\mathbf{p}}}$ and its component ν_l represent the distortion of the Fermi surface. We assume that the modes with high angular momentum l are negligible, and keep only the modes with $l = 0, 1, 2$. The dispersion relation of the wave can be obtained from the parameter s, which is the eigenvalue given by the above eigenvalue problem (2.41). Even with the above approximation, the eigenvalue problem can be solved only by numerical methods in general. When the first few Landau parameters are large, the solution $s \equiv \omega / v_{\mathrm{F}} k$ is also large, and an asymptotic expression can be obtained in this case. The zero sound velocity is given by $c_0 = v_{\mathrm{F}} s$, and the difference from the normal sound velocity can

be expressed as

$$\frac{c_0^2 - c_1^2}{c_1^2} = \frac{4}{5}\frac{1 + F_2^s/5}{1 + F_0^s} + \mathcal{O}(v_F^2/c_1^2)\,, \tag{2.42}$$

where $c_1^2 = (1 + F_0^s)(1 + F_1^s/3)v_F^2/3$ is the square of the normal sound velocity. For a detailed analysis, the reader can refer to the review articles by Abrikosov and Khalatnikov [27], and Baym and Pethick [22]. For liquid ^{3}He, the first Landau parameter F_0^s is much larger than unity [23] and thus the above expression can be applied.

As shown in the expression (2.42), the velocity difference between zero sound and normal sound is small if F_0^s is large. The more distinctive feature of zero sound is its attenuation. Consider a plain wave propagating along the x direction and thus the spatial dependence is e^{ikx}. To include the attenuation, the wave number k must have an imaginary part, and thus the attenuation coefficient α is defined by the expression

$$k = \frac{\omega}{c} + i\alpha\,. \tag{2.43}$$

As mentioned at the beginning of this section, the attenuation of zero sound scales as T^2, while the attenuation of first sound is proportional to $1/T^2$. Experimentally, this behavior has been shown by Abel, Anderson, and Wheatley [9]. As shown in the upper part of Figure 2.1, the attenuation of zero sound and normal sound are qualitatively different. We can see from the log scales of the plot that the sound attenuation scales as $1/T^2$ in the high-temperature regime, and as T^2 in the low-temperature regime. The lower part of Figure 2.1 gives the zero sound velocity. The velocity difference is consistent with the expression (2.42), where the Landau parameter F_0^s can be measured from the compressibility [23], and the parameter F_2^s can be assumed

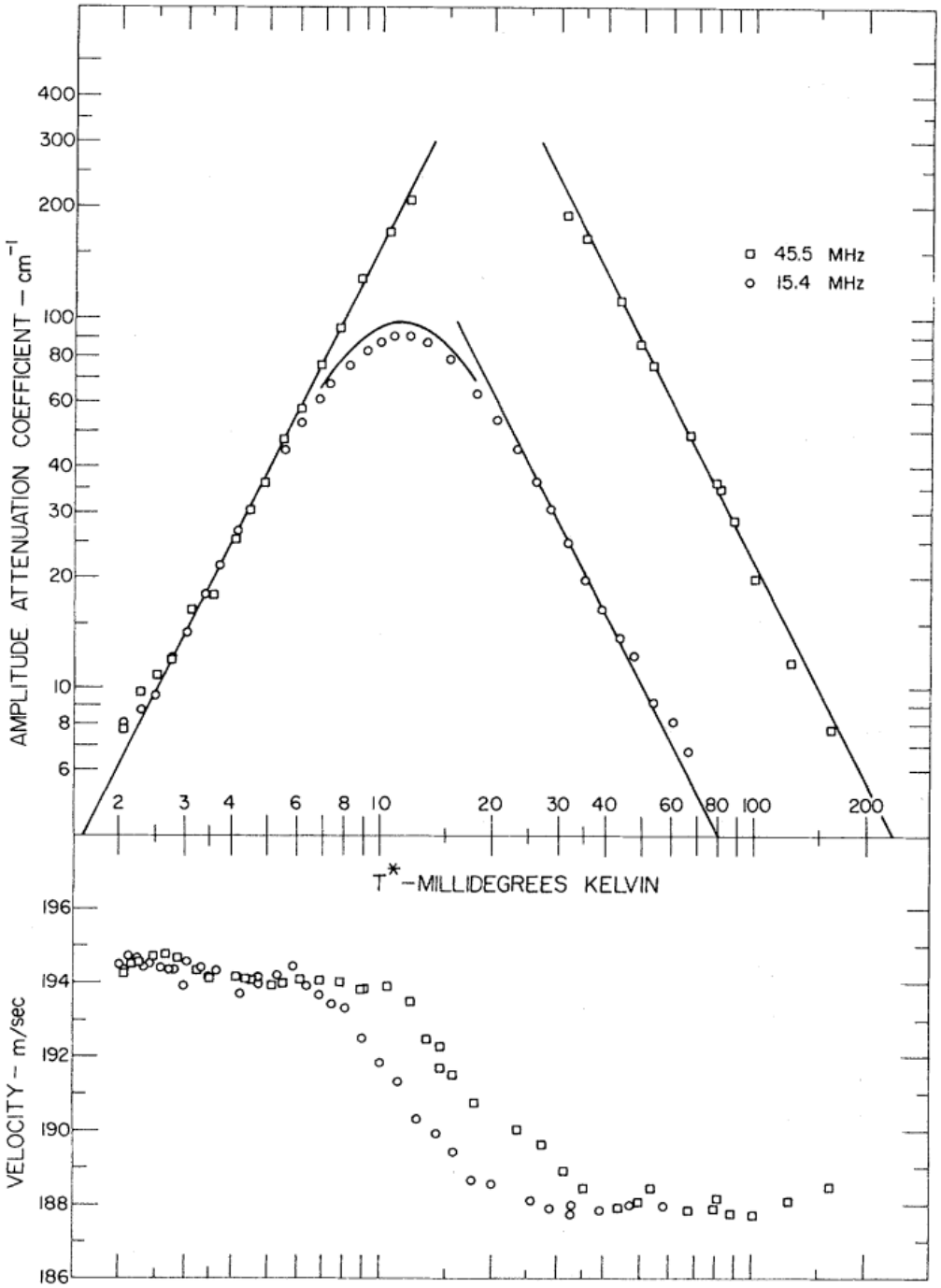


Figure 2.1. This figure is taken from Abel, Anderson, and Wheatley [9]. The experiment was done at 0.32 atm, with two frequencies 15.4 MHz (denoted by open circles) and 45.5 MHz (denoted by open squares). The upper part shows the zero sound attenuation in log scales. The lower part gives the zero sound velocity. The upper and lower part share the horizontal axis, which represents the temperature.

to be small.[15] Moreover, we can see the crossover between normal sound and zero sound happens at the same temperature for both the velocity and the attenuation. The crossover temperature, which is roughly determined by the condition $\omega\tau \sim 1$, depends on the sound frequency.

Zero sound exists in the collisionless regime. In the presence of quasiparticle collisions, the zero sound attenuation behaves as T^2, distinct from the normal sound. In the above discussion, we ignored the collision integral in the kinetic equation, and only discussed the attenuation qualitatively at the beginning of this section. In the next section, we will give a detailed derivation of zero sound attenuation based on the collision integral.

2.7. Collision integral and zero sound attenuation

In this section, we will discuss the collision integral in detail. In particular, we will derive an expression for the zero sound attenuation using perturbation methods [28, 29]. Consider the linearized kinetic equation (2.34). Using the Fourier component $\delta n(\mathbf{r}, t) = \delta n(\mathbf{k}, \omega) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$, the equation can be written as

$$(\omega - \mathbf{v}_\mathrm{F}\cdot\mathbf{k})\delta n + \frac{\partial n^\mathrm{eq}}{\partial \epsilon^\mathrm{eq}} \mathbf{v}_\mathrm{F}\cdot\mathbf{k} \frac{1}{V}\sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}\delta n_{\mathbf{p}'\sigma'} = iI[\delta n]\,. \tag{2.44}$$

Note that the collision integral acquires an imaginary prefactor, which signifies that the dispersion relation given in (2.43) will have nonvanishing attenuation. We remark that although we formally write the linearized collision integral here as a functional

[15] The precise value of F_2^s is still unknown, but most experiments give a value less than unity. See the plot given by Halperin's group in http://spindry.phys.northwestern.edu/3HeCalculator/F2s_plot.html.

of the deviation from global equilibrium δn, and in the following in terms of its component $\nu_{\hat{\mathbf{p}}}$, the most natural expansion variable for the collision integral is the deviation from *local equilibrium*. This point will be discussed in detail in the next section.

Since the collision integral should give rise to a relaxation time which is much longer than the period of zero sound, the collision integral can be considered to be of order $1/\omega\tau$, compared to the terms in the left-hand side. Using the ansatz (2.37) and integrating over the energy $\epsilon^{\mathrm{eq}}_{\mathbf{p}}$, we have

$$(\omega - \mathbf{v}_{\mathrm{F}} \cdot \mathbf{k})\nu_{\hat{\mathbf{p}}} - \mathbf{v}_{\mathrm{F}} \cdot \mathbf{k} \int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} F^{\mathrm{s}}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\nu_{\hat{\mathbf{p}}'} = i \int d\epsilon^{\mathrm{eq}}_{\mathbf{p}}\, I[\nu_{\hat{\mathbf{p}}}]\,. \tag{2.45}$$

Assume that the distribution ν and the solution $s \equiv \omega/v_{\mathrm{F}}k$ can be expanded in the asymptotic series

$$\nu = \nu^{(0)} + \nu^{(1)} + \cdots\,, \tag{2.46}$$

$$s = s_0 + s_1 + \cdots\,, \tag{2.47}$$

where the ratio of the $(i+1)$-th term to the i-th term is of the order $1/\omega\tau$, which is small for zero sound. Plugging the above asymptotic expansion into equation (2.45) and keeping only terms to the first order, we obtain

$$(s_0 - \cos\theta)\nu^{(0)}_{\hat{\mathbf{p}}} - \cos\theta \int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} F^{\mathrm{s}}(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}')\nu^{(0)}_{\hat{\mathbf{p}}'} = 0\,, \tag{2.48}$$

which is the collisionless kinetic equation, and

$$(s_0 - \cos\theta)\nu^{(1)}_{\hat{\mathbf{p}}} + s_1\nu^{(0)}_{\hat{\mathbf{p}}} - \cos\theta \int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} F^{\mathrm{s}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\nu^{(1)}_{\hat{\mathbf{p}}'} = i\frac{\int d\epsilon^{\mathrm{eq}}_{\mathbf{p}}\, I[\nu^{(0)}_{\hat{\mathbf{p}}}]}{v_{\mathrm{F}}k}\,, \tag{2.49}$$

which is the equation for the first-order correction. Note that the angle θ is given by $\cos\theta = \hat{\mathbf{p}}\cdot\hat{\mathbf{k}}$. The zeroth-order equation has been solved in the previous section. To solve for the first-order correction, we multiply the zeroth-order equation by $\nu^{(1)}/\cos\theta$ and the first-order equation by $\nu^{(0)}/\cos\theta$, and then integrate both equations over $\Omega_{\hat{\mathbf{p}}}$. Since the Landau f function is symmetric in the two momentum variables, the difference between the two equations gives

$$s_1 = \frac{\dfrac{i}{v_{\mathrm{F}}k}\displaystyle\int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\frac{\nu^{(0)}_{\hat{\mathbf{p}}}}{\cos\theta}\int d\epsilon^{\mathrm{eq}}_{\mathbf{p}}\, I[\nu^{(0)}_{\hat{\mathbf{p}}}]}{\displaystyle\int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\frac{(\nu^{(0)}_{\hat{\mathbf{p}}})^2}{\cos\theta}}\,. \tag{2.50}$$

This expression gives the leading-order correction to the collisionless kinetic equation. The magnitude of this leading-order correction can be estimated by assuming that the collision integral gives rise to a time scale τ for quasiparticle relaxation. Then from dimensional analysis, we can see

$$s_1 \sim \frac{1}{v_{\mathrm{F}}k}\frac{1}{\tau} \sim s_0\frac{1}{\omega\tau}\,. \tag{2.51}$$

This is indeed consistent with our initial assumption that the perturbation series is based on the parameter $1/\omega\tau$, which is small in the collisionless regime.

We now study the relation between the correction s_1 given in (2.50) and the zero sound attenuation α_0. From the definition (2.43), we have

$$s \equiv \frac{\omega}{v_{\rm F} k(\omega)} = \frac{\omega}{v_{\rm F}(\frac{\omega}{c_0} + i\alpha_0)} \approx \frac{c_0}{v_{\rm F}}(1 - i\frac{\alpha_0 c_0}{\omega}) = s_0(1 - i\frac{\alpha_0 c_0}{\omega}), \tag{2.52}$$

where c_0 is the zero sound velocity, and $s_0 = c_0/v_{\rm F}$ is the zeroth-order solution. Since the expression (2.50) is purely imaginary and the attenuation coefficient has to be real, the leading-order contribution to the attenuation is obtained from the relation $-is_0\alpha_0 c_0/\omega = s_1$, or

$$\alpha_0 = i\frac{s_1}{s_0}k, \tag{2.53}$$

where $k = \omega/c_0$ is the wave number.[16]

2.7.1. Simplification of the expression (2.50)

The correction s_1 directly gives the zero sound attenuation to leading order in $1/\omega\tau$. In the following we simplify the expression given in Eq. (2.50) for later use. For the denominator, we first write the zeroth-order equation (2.48) as

$$\frac{\nu^{(0)}_{\hat{\mathbf{p}}}}{\cos\theta} = \frac{1}{s_0}\Big[\nu^{(0)}_{\hat{\mathbf{p}}} + \int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} F^{\rm s}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\nu^{(0)}_{\hat{\mathbf{p}}'}\Big]. \tag{2.54}$$

Multiplying both sides by $\nu^{(0)}_{\hat{\mathbf{p}}}$ and integrating over $\Omega_{\hat{\mathbf{p}}}$, we obtain the denominator

$$\int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\frac{(\nu^{(0)}_{\hat{\mathbf{p}}})^2}{\cos\theta} = \frac{1}{s_0}\sum_l \frac{(\nu^{(0)}_l)^2}{(2l+1)}\Big(1 + \frac{F^{\rm s}_l}{2l+1}\Big). \tag{2.55}$$

[16] Note that the wave number k here is the real part of the complex wave number $k(\omega)$ given in the dispersion relation (2.43). We use the same symbol for both since they are closely related. The meaning of the symbol should be clear from the context.

For the numerator in (2.50), we have to take care of the collision integral. We linearize every term in the kinetic equation by the deviation from the global equilibrium,

$$\delta n_i = n_i - n_{\rm F}(\epsilon_i^{\rm eq})\,. \tag{2.56}$$

This quantity, however, is not the most natural way to express the collision integral, which contains a delta function to ensure energy conservation. The conserved energy involves the quasiparticle energies *including the Landau interaction energy due to the nonequilibrium local distribution*. More precisely, the collision integral contains a delta function together with the occupation numbers,

$$\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)\big[n_3n_4(1-n_1)(1-n_2)-n_1n_2(1-n_3)(1-n_4)\big]\,, \tag{2.57}$$

where the quasiparticle energy is

$$\epsilon_i = \epsilon_i^0 + \frac{1}{V}\sum_j f_{i,j}(n_j - n_j^0)\,. \tag{2.58}$$

The quasiparticle energy can be separated into two parts: $\epsilon_i = \epsilon_i^{\rm eq} + \delta\epsilon_i$, where the first part is given by the equilibrium distribution, as shown in (2.35), and the second part is given by the deviation from the equilibrium,

$$\delta\epsilon_i = \frac{1}{V}\sum_j f_{i,j}\delta n_j\,. \tag{2.59}$$

Since the energies in the delta function are the quasiparticle energies ϵ_i including the contribution from local nonequilibrium quasiparticles, it is better to linearize the collision integral using the deviation from the *local equilibrium*, which is given by

$n_{\mathrm{F}}(\epsilon_i)$,

$$\delta\bar{n}_i \equiv n_i - n_{\mathrm{F}}(\epsilon_i)\,. \tag{2.60}$$

With the identity

$$\begin{aligned}\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)\big[&n_{\mathrm{F}}(\epsilon_3)n_{\mathrm{F}}(\epsilon_4)(1-n_{\mathrm{F}}(\epsilon_1)(1-n_{\mathrm{F}}(\epsilon_2))\\ &-n_{\mathrm{F}}(\epsilon_1)n_{\mathrm{F}}(\epsilon_2)(1-n_{\mathrm{F}}(\epsilon_3))(1-n_{\mathrm{F}}(\epsilon_4))\big]=0\,,\end{aligned} \tag{2.61}$$

the collision integral is proportional to $\delta\bar{n}$ to leading order.

There are thus two ways to linearize the theory: One is using the deviation from the global equilibrium, given by (2.56), and the other one is using the deviation from the local equilibrium,[17] given by (2.60). These two ways must be related in the linear response regime. Consider

$$\begin{aligned}\delta\bar{n}_i &\equiv n_i - n_{\mathrm{F}}(\epsilon_i)\\ &= n_i - n_{\mathrm{F}}(\epsilon_i^{\mathrm{eq}}+\delta\epsilon_i)\\ &\approx n_i - n_{\mathrm{F}}(\epsilon_i^{\mathrm{eq}}) - \frac{\partial n_{\mathrm{F}}}{\partial\epsilon_i^{\mathrm{eq}}}\delta\epsilon_i\\ &= \delta n - \frac{\partial n_{\mathrm{F}}}{\partial\epsilon_i^{\mathrm{eq}}}\delta\epsilon_i\,.\end{aligned} \tag{2.62}$$

Following the definition of $\nu_{\hat{\mathbf{p}}}$ given in (2.37), we define a function $\psi_{\hat{\mathbf{p}}}$ by

$$\delta\bar{n}_{\mathbf{p}} = -\frac{\partial n_{\mathrm{F}}}{\partial\epsilon_{\mathbf{p}}}\psi_{\hat{\mathbf{p}}}\,, \tag{2.63}$$

[17] This description is somewhat misleading. The "local equilibrium" here means the Fermi-Dirac distribution determined by the local quasiparticle energy, which however includes nonequilibrium contributions.

and to the leading order in $\delta\epsilon_{\hat{\mathbf{p}}}$ the above relation is reduced to

$$\psi_{\hat{\mathbf{p}}} = \nu_{\hat{\mathbf{p}}} + \delta\epsilon_{\hat{\mathbf{p}}}\,, \tag{2.64}$$

where all the momenta are restricted to the Fermi surface. Using (2.59) and the result at the end of Section 2.3, we have

$$\psi_{\hat{\mathbf{p}}} = \nu_{\hat{\mathbf{p}}} + \int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} F^{\mathrm{s}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\nu_{\hat{\mathbf{p}}'}\,, \tag{2.65}$$

or with spherical harmonics,

$$\psi_l = \nu_l\Big(1 + \frac{F_l^{\mathrm{s}}}{2l+1}\Big)\,. \tag{2.66}$$

This simple relation between δn and $\delta\bar{n}$ will be used later.

Using the above relation between ψ and ν, the zeroth-order equation (2.54) can be written as

$$\frac{\nu_{\hat{\mathbf{p}}}^{(0)}}{\cos\theta} = \frac{\psi_{\hat{\mathbf{p}}}^{(0)}}{s_0}\,, \tag{2.67}$$

and thus the collision integral (2.50) becomes

$$s_1 = \Big(\frac{i}{v_{\mathrm{F}}k}\Big)\frac{\displaystyle\int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\Big[\int d\epsilon_{\mathbf{p}}^{\mathrm{eq}}\,\delta I[\psi_{\hat{\mathbf{p}}}^{(0)}]\Big]\psi_{\hat{\mathbf{p}}}^{(0)}}{\displaystyle\sum_{l\geq 0}\frac{(\nu_l^{(0)})^2}{2l+1}\Big(1+\frac{F_l^{\mathrm{s}}}{2l+1}\Big)}\,, \tag{2.68}$$

where we have used the result (2.55). Inserting the above result into the expression (2.53), the zero sound attenuation can be expressed as

$$\alpha_0 = \Big(\frac{-1}{c_0}\Big)\frac{\int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}\Big[\int d\epsilon_{\mathbf{p}}^{\rm eq}\ \delta I[\psi_{\hat{\mathbf{p}}}^{(0)}]\Big]\psi_{\hat{\mathbf{p}}}^{(0)}}{\sum_{l\geq 0}\frac{(\nu_l^{(0)})^2}{2l+1}\Big(1+\frac{F_l^{\rm s}}{2l+1}\Big)}\,. \tag{2.69}$$

For a collision integral with the form

$$I = \int_{\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} W(1,2;3,4)\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)\delta(\mathbf{p}_1+\mathbf{p}_2-\mathbf{p}_3-\mathbf{p}_4) \times\big[(1-n_1)(1-n_2)n_3n_4 - n_1n_2(1-n_3)(1-n_4)\big]\,, \tag{2.70}$$

the linear approximation can be written as

$$\delta I_{\mathbf{p}_1}[\psi_{\hat{\mathbf{p}}}^{(0)}] = -\beta\int\frac{d^3p_2}{(2\pi)^3}\int\frac{d^3p_3}{(2\pi)^3}W(\mathbf{p}_1,\mathbf{p}_2;\mathbf{p}_3,\mathbf{p}_4)\delta(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4) \times\Big(\psi_{\hat{\mathbf{p}}_1}^{(0)}+\psi_{\hat{\mathbf{p}}_2}^{(0)}-\psi_{\hat{\mathbf{p}}_3}^{(0)}-\psi_{\hat{\mathbf{p}}_4}^{(0)}\Big)\times\big[n_{\rm F}(\epsilon_1)n_{\rm F}(\epsilon_2)\big(1-n_{\rm F}(\epsilon_3)\big)\big(1-n_{\rm F}(\epsilon_4)\big)\big]\,, \tag{2.71}$$

where we have used $n_i = n_{\rm F}(\epsilon_i) + \delta\bar{n}_i$ (see (2.62)) and

$$\delta\bar{n}_{\mathbf{p}} = -\frac{\partial n_{\rm F}}{\partial\epsilon_{\mathbf{p}}}\psi_{\hat{\mathbf{p}}} = \beta n_{\rm F}(\epsilon_{\mathbf{p}})(1-n_{\rm F}(\epsilon_{\mathbf{p}}))\psi_{\hat{\mathbf{p}}} \tag{2.72}$$

from the definition (2.63). These results will be used later.

2.8. Summary

In this chapter, we have reviewed Landau's Fermi liquid theory, and introduced the theory of zero sound. The existence of zero sound relies on the Landau interaction, which can be characterized by the Fermi liquid parameters. Zero sound attenuation follows the T^2 scaling,[18] which is qualitatively different from the $1/T^2$ behavior of normal sound. This T^2 scaling has been verified experimentally [9]. The reader can find more details about the experiments in [23]. The theory of zero sound thus demonstrates that for liquid ^{3}He, Landau's Fermi liquid theory is indeed an effective theory for the low-energy behavior.

[18] There is a so-called quantum correction to this T^2 behavior, which is not important when the sound frequency is low compared to the temperature, or more precisely when $\omega/2\pi T \ll 1$. See Landau's paper [26].

CHAPTER 3

Microscopic Description of Fermi Liquids

Landau's Fermi liquid theory has been successfully applied to normal-state liquid ^{3}He. In this thesis, we want to understand the influence of pair fluctuations for normal liquid ^{3}He near the superfluid transition. As seen in the last chapter, Landau obtained the Fermi liquid theory from adiabatic continuity, which *does not allow any phase transition* by assumption. In order to describe the superfluid transition and the pair fluctuations, we need a more microscopic description, which can accommodate the pairing interaction, as well as the Fermi liquid theory.

Microscopic descriptions for Fermi liquids have been studied for a long time, initiated by Landau himself [30]. Landau derived the relations between the $f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}$ function and the two-particle scattering amplitude using Feynman diagrams. His approach was followed by other researchers and the field theory of interacting fermions was advanced a lot in the 1960s [31, 32]. In the 1970s, quantum and statistical field theories underwent important progress, driven by Wilson's renormalization group (RG). RG is the appropriate tool for analyzing low-energy effective theories, including Landau's Fermi liquid theory for interacting fermions. The RG analysis of Fermi liquid theory started in the 1990s. For a pedagogical introduction to the early results, the reader can refer to the lectures by Polchinski [33] and the review article by Shankar [34]. The results of the RG analysis confirm that the Fermi liquid state is indeed a stable state in the long-wavelength limit, while the pairing interaction is a

(marginally) relevant perturbation, leading to the BCS transition. Another approach to formulate the Fermi liquid theory makes use of quasiclassical methods, which were developed in the community of superconductivity and superfluid ^{3}He [35–37], and later shown to be closely related to the RG idea [38].

In this chapter we introduce the idea of quasiclassical methods, and then derive the kinetic equation using the Keldysh method. The result is identified with Landau's kinetic equation with subleading corrections. We will consider the correction from pair fluctuations in liquid ^{3}He in later chapters.

3.1. Low-energy effective theory for Fermi liquids

In principle, the bare mass and the interaction between two bare ^{3}He particles can be measured experimentally, and we can build a field theory using these parameters, with some microscopic cutoff Λ_0 on the atomic scale. It is, however, almost impossible to analytically calculate macroscopic quantities from the theory with microscopic parameters. Instead of directly working with the microscopic theory, we transform it into a low-energy effective theory. This reduction can be performed in two equivalent approaches. The first one, developed and widely used in the community of superconductors and superfluids, is based on diagrams [35–37]. Consider a formal microscopic theory formulated in Feynman diagrams. We can separate each fermion propagator into a low and high energy part by some low-energy scale $\Lambda \ll \epsilon_{\mathrm{F}}$. A diagram with n internal lines becomes 2^n diagrams with low-energy and high-energy lines. The perturbation expansion can thus be reorganized as diagrams with the vertices containing only high-energy lines, and the low-energy propagators. *Assuming*

there are no singularities in these vertices, the perturbation expansion of the self-energy can be organized according to the number of low-energy lines and loops. As realized by Landau himself [2], the leading-order contribution must contain the two-particle vertex. This approach is called the **quasiclassical method**, where the perturbation expansion relies on the low-energy scale $\Lambda \ll \epsilon_{\mathrm{F}}$.

Equivalently, we can obtain the low-energy theory by integrating out the high-energy degrees of freedom. This approach is based on Wilson's RG [38]. Obviously, the difference between the two approaches is formal, and the essence of the two methods is the same. From the viewpoint of RG, the action depends on the energy scale on which the theory is defined. In particular, the low-energy action for interacting fermions can be written as [38]

$$S^{\Lambda}[\psi,\psi^*] = \int_p^{\Lambda} \psi_p^*[-G^{\Lambda}(p)]^{-1}\psi_p + \frac{1}{4}\int_{p,p',q}^{\Lambda} \Gamma^{\Lambda}(p,p';q)\psi^*_{p-q/2}\psi^*_{p'+q/2}\psi_{p'-q/2}\psi_{p+q/2} + \mathcal{O}[(\psi^*\psi)^3]\,, \quad (3.1)$$

where the ψ and ψ^* are the fermion fields, the subscripts include all possible indices, and the superscript Λ means the integrals have a ultraviolet cutoff. The function G^{Λ} and Γ^{Λ} are the propagator and the two-particle interaction at the energy scale Λ.

The three-particle and higher-order interactions contribute only subleading terms to the self-energy. Consider the self-energy from a two-particle vertex, which has one internal line and one loop, as shown in Figure 3.1. Assume the two-particle interaction has weak energy dependence when its variables are near the Fermi surface. Then the

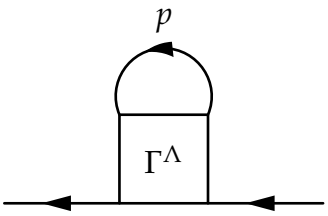


Figure 3.1. The self-energy diagram with one internal line, corresponding to mean-field contribution. Note that the vertex Γ^Λ is of order s^0.

self-energy can be estimated by

$$\Sigma \sim \int^\Lambda d\epsilon \int^\Lambda d\xi_{\mathbf{p}} \frac{1}{\epsilon - \xi_{\mathbf{p}}} \sim \Lambda\,, \tag{3.2}$$

since both integrals are restricted to the low-energy scale. On the other hand, the same estimate for the self-energy diagram from a three-particle vertex gives Λ^2, since the diagram has two internal lines and two loops. In the same way, higher-order interactions have subleading contributions to the self-energy, assuming they have weak energy dependence near the Fermi surface. In the limit $\Lambda/\epsilon_F \to 0$, we can neglect the higher-order interactions in the effective action (3.1).

The above analysis shows that the theory can be expanded in terms of the small parameter Λ/ϵ_F. The energy scale Λ is determined by the low-energy processes, like the transition temperature T_c. In the following, we call the small parameter s.

Higher-order processes in terms of the two-particle vertex also give subleading contributions. For, example, consider the diagram shown in Figure 3.2, which corresponds to normal collisions of quasiparticles. This diagram contains an additional phase-space factor from the constraint $\mathbf{p}''' = \mathbf{p} + \mathbf{p}'' - \mathbf{p}'$, and has order of magnitude s^2.

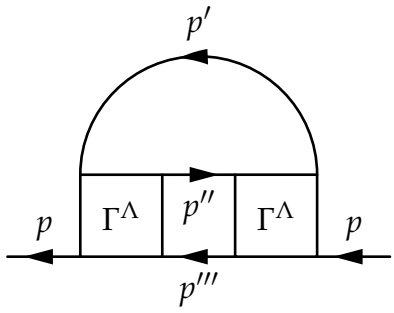


Figure 3.2. The diagram for collisions.

The above discussion gives a brief summary of the idea of quasiclassical methods. More details are given in the references [35–38]. We focus on the low-energy theory in the following.

3.2. Quantum kinetic equation

In order to describe nonequilibrium phenomena, we use the Keldysh method to formulate the field theory [39]. For a detailed introduction to the Keldysh method, the reader can refer to the excellent article by Kita [40]. The contour-ordered Green function is defined by

$$G(1^a, 2^b) \equiv -i\langle \mathsf{T}_{\mathcal{C}} \psi(1^a)\psi^\dagger(2^b)\rangle\,, \tag{3.3}$$

where the expression $\psi(i)$ represents $\psi(x_i, t_i)$ with x including the spatial and spin variables, and the superscripts a, b represent the forward and backward branches of the Keldysh contour (see Figure 3.3), indexed by 1 and 2 respectively. Similar to the Green functions in the equilibrium theory, the contour-ordered Green function also

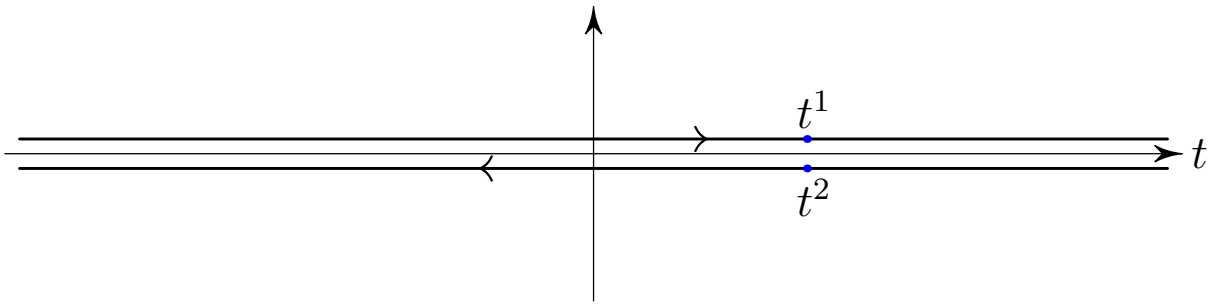


Figure 3.3. The Keldysh contour. Time coordinates shown on the upper (lower) branch with Keldysh index 1 (2).

satisfies the Dyson equation,

$$\left(i\frac{\partial}{\partial t_1^a} - K_1\right) G(1^a, 2^b) - \int_{3^c} \Sigma(1^a, 3^c) G(3^c, 2^b) d3^c = \delta^{\mathrm{C}}(1^a, 2^b)\,, \tag{3.4}$$

where K_1 represents the single-particle Hamiltonian, $K_1 \equiv \nabla_1^2/2m - \mu$, operating on the first variable 1, and Σ is the self-energy given by interactions. The contour delta function is defined as

$$\delta^{\mathrm{C}}(t^a, t'^b) = \begin{cases} \delta(t - t')\,, & a = b = 1 \\ -\delta(t - t')\,, & a = b = 2 \\ 0, & a \neq b \end{cases} \cdot \tag{3.5}$$

The minus sign in this definition is due to the Heaviside step function in the anti-time-ordered Green function $G^{\tilde{\mathrm{T}}}$. Note that the notation $\int_{3^c}$ represents the summation over all dummy variables, including a time integral along the Keldysh contour.

Although the contour Dyson equation (3.4) is mathematically elegant, it is hard to be used for practical calculations and interpretation. We can distinguish the time variables in the forward and backward branches, and define the matrix Green function

$G^{ab}(1,2) \equiv G(1^a, 2^b)$. Using the definition of contour-ordering, it is easy to see

$$[G^{ab}] \equiv \begin{pmatrix} G^{11} & G^{12} \\ G^{21} & G^{22} \end{pmatrix} = \begin{pmatrix} G^{\mathrm{T}} & G^{<} \\ G^{>} & G^{\tilde{\mathrm{T}}} \end{pmatrix} , \tag{3.6}$$

where the *time-ordered, anti-time-ordered, lesser, and greater Green functions* are defined by

$$G^{\mathrm{T}}(1,2) \equiv -i\big[\theta(t_1 - t_2)\langle\psi(1)\psi^{\dagger}(2)\rangle - \theta(t_2 - t_1)\langle\psi^{\dagger}(2)\psi(1)\rangle\big] , \tag{3.7}$$

$$G^{\tilde{\mathrm{T}}}(1,2) \equiv -i\big[\theta(t_2 - t_1)\langle\psi(1)\psi^{\dagger}(2)\rangle - \theta(t_1 - t_2)\langle\psi^{\dagger}(2)\psi(1)\rangle\big] , \tag{3.8}$$

$$G^{<}(1,2) \equiv i\langle\psi^{\dagger}(2)\psi(1)\rangle , \tag{3.9}$$

$$G^{>}(1,2) \equiv -i\langle\psi(1)\psi^{\dagger}(2)\rangle , \tag{3.10}$$

respectively. The matrix self-energy can be defined in the same way. From the above definition, it is obvious that under complex conjugation the Green functions have the following relations

$$G^{>}(1,2)^{*} = -G^{>}(2,1) , \tag{3.11}$$

$$G^{<}(1,2)^{*} = -G^{<}(2,1) , \tag{3.12}$$

and

$$G^{\mathrm{T}}(1,2)^{*} = -G^{\tilde{\mathrm{T}}}(2,1) . \tag{3.13}$$

It is less obvious that the self-energies satisfy the same relations

$$\Sigma^{21}(1,2)^* = -\Sigma^{21}(2,1)\,, \tag{3.14}$$

$$\Sigma^{12}(1,2)^* = -\Sigma^{12}(2,1)\,, \tag{3.15}$$

$$\Sigma^{11}(1,2)^* = -\Sigma^{22}(2,1)\,. \tag{3.16}$$

See Kita [40] for more details about these relations.

Using the components defined above, the Dyson equation can be written as

$$\Big(i\frac{\partial}{\partial t_1} - K_1\Big)G^{11} - \int \big[\Sigma^{11}G^{11} - \Sigma^{12}G^{21}\big] = \delta(1,2)\,, \tag{3.17}$$

$$\Big(i\frac{\partial}{\partial t_1} - K_1\Big)G^{12} - \int \big[\Sigma^{11}G^{12} - \Sigma^{12}G^{22}\big] = 0\,, \tag{3.18}$$

$$\Big(i\frac{\partial}{\partial t_1} - K_1\Big)G^{21} - \int \big[\Sigma^{21}G^{11} - \Sigma^{22}G^{21}\big] = 0\,, \tag{3.19}$$

$$\Big(i\frac{\partial}{\partial t_1} - K_1\Big)G^{22} - \int \big[\Sigma^{21}G^{12} - \Sigma^{22}G^{22}\big] = -\delta(1,2)\,, \tag{3.20}$$

where the symbol $\int$ represents convolutions for all intermediate variables, including spins, with the standard time integral from $-\infty$ to ∞. The operators $i\partial_1 - K_1$ in the above equations are applied to the first variable of the Green functions. We can also consider the equations where the operators are applied to the second variable of the Green functions. This can be obtained by taking the complex conjugate of the above equations. In particular, for the (1,2)-component we have

$$\Big(-i\frac{\partial}{\partial t_2} - K_2\Big)G^{12} - \int \big[G^{11}\Sigma^{12} - G^{12}\Sigma^{22}\big] = 0\,. \tag{3.21}$$

To get the quantum kinetic equation, we focus on the lesser Green function $G^< \equiv G^{12}$, which is closely related to the occupation numbers in single-particle states [41]. Taking the difference between (3.18) and (3.21), we obtain

$$\Big(-i(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}) + \frac{1}{2m}(\nabla_2^2 - \nabla_1^2)\Big) G^{12} \\ - \int \Big[G^{11}\Sigma^{12} - G^{12}\Sigma^{22} - \Sigma^{11}G^{12} + \Sigma^{12}G^{22} \Big] = 0\,. \quad (3.22)$$

We have not made any approximation so far, and the above equation is exact. To connect this equation with the kinetic theory, we want to express it in terms of the position and momentum variables. Indeed, the distribution function in the classical kinetic theory depends on both position and momentum of particles. When the system is homogeneous in both space and time, all Green functions depend only on the differences $\mathbf{r}_1 - \mathbf{r}_2$ and $t_1 - t_2$, and it is natural to describe the system in Fourier space, that is, in the momentum (and energy) variables. For a Fermi liquid, the characteristic scale is given by the Fermi momentum and Fermi energy. If the system is influenced by a long-wavelength and low-energy perturbation, compared to the Fermi momentum and Fermi energy, then we should still be able to describe the system by momentum variables locally, assuming the momenta are near the Fermi surface.[1] This idea can be formulated mathematically by the Wigner transformation as follows. Consider the mixed spatial coordinates $\mathbf{R} \equiv (\mathbf{r}_1 + \mathbf{r}_2)/2$, $\mathbf{r} \equiv \mathbf{r}_1 - \mathbf{r}_2$, and the mixed temporal coordinates $T \equiv (t_1 + t_2)/2$, $t \equiv t_1 - t_2$. More compactly, we write $X \equiv (T, \mathbf{R})$, $x \equiv (t, \mathbf{r})$. For a homogeneous system, the Green functions $G(1,2) = G(X + x/2, X - x/2)$ are

[1] Note that we used the same argument for the phenomenological kinetic equation, see Section 2.5.

independent of X, and the Fourier transform of x gives the momentum variables. Similarly, we can take the Fourier transform with respect to the relative coordinate x for an inhomogeneous system,

$$G(X,p) = \int dx\, e^{-ipx} G(X + \frac{x}{2}, X - \frac{x}{2}) \,, \tag{3.23}$$

where $p \equiv (\epsilon, \mathbf{p})$ is the corresponding Fourier variable, and $px \equiv -\epsilon t + \mathbf{p}\cdot\mathbf{r}$ is the inner product between p and x. Assuming that the center-of-mass coordinate X varies on a length scale much larger than the Fermi wavelength, we see that the coordinate X and the momentum p have similar meaning as the variables $\mathbf{r}$ and $\mathbf{p}$ in the classical kinetic theory.

Derivatives can be expressed in the mixed coordinates,

$$\begin{aligned} \partial_{1,2} &= \frac{\partial X}{\partial x_{1,2}}\partial_X + \frac{\partial x}{\partial x_{1,2}}\partial_x \\ &= \frac{1}{2}\partial_X \pm \partial_x \,. \end{aligned} \tag{3.24}$$

In particular, we have

$$\nabla_2^2 - \nabla_1^2 = -2\nabla_{\mathbf{R}}\cdot\nabla_{\mathbf{r}} \,. \tag{3.25}$$

Using the Wigner representation of the Green functions, the first term in (3.22) can be written as

$$\Big(-i(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}) + \frac{1}{2m}(\nabla_2^2 - \nabla_1^2)\Big) G^{12} \xrightarrow{\text{W.T.}} (\partial_T + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{R}})[-iG^{12}] \,. \tag{3.26}$$

The second term contains convolutions, whose spatial-temporal part in the mixed coordinates is given by the Moyal product [42]

$$(A \otimes B)(X,p) = \exp\left[\frac{i}{2}\left(\partial_X^A \partial_p^B - \partial_p^A \partial_X^B\right)\right] A(X,p) B(X,p)\,, \tag{3.27}$$

where

$$\partial_X^A \partial_p^B \equiv -\partial_T^A \partial_\epsilon^B + \partial_{\mathbf{R}}^A \partial_{\mathbf{p}}^B\,, \tag{3.28}$$

and the superscripts A and B of the differential operators refers to the functions that are being differentiated. We will focus on the system in the absence of magnetic field, so the spin structure is trivial and have been dropped from our expressions. Since we are interested in the long-wavelength behavior, we keep only the zeroth and first-order terms in the expansion of Moyal product,

$$\begin{aligned}(A \otimes B) &\approx AB + \frac{i}{2}\left[\partial_X A \partial_p B - \partial_p A \partial_X B\right] \\ &\equiv AB + \frac{i}{2}\{A,B\}_{\mathrm{PB}}\,,\end{aligned} \tag{3.29}$$

where in the second line we have used the definition

$$\{A,B\}_{\mathrm{PB}} \equiv \left[\partial_X A \partial_p B - \partial_p A \partial_X B\right]. \tag{3.30}$$

Using (3.26), (3.29) and the relations $G^{11} + G^{22} = G^{12} + G^{21}$, $\Sigma^{11} + \Sigma^{22} = \Sigma^{12} + \Sigma^{21}$, the equation (3.22) can be written as

$$\begin{aligned}&\left(\partial_T + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{R}}\right)[-iG^{12}] \\ &\quad - \left[G^{21}\Sigma^{12} - G^{12}\Sigma^{21}\right] - \frac{i}{2}\{G^{11} - G^{22}, \Sigma^{12}\}_{\mathrm{PB}} + \frac{i}{2}\{\Sigma^{11} - \Sigma^{22}, G^{12}\}_{\mathrm{PB}} = 0\,.\end{aligned} \tag{3.31}$$

The first term in this equation has the form of the drift term in the classical kinetic equation, and thus the whole equation can be thought of as the kinetic equation for a quantum system.

To understand the meaning of the terms with the self-energy in (3.31), we write the drift term in the above equation as a Poisson bracket

$$(\partial_T + \frac{\mathbf{p}}{m}\cdot\nabla_{\mathbf{R}})[-iG^{12}] = \{\epsilon - \frac{\mathbf{p}^2}{2m}, -iG^{12}\}_{\mathrm{PB}}\,. \tag{3.32}$$

Thus the drift term and the last bracket can be combined into a single term

$$\{\epsilon - \frac{\mathbf{p}^2}{2m} - \frac{\Sigma^{11}-\Sigma^{22}}{2}, -iG^{12}\}_{\mathrm{PB}}\,, \tag{3.33}$$

and thus the quantum kinetic equation (3.31) becomes

$$\{\epsilon - \frac{\mathbf{p}^2}{2m} - \frac{\Sigma^{11}-\Sigma^{22}}{2}, -iG^{12}\}_{\mathrm{PB}} - \frac{i}{2}\{G^{11}-G^{22}, \Sigma^{12}\}_{\mathrm{PB}} = G^{21}\Sigma^{12} - G^{12}\Sigma^{21}\,. \tag{3.34}$$

The first bracket in this equation gives the drift term, including the force generated by the interaction and external field. The second bracket in the left-hand side is a quantum correction, which does not have a classical correspondence. The two terms in the right-hand side correspond to the collision integral, as we will see later.

3.3. Quasiparticle approximation

The key feature of Landau's Fermi liquid theory is that the low-energy states of the system can be described by a collection of quasiparticles, as discussed in the last chapter. In particular, the lifetime of the quasiparticles is long enough and the quasiparticle distribution function $n_{\mathbf{p}}$ is valid for describing phenomena happening

on shorter time scales, like the zero sound oscillation. To obtain the Landau's kinetic equation (2.33) from the quantum kinetic equation (3.34), we need to express the quasiparticle distribution using the Green function. From the definition of the lesser Green function $G^< \equiv G^{12}$ and the Wigner transformation, we have

$$n_{\mathbf{p}}(\mathbf{R}, T) = \int d^3r\, e^{-i\mathbf{p}\cdot\mathbf{r}}[-iG^<(x_1, x_2)]\Big|_{t_1=t_2} = \int_{-\infty}^{\infty} d\epsilon[-iG^<(\mathbf{R}, T; \mathbf{p}, \epsilon)]\,. \tag{3.35}$$

In Fermi liquids, we can assume that the spectral function is approximately a peak near the Fermi surface, $A \approx 2\pi\delta(\epsilon - \epsilon_{\mathbf{p}})$, characterizing the quasiparticles. This is the *quasiparticle approximation*. The lesser Green function can be written as [40]

$$-iG^< = A(X, p)\phi(X, p)\,, \tag{3.36}$$

where A is the nonequilibrium spectral function and satisfies

$$\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} A(X, p) = 1. \tag{3.37}$$

The function $\phi(X, p)$ reduces to Fermi-Dirac distribution function in equilibrium. Using the quasiparticle approximation and (3.35), we have

$$\phi(X, p)\Big|_{\epsilon=\epsilon_{\mathbf{p}}} \approx n_{\mathbf{p}}\,. \tag{3.38}$$

Note that the quasiparticle energy $\epsilon_{\mathbf{p}}$ can depend on space and time. Performing the integration over ϵ on the kinetic equation (3.34), we obtain an equation for $n_{\mathbf{p}}(\mathbf{R}, T)$. Due to the delta peak contained in the $-iG^<$, all functions are evaluated at $\epsilon = \epsilon_{\mathbf{p}}$

after taking the differentiation given by the Poisson bracket. The first bracket gives

$$(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{R}})n - \partial_{\mathbf{R}}\mathfrak{Re}\Sigma^{11}\big|_{\epsilon=\epsilon_{\mathbf{p}}}\cdot\partial_{\mathbf{p}}n + \partial_{\mathbf{p}}\mathfrak{Re}\Sigma^{11}\big|_{\epsilon=\epsilon_{\mathbf{p}}}\cdot\partial_{\mathbf{R}}n - \partial_{\epsilon}\mathfrak{Re}\Sigma^{11}\big|_{\epsilon=\epsilon_{\mathbf{p}}}\partial_t n\,, \quad (3.39)$$

where we have used the identity $(\Sigma^{11}-\Sigma^{22})/2 = \mathfrak{Re}\Sigma^{11}$. Note that

$$\int d\epsilon\,\partial_T\mathfrak{Re}\Sigma^{11}\partial_\epsilon(-iG^<) \approx \partial_T\mathfrak{Re}\Sigma^{11}\Big|_{\epsilon=\epsilon_{\mathbf{p}}}\int d\epsilon\,\partial_\epsilon(-iG^<) = 0\,, \quad (3.40)$$

since the spectral function A vanishes as $\epsilon\to\pm\infty$.

The second term in the left-hand side of (3.34) is

$$-\frac{i}{2}\{G^{11}-G^{22},\Sigma^{12}\}_{\mathrm{PB}}\,. \quad (3.41)$$

Note that $G^{11}-G^{22} = G^{\mathrm{R}}+G^{\mathrm{A}} = 2\mathfrak{Re}G^{\mathrm{R}}$ and

$$\mathfrak{Re}G^{\mathrm{R}} = \mathrm{P.V.}\int_{-\infty}^{\infty}\frac{d\epsilon'}{2\pi}\frac{A(\epsilon',\mathbf{p})}{\epsilon-\epsilon'}\,. \quad (3.42)$$

Because the spectral function A is approximately a delta function around the Fermi surface and the self-energy depends weakly on the energy variable ϵ according to the quasiclassical method, the main contribution of the ϵ integral is from the $\mathfrak{Re}G^{\mathrm{R}}$. Since

$$\mathrm{P.V.}\int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\frac{1}{\epsilon-\epsilon'} = 0\,, \quad (3.43)$$

the second bracket is approximately zero after the ϵ integration.

For the right-hand side, note that $G^{21} \equiv G^{>} = -iA(1-\phi)$[2] and thus we obtain

$$-i\Sigma^{12}\big|_{\epsilon=\epsilon_{\mathbf{p}}}(1-n) - i\Sigma^{21}\big|_{\epsilon=\epsilon_{\mathbf{p}}} n \tag{3.44}$$

after the ϵ integration. Hence from (3.39) and (3.44), we have

$$\begin{aligned}(\partial_t + \mathbf{v}\cdot\partial_{\mathbf{R}})n_{\mathbf{p}} - \partial_{\mathbf{R}}\mathfrak{Re}\Sigma^{11}\big|_{\epsilon=\epsilon_{\mathbf{p}}}\cdot\partial_{\mathbf{p}}n_{\mathbf{p}} + \partial_{\mathbf{p}}\mathfrak{Re}\Sigma^{11}\big|_{\epsilon=\epsilon_{\mathbf{p}}}\cdot\partial_{\mathbf{R}}n_{\mathbf{p}} - \partial_{\epsilon}\mathfrak{Re}\Sigma^{11}\big|_{\epsilon=\epsilon_{\mathbf{p}}}\partial_t n_{\mathbf{p}} \\ = -i\Sigma^{12}\big|_{\epsilon=\epsilon_{\mathbf{p}}}(1-n_{\mathbf{p}}) - i\Sigma^{21}\big|_{\epsilon=\epsilon_{\mathbf{p}}} n_{\mathbf{p}}\,.\end{aligned} \tag{3.45}$$

This equation has the form similar to Landau's kinetic equation (2.33). The leading-order contribution of the self-energy gives rise to the Landau interaction, which can be described by Landau's f function.

3.4. Summary

We have discussed how to describe Landau's Fermi liquid theory in terms of quantum field theory. The description relies on a reorganization of the perturbation series, leading to a low-energy effective theory. We can then derive the quantum kinetic equation from the effective theory. Finally, a kinetic equation for the quasiparticle distribution function is obtained. We will use the kinetic equation (3.45) to study the corrections from pair fluctuations.

[2]This can be seen from the relation $A(X,p) \equiv i(G^{\mathrm{R}} - G^{\mathrm{A}}) \equiv i(G^{>} - G^{<})$.

CHAPTER 4

Pair-Fluctuations Corrections to Kinetic Equation and Zero Sound

In the previous chapters, we reviewed the theories for describing Fermi liquids. The goal of this thesis is to study the influence of pair fluctuations on a Fermi liquid. We focus on liquid ^{3}He and the zero sound in it, since they are best understood [23]. In this chapter we first calculate the vertex function for the Cooper instability, and then consider its contribution to the self-energy. Using the result from the previous chapter, we can obtain the corrections to Landau's kinetic equation, and thus we can calculate the zero sound properties. We then compare our result with the existing experimental and theoretical work.

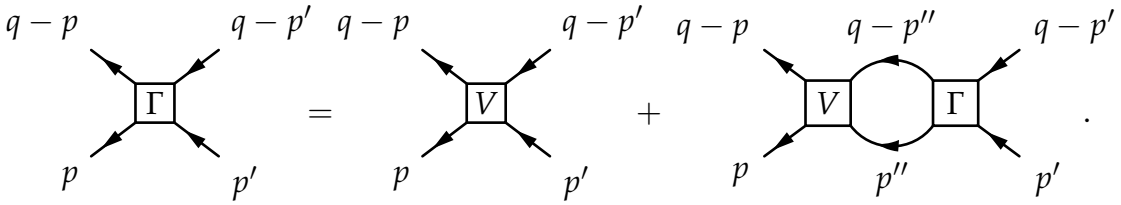
4.1. Cooper pair fluctuations

The Cooper instability is due to the attractive interaction between quasiparticles with opposite momenta on the Fermi surface [31]. Since the liquid ^{3}He is a p-wave superfluid, we consider the pairing interaction in the p-wave channel [3],

$$V_{\alpha\beta,\gamma\delta}(p,p') = 3V\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}' \times \frac{1}{2}\mathbf{g}_{\alpha\beta}\cdot\mathbf{g}^{\dagger}_{\gamma\delta}\,, \tag{4.1}$$

where the coupling constant is negative, $V < 0$, and $\mathbf{g} \equiv i\sigma\sigma_y$ is the matrix for the spin-triplet structure, with $\sigma = (\sigma_x, \sigma_y, \sigma_z)$. Note that the pairing interaction V is determined by the interaction Γ^{Λ} in the effective action (3.1), where two incoming particles have almost opposite momenta on the Fermi surface, i.e., the momenta are

$\mathbf{p}+\mathbf{q}/2$ and $-\mathbf{p}+\mathbf{q}/2$ with $|\mathbf{q}| \ll p_{\mathrm{F}}$. The vertex function in the Cooper channel is given by this Bethe-Salpeter equation:



(4.2)

The vertex Γ, as shown in the diagrammatic equation, describes two fermions with almost opposite momenta. When the temperature is above but near T_{c}, the dynamics of two particles is near a bound state. In other words, the two particles are like a Cooper pair with long but finite lifetime, which is *Cooper pair fluctuations*. We calculate the vertex Γ near T_{c} in the following, which characterizes the Cooper pair fluctuations.[1]

From the Feynman rule of the Keldysh method [40], the above diagrammatic equation is translated to

$$\begin{aligned}\Gamma^{ab}_{\alpha\beta,\gamma\delta}(p,p';q) = {} & iV_{\alpha\beta,\gamma\delta}(p,p')\check{\tau}_3^{ab} \\ & + \sum iV_{\alpha\beta,\alpha'\beta'}(p,p'')\check{\tau}_3^{ac}G^{cd}(p'')G^{cd}(q-p'')\Gamma^{db}_{\alpha'\beta',\gamma\delta}(p'',p';q)\,, \end{aligned} \tag{4.3}$$

where the superscripts a, b, c, d are the Keldysh indices, the matrix $\check{\tau}_3$ is the Pauli matrix in Keldysh space, and the summation includes all dummy indices. The equation can be simplified by the ansatz

$$\Gamma^{ab}_{\alpha\beta,\gamma\delta}(p,p';q) = \Gamma^{ab}(p,p';q) \times \frac{1}{2}\mathbf{g}_{\alpha\beta}\cdot\mathbf{g}^{\dagger}_{\gamma\delta}\,, \tag{4.4}$$

[1] If we consider the two incoming particles and the two outgoing particles to be Cooper pairs, then the vertex given above is like the pair-pair propagator.

which separates the spin part from the orbital part. The equation can thus be reduced to

$$\Big[\frac{1}{3iV}\check{\tau}_3^{ac}\Gamma^{cb} - \sum \hat{\mathbf{p}}\cdot\hat{\mathbf{p}}'' G^{ac}(p'')G^{ac}(q-p'')\Gamma^{cb}(p'',p';q)\Big] = \hat{\mathbf{p}}\cdot\hat{\mathbf{p}}'\delta^{ab}\,. \tag{4.5}$$

From rotation symmetry, the vertex function can be decomposed into

$$\Gamma^{ab} = \sum_{\lambda=\parallel,\perp} \hat{p}_i[\Gamma_\lambda^{ab}P_{ij}^\lambda]\hat{p}'_j \tag{4.6}$$

with the projections $P_{ij}^{\parallel} = \hat{q}_i\hat{q}_j$ and $P_{ij}^{\perp} = \delta_{ij} - \hat{q}_i\hat{q}_j$. In the quasiclassical approximation, the vertex function Γ can be assumed to depend only on the direction of fermion momenta, but not fermion energies. Then the summation over $p'' \equiv (\epsilon'', \mathbf{p}'')$ can be reduced to the following expression. For convenience, we define

$$(GG)_{ik}^{ac} \equiv \sum_{p''} \hat{p}_i'' G^{ac}(p'')G^{ac}(q-p'')\hat{p}_k''\,. \tag{4.7}$$

Integrating out the frequency ϵ'', we have

$$(GG)^{11} = i\int_{\mathbf{p}} \frac{n_{\mathbf{p}} + n_{\mathbf{q}-\mathbf{p}} - 1}{\omega - \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{q}-\mathbf{p}} + i0}\hat{p}_i\hat{p}_k + (GG)^{12}\,, \tag{4.8}$$

$$(GG)^{12} = -2\pi\int_{\mathbf{p}} \delta(\omega - \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{q}-\mathbf{p}})n_{\mathbf{p}}n_{\mathbf{q}-\mathbf{p}}\hat{p}_i\hat{p}_k\,, \tag{4.9}$$

$$(GG)^{21} = -2\pi\int_{\mathbf{p}} \delta(\omega - \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{q}-\mathbf{p}})\bar{n}_{\mathbf{p}}\bar{n}_{\mathbf{q}-\mathbf{p}}\hat{p}_i\hat{p}_k\,, \tag{4.10}$$

$$(GG)^{22} = -i\int_{\mathbf{p}} \frac{n_{\mathbf{p}} + n_{\mathbf{q}-\mathbf{p}} - 1}{\omega - \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{q}-\mathbf{p}} + i0}\hat{p}_i\hat{p}_k + (GG)^{21}\,, \tag{4.11}$$

where we have changed the dummy variable $\mathbf{p}'' \to \mathbf{p}$ in the integrals for simplicity, and $\bar{n}_{\mathbf{p}} \equiv 1 - n_{\mathbf{p}}$. The distribution function $n_{\mathbf{p}}$ can be a nonequilibrium distribution.

We also decompose the $(GG)^{ac}$ into the parallel ($\|$) and the perpendicular ($\perp$) components, $(GG)^{ac}_{ik} = \sum_{\lambda=\|,\perp}(GG)^{ac}_{\lambda} P^{\lambda}_{ik}$. Thus the above equation (4.5) can be decomposed and written as

$$\left[\frac{1}{3iV}\check{\tau}_3^{ac} - (GG)^{ac}_{\lambda}\right]\Gamma^{cb}_{\lambda} = 1\delta^{ab} \tag{4.12}$$

for $\lambda = \|, \perp$. Define

$$L^{\mathrm{GL}}_{ik} \equiv \frac{1}{3iV} - i\int_{\mathbf{p}} \frac{n_{\mathbf{p}} + n_{\mathbf{q}-\mathbf{p}} - 1}{\omega - \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{q}-\mathbf{p}} + i0}\hat{p}_i\hat{p}_k, \tag{4.13}$$

and decompose it to $L^{\mathrm{GL}}_{ij} = \sum_{\lambda} L^{\mathrm{GL}}_{\lambda} P^{\lambda}_{ij}$ using the same decomposition. In equilibrium (see Appendix A), we have

$$L^{\mathrm{GL}}_{\lambda} = i\frac{N(0)}{3}\left(\theta + \xi^2_{\lambda} q^2 - i\frac{\pi}{8}\frac{\omega}{T}\right), \tag{4.14}$$

with the reduced temperature $\vartheta \equiv T/T_{\mathrm{c}} - 1$ and the length scales $\xi^2_{\|} = \frac{9}{5}\xi^2_0$ and $\xi^2_{\perp} = \frac{3}{5}\xi^2_0$, where $\xi^2_0 = \frac{7\zeta(3)}{48\pi^2}\frac{v_{\mathrm{F}}^2}{T_{\mathrm{c}}^2}$ is the s-wave coherence length. This equilibrium expression is the same as the result obtained from the Matsubara method [20]. The above matrix equation can be written as $L_{\lambda}\Gamma_{\lambda} = I$ with

$$L_{\lambda} \equiv \begin{bmatrix} L^{\mathrm{GL}}_{\lambda} - (GG)^{12}_{\lambda} & -(GG)^{12}_{\lambda} \\ -(GG)^{21}_{\lambda} & -L^{\mathrm{GL}}_{\lambda} - (GG)^{21}_{\lambda} \end{bmatrix}. \tag{4.15}$$

The inverse gives the vertex function

$$\begin{aligned}\Gamma_\lambda =& L_\lambda^{-1} \\ =& \frac{1}{|L_\lambda^{\mathrm{GL}}|^2}\begin{bmatrix} -L_\lambda^{\mathrm{GL}} - (GG)_\lambda^{21} & (GG)_\lambda^{12} \\ (GG)_\lambda^{21} & L_\lambda^{\mathrm{GL}} - (GG)_\lambda^{12}\end{bmatrix},\end{aligned} \tag{4.16}$$

where we have used the identity

$$(GG)_\lambda^{12} - (GG)_\lambda^{21} = L_\lambda^{\mathrm{GL}} + (L_\lambda^{\mathrm{GL}})^*, \tag{4.17}$$

which can be seen from (4.8)–(4.11) and (4.13). The factor $1/|L_\lambda^{\mathrm{GL}}|^2$ gives a singularity at the reduced temperature $\vartheta = 0$, signifying the onset of the phase transition.

4.2. Self-energy and the collision Integral

In this section, we consider the right-hand side of the kinetic equation (3.45). We will see that the two terms in the right-hand side give rise to the collision integral with the transition probability determined by the pair fluctuations. For the left-hand side of the kinetic equation (3.45), we keep only the self-energy from Landau interaction. Thus for the left-hand side we have the standard expression

$$\mathrm{LHS} = \partial_t n + \frac{\mathbf{p}}{m^*}\cdot\nabla_{\mathbf{R}} n - \nabla_{\mathbf{R}}\epsilon_{\mathbf{p}}\cdot\nabla_{\mathbf{p}} n, \tag{4.18}$$

where m^* is the quasiparticle mass, and

$$\epsilon_{\mathbf{p}} = \epsilon_{\mathbf{p}}^0 + \frac{1}{V}\sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma}\delta n_{\mathbf{p}'\sigma'} \tag{4.19}$$

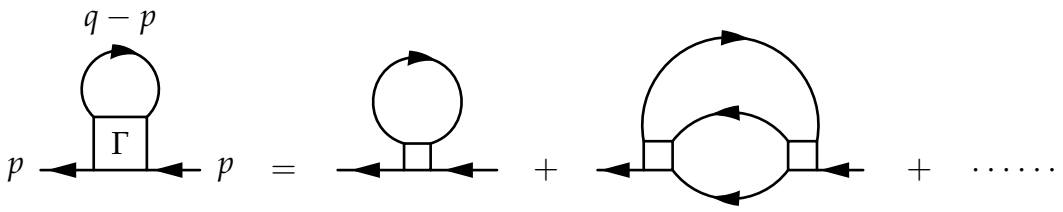


Figure 4.1. The fluctuation-induced self-energy, which is obtained from connecting an incoming and an outgoing line of the vertex Γ. On the right-hand side, the vertex Γ is expanded as a series of ladder diagrams, as defined in (4.2) (we neglect the label V in the small squares here). Note that the vertex Γ not only contains the high-energy contribution, but also includes the low-energy propagators in the particle-particle channel. The magnitude of the vertex Γ is of order s^2.

with the Landau function $f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}$. For the right-hand side, we consider Σ^{12} and Σ^{21} by connecting an incoming and an outgoing lines of the vertex Γ obtained in the previous section, as shown in Figure 4.1. Note that the vertex Γ is obtained from summing over scattering processes in the Cooper channel, as shown in the Bethe-Salpeter equation (4.2). The order of magnitude of this diagram is discussed at the end of this section.

The self-energy is given by

$$\Sigma^{12}(p) = \sum_q \Gamma^{12}_{\alpha\beta,\gamma\delta}(p,p;q) G^{21}_{\beta\delta}(q-p) \tag{4.20}$$

and

$$\Sigma^{21}(p) = \sum_q \Gamma^{21}_{\alpha\beta,\gamma\delta}(p,p;q) G^{12}_{\beta\delta}(q-p)\,. \tag{4.21}$$

Note that the vertex and the Green functions are nonequilibrium. From Section 4.1, the Γ^{12} sector of the vertex without spin dependence is given by

$$\Gamma^{12} = \sum_{\lambda=\parallel,\perp} \frac{1}{|L_\lambda^{\mathrm{GL}}|^2} \hat{p}_i (GG)_\lambda^{12} P_{ij}^\lambda \hat{p}'_j \tag{4.22}$$

with $(GG)_\parallel = \sum_{i,j} (GG)_{ij} P_{ji}^\parallel$ and $(GG)_\perp = \frac{1}{2}\sum_{i,j} (GG)_{ij} P_{ji}^\perp$. Using the original expression of $(GG)^{12}$, we have

$$(GG)_\lambda^{12} = -2\pi \int_{\mathbf{p}''} \delta(\omega - \epsilon_{\mathbf{p}''} - \epsilon_{\mathbf{q}-\mathbf{p}''}) n(\epsilon_{\mathbf{p}''}) n(\omega - \epsilon_{\mathbf{p}''}) \begin{cases} (\hat{\mathbf{p}}'' \cdot \hat{\mathbf{q}})^2, & \lambda = \parallel \\ \frac{1}{2}(1 - (\hat{\mathbf{p}}'' \cdot \hat{\mathbf{q}})^2), & \lambda = \perp \end{cases} . \tag{4.23}$$

To compare the result with Emery's expression [5], we take $\hat{\mathbf{q}}$ as the z direction, and write the expression in terms of spherical harmonics. Using

$$\hat{p}_i P_{ij}^\parallel \hat{p}'_j = (\hat{\mathbf{p}} \cdot \hat{\mathbf{q}})(\hat{\mathbf{p}}' \cdot \hat{\mathbf{q}}) = \frac{4\pi}{3} Y_{10}(\hat{\mathbf{p}}) Y_{10}^*(\hat{\mathbf{p}}') \tag{4.24}$$

and

$$\hat{p}_i P_{ij}^\perp \hat{p}'_j = (\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') - (\hat{\mathbf{p}} \cdot \hat{\mathbf{q}})(\hat{\mathbf{p}}' \cdot \hat{\mathbf{q}}) = \frac{4\pi}{3} \left[Y_{11}(\hat{\mathbf{p}}) Y_{11}^*(\hat{\mathbf{p}}') + Y_{1,-1}(\hat{\mathbf{p}}) Y_{1,-1}^*(\hat{\mathbf{p}}') \right], \tag{4.25}$$

we have

$$\hat{p}_i(GG)^{12}_{\lambda}P^{\lambda}_{ij}\hat{p}'_j = -2\pi\int_{\mathbf{p}''}\delta(\omega-\epsilon_{\mathbf{p}''}-\epsilon_{\mathbf{q}-\mathbf{p}''})f(\epsilon_{\mathbf{p}''})f(\omega-\epsilon_{\mathbf{p}''}) \times\begin{cases}\left(\frac{4\pi}{3}\right)^2 Y_{10}(\hat{\mathbf{p}}'')Y^*_{10}(\hat{\mathbf{p}}'')Y_{10}(\hat{\mathbf{p}})Y^*_{10}(\hat{\mathbf{p}}'), & \lambda=\| \\ \frac{1}{2}\left(\frac{4\pi}{3}\right)^2[Y_{11}Y^*_{11}+Y_{1,-1}Y^*_{1,-1}][Y_{11}Y^*_{11}+Y_{1,-1}Y^*_{1,-1}], & \lambda=\perp\end{cases}, \quad (4.26)$$

where the spherical harmonics in the second component has the same momentum variables as the first component. For the denominator $|L^{\mathrm{GL}}_{\lambda}|^2$, we consider only the singular part, which is obtained from its equilibrium value, given by the expression (4.14). For the self-energy, the incoming momentum must equal the outgoing momentum, $\mathbf{p}=\mathbf{p}'$. Thus using $Y_{11}(\hat{\mathbf{p}})Y^*_{11}(\hat{\mathbf{p}}) = Y_{1,-1}(\hat{\mathbf{p}})Y^*_{1,-1}(\hat{\mathbf{p}})$ we have

$$\hat{p}_i(GG)^{12}_{\lambda}P^{\lambda}_{ij}\hat{p}'_j = -2\pi\int_{\mathbf{p}''}\delta(\omega-\epsilon_{\mathbf{p}''}-\epsilon_{\mathbf{q}-\mathbf{p}''})n(\epsilon_{\mathbf{p}''})n(\omega-\epsilon_{\mathbf{p}''}) \times\begin{cases}\left(\frac{4\pi}{3}\right)^2 Y_{10}(\hat{\mathbf{p}}'')Y^*_{10}(\hat{\mathbf{p}}'')Y_{10}(\hat{\mathbf{p}})Y^*_{10}(\hat{\mathbf{p}}), & \lambda=\| \\ 2\left(\frac{4\pi}{3}\right)^2 Y_{11}(\hat{\mathbf{p}}'')Y^*_{11}(\hat{\mathbf{p}}'')Y_{11}(\hat{\mathbf{p}})Y^*_{11}(\hat{\mathbf{p}}), & \lambda=\perp\end{cases}. \quad (4.27)$$

Note the similarity between the expressions, except for the factor of 2 in the $\perp$ component. We can define

$$L^{\mathrm{GL}}_m = \begin{cases} L^{\mathrm{GL}}_{\|}, & m=0 \\ L^{\mathrm{GL}}_{\perp}, & m=\pm 1\end{cases}. \quad (4.28)$$

The factor of 2 in the $\lambda = \perp$ component is thus naturally split for the two $m = \pm 1$ modes. Thus we have

$$\begin{aligned}\Gamma^{12}(p,p;q) &= \sum_{\lambda=\parallel,\perp} \frac{1}{|L_\lambda^{\mathrm{GL}}|^2} \hat{p}_i (GG)_\lambda^{12} P_{ij}^\lambda \hat{p}_j \\ &= -2\pi \Big(\frac{4\pi}{3}\Big)^2 \int_{\mathbf{p}''} \delta(\omega - \epsilon_{\mathbf{p}''} - \epsilon_{\mathbf{q}-\mathbf{p}''}) f(\epsilon_{\mathbf{p}''}) f(\omega - \epsilon_{\mathbf{p}''}) \sum_m \left| \frac{1}{L_m^{\mathrm{GL}}} Y_{1m}(\hat{\mathbf{p}}) Y_{1m}^*(\hat{\mathbf{p}}'') \right|^2 .\end{aligned} \tag{4.29}$$

Including the spin part, the full vertex is

$$\Gamma^{ab}_{\alpha\beta,\gamma\delta}(p,p';q) = \Gamma^{ab}(p,p';q) \times \frac{1}{2} \mathbf{g}_{\alpha\beta} \cdot \mathbf{g}^\dagger_{\gamma\delta} , \tag{4.30}$$

where the vector $\mathbf{g}$, given by $g_i = i\sigma_i\sigma_y$, represents the spin-triplet component. Assuming the spectral function can be approximated by a delta peak, we have

$$G^{21}_{\sigma\sigma'}(q-p) \approx -2\pi i \delta(\omega - \epsilon - \epsilon_{\mathbf{q}-\mathbf{p}}) \big(1 - n(\omega - \epsilon)\big) \delta_{\sigma\sigma'} . \tag{4.31}$$

Plugging (4.30) and (4.31) into (4.20) and integrating out the frequency ω, we have

$$\begin{aligned}\Sigma^{12}_{\alpha\gamma}(p) = i \int_{\mathbf{q}} \int_{\mathbf{p}''} W(\epsilon_{\mathbf{p}''} + \epsilon_{\mathbf{q}-\mathbf{p}''}, \mathbf{q}) n(\epsilon_{\mathbf{p}''}) n(\epsilon_{\mathbf{q}-\mathbf{p}''}) \big(1 - n(\epsilon_{\mathbf{p}''} + \epsilon_{\mathbf{q}-\mathbf{p}''} - \epsilon)\big) \\ \times \delta(\epsilon_{\mathbf{p}''} + \epsilon_{\mathbf{q}-\mathbf{p}''} - \epsilon - \epsilon_{\mathbf{q}-\mathbf{p}}) \times \delta_{\alpha\gamma}\end{aligned} \tag{4.32}$$

with

$$\begin{aligned} W(\omega,\mathbf{q}) &= 3\pi\Big(\frac{4\pi}{3}\Big)^2\sum_m\left|\frac{1}{L_m^{\mathrm{GL}}(\omega,\mathbf{q})}Y_{1m}(\hat{\mathbf{p}})Y_{1m}^*(\hat{\mathbf{p}}'')\right|^2 \\ &= 3\pi\Big(\frac{4\pi}{N(0)}\Big)^2\sum_m\left|\frac{1}{\vartheta+\xi_m^2q^2-i\frac{\pi\omega}{8T}}Y_{1m}(\hat{\mathbf{p}})Y_{1m}^*(\hat{\mathbf{p}}'')\right|^2. \end{aligned} \tag{4.33}$$

The factor of 3 in front of the above expression comes from the spin part:

$$\frac{1}{2}\mathbf{g}_{\alpha\beta}\cdot\mathbf{g}^\dagger_{\gamma\beta}=\frac{3}{2}\delta_{\alpha\gamma}\,, \tag{4.34}$$

where the factor of 3 is related to the spin multiplicity $2S+1$ with $S=1$. For spin singlet ($S=0$), we have $\frac{1}{2}(i\sigma_y)(-i\sigma_y)=\frac{1}{2}$, corresponding to $2S+1=1$. Note that the spin structure is trivial, and so we neglect the $\delta_{\alpha\gamma}$ in the following equations.

The calculation for Σ^{21} is the same. We obtain

$$\begin{aligned} \Sigma^{21}_{\alpha\gamma}(p) = -i\int_{\mathbf{q}}\int_{\mathbf{p}''} W(\epsilon_{\mathbf{p}''}+\epsilon_{\mathbf{q}-\mathbf{p}''},\mathbf{q})(1-n(\epsilon_{\mathbf{p}''}))(1-n(\epsilon_{\mathbf{q}-\mathbf{p}''}))n(\epsilon_{\mathbf{p}''}+\epsilon_{\mathbf{q}-\mathbf{p}''}-\epsilon) \\ \times\,\delta(\epsilon_{\mathbf{p}''}+\epsilon_{\mathbf{q}-\mathbf{p}''}-\epsilon-\epsilon_{\mathbf{q}-\mathbf{p}})\times\delta_{\alpha\gamma}\,. \end{aligned} \tag{4.35}$$

Recall that the right-hand side of the kinetic equation (3.45) is

$$\mathrm{RHS} = -i\Sigma^{12}\big|_{\epsilon=\epsilon_{\mathbf{p}}}(1-n) - i\Sigma^{21}\big|_{\epsilon=\epsilon_{\mathbf{p}}}n\,. \tag{4.36}$$

Inserting the above results into this expression, the right-hand side has the form of collision integral,

$$\mathrm{RHS} = \int_{\mathbf{q}} \int_{\mathbf{p}'} W(\epsilon_{\mathbf{p}'} + \epsilon_{\mathbf{q}-\mathbf{p}'}, \mathbf{q}) \Big[n(\epsilon_{\mathbf{p}'}) n(\epsilon_{\mathbf{q}-\mathbf{p}'}) \big(1 - n(\epsilon_{\mathbf{p}'} + \epsilon_{\mathbf{q}-\mathbf{p}'} - \epsilon_{\mathbf{p}})\big) \big(1 - n(\epsilon_{\mathbf{p}})\big) - \big(1 - n(\epsilon_{\mathbf{p}'})\big) \big(1 - n(\epsilon_{\mathbf{q}-\mathbf{p}'})\big) n(\epsilon_{\mathbf{p}'} + \epsilon_{\mathbf{q}-\mathbf{p}'} - \epsilon_{\mathbf{p}}) n(\epsilon_{\mathbf{p}}) \Big] \delta(\epsilon_{\mathbf{p}'} + \epsilon_{\mathbf{q}-\mathbf{p}'} - \epsilon_{\mathbf{p}} - \epsilon_{\mathbf{q}-\mathbf{p}}) \,. \tag{4.37}$$

The transition probability W is given in Eq. (4.33).

The transition probability proposed by Emery [5, 10] using a heuristic argument is

$$W^{\mathrm{Emery}}(\omega, \mathbf{q}) = 6\pi \Big(\frac{4\pi}{N(0)} \Big)^2 \Big| \sum_m \frac{1}{\vartheta + \xi_m^2 q^2 - i \frac{\pi\omega}{8T}} Y_{1m}(\hat{\mathbf{p}}) Y_{1m}^*(\hat{\mathbf{p}}'') \Big|^2 . \tag{4.38}$$

In comparison with Emery's heuristic result, our collision integral has a smaller prefactor and *no interference* terms. Although the expressions of us and Emery seem similar, they have different origins. Emery's collision integral was obtained from application of Fermi's golden rule, while our expression is obtained from the self-energy given by the diagram in Figure 4.1. The golden rule can be associated to a diagram with two vertices, similar to the second-order diagram which gives rise to the collision integral in a perturbation expansion in terms of high-energy vertices [35]. Our vertex Γ has summed over all repeated scattering in the particle-particle channel, and thus calculating the second-order diagram using the effective vertex Γ, as done by Emery, doubly counts the ladder diagrams. The double counting may explain why Emery's expression has a prefactor twice larger than our result. More seriously,

Emery's expression contains the interference terms between different modes of fluctuations, which suggest different modes are added coherently. This is not reasonable since there is *no phase coherence* above the transition temperature, and incoherent pair fluctuations should not be added in this manner.[2] In contrast, our theory naturally generates the transition probability without interference terms, consistent with the nature of incoherent fluctuations. Although our expression (4.33) seems similar to Emery's (4.38), we emphasize that they are fundamentally different and show the importance of the microscopic theory.

4.2.1. Magnitude of the correction

The magnitude of the self-energy can be estimated as follows. From the expressions (4.32), the integral over Cooper pair momentum $\int d^3q$ gives rise to a dimensional factor $1/\xi_0^3$, and the integral $\int_{\mathbf{p}''} \sim N(0) \int d\epsilon_{\mathbf{p}''}$ together with a delta function gives a factor $N(0)$. Recall that the transition probability (4.33) has magnitude $W \sim 1/N(0)^2$. The overall magnitude for Σ^{12} is thus given by

$$\Sigma^{12} \sim \frac{1}{N(0)\xi_0^3} = \frac{T_c}{N(0)\xi_0^3 T_c}, \tag{4.39}$$

where the transition temperature T_c gives an estimate of the typical energy scale when the temperature is near T_c. The dimensionless parameter $\frac{1}{N(0)\xi_0^3 T_c}$ is a small number, as shown in Figure 4.2. The parameter $\frac{1}{N(0)\xi_0^3 T_c}$ is $\mathcal{O}(s^2)$, and the self-energy Σ^{12} is $\mathcal{O}(s^3)$ because T_c is of order s. *The estimate also applies to other self-energies, and thus the parameter controls all phenomena related to pair fluctuations*. The smallness of this

[2] This observation was pointed by Prof. Jim Sauls.

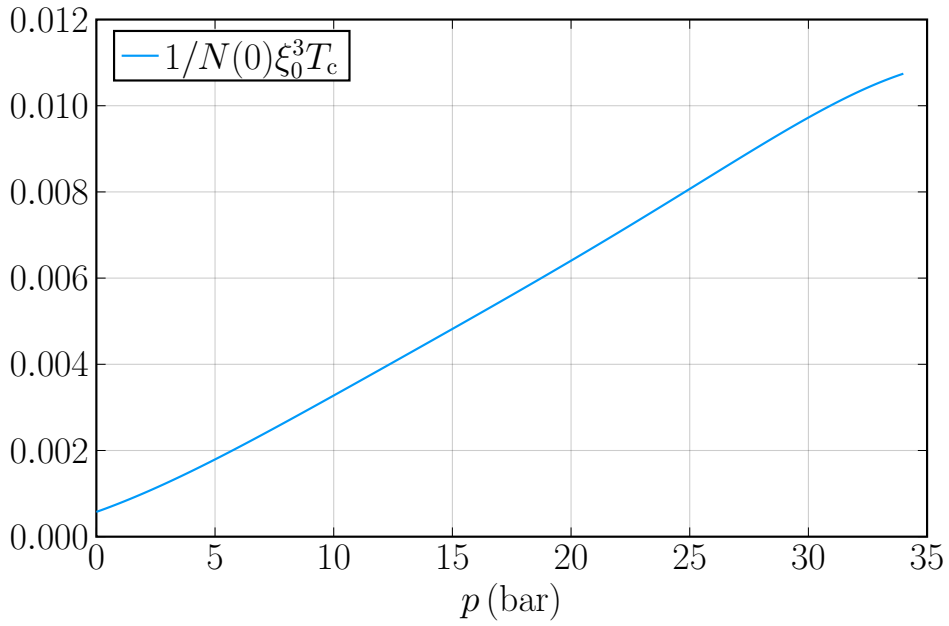


Figure 4.2. This figure shows the dimensionless parameter $\frac{1}{N(0)\xi_0^3 T_c}$. The value varies with the pressure. At low pressures, the value is on the order of 10^{-3}, and monotonically increases to about 10^{-2} at high pressures.

parameter makes all corrections due to pair fluctuations small. Indeed, the increase of zero sound attenuation near the superfluid transition is only a few percent in Paulson and Wheatley's experiment [8], as we shall see in the next section.

4.3. Zero sound attenuation from pair fluctuations

In this section, we calculate the zero sound attenuation caused by pair fluctuations. The estimate at the end of the last section shows that the magnitude of the collision integral is controlled by a small parameter, which means that the overall relaxation time is still much longer than the period of zero sound oscillation. Thus we can use the perturbation method developed in Section 2.7 to calculate the attenuation.

For the two incoming particles (and thus the two outgoing particles) comprising a pair fluctuation, their momenta are roughly opposite, that is, we can assume the total momentum $\mathbf{q} = 0$, except for the q^2 dependence in the transition probability W (see (4.33)). Hence the two particles have the same energy, and the collision integral (2.71) becomes

$$\delta I_{\mathbf{p}_1} = -\beta \int \frac{d^3p_2}{(2\pi)^3} \int \frac{d^3p_3}{(2\pi)^3} W(2\epsilon_1, \mathbf{q}) \left(\psi^{(0)}_{\hat{\mathbf{p}}_1} + \psi^{(0)}_{-\hat{\mathbf{p}}_1} - \psi^{(0)}_{\hat{\mathbf{p}}_3} - \psi^{(0)}_{-\hat{\mathbf{p}}_3} \right) \times \delta(2\epsilon_1 - 2\epsilon_3) n_{\mathrm{F}}(\epsilon_1)^2 \big(1 - n_{\mathrm{F}}(\epsilon_3)\big)^2 . \quad (4.40)$$

By carrying out the energy integral $\int d\epsilon_3$, we obtain

$$\delta I_{\mathbf{p}_1} = -\beta \int \frac{d^3q}{(2\pi)^3} N(0) \int \frac{d\Omega_{\hat{\mathbf{p}}_3}}{4\pi} W(2\epsilon_1, \mathbf{q}) \left(\psi^{(0)}_{\hat{\mathbf{p}}_1} + \psi^{(0)}_{-\hat{\mathbf{p}}_1} - \psi^{(0)}_{\hat{\mathbf{p}}_3} - \psi^{(0)}_{-\hat{\mathbf{p}}_3} \right) \times \frac{1}{2} n_{\mathrm{F}}(\epsilon_1)^2 \big(1 - n_{\mathrm{F}}(\epsilon_1)\big)^2 , \quad (4.41)$$

where we have changed the integral variable from $\mathbf{p}_2$ to $\mathbf{q} = \mathbf{p}_2 + \mathbf{p}_1$. The $\psi^{(0)}_{\hat{\mathbf{p}}}$ can be expanded as

$$\psi^{(0)}_{\hat{\mathbf{p}}} = \sum_{l \geq 0} \psi^{(0)}_l P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{k}}) , \quad (4.42)$$

where $\hat{\mathbf{k}}$ is the propagation direction of zero sound. We neglect the components with $l \geq 3$, and the $l = 0, 1$ components vanish because of particle number conservation and momentum conservation, so

$$\left(\psi^{(0)}_{\hat{\mathbf{p}}_1} + \psi^{(0)}_{-\hat{\mathbf{p}}_1} - \psi^{(0)}_{\hat{\mathbf{p}}_3} - \psi^{(0)}_{-\hat{\mathbf{p}}_3} \right) = 2\psi^{(0)}_2 \left[P_2(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{k}}) - P_2(\hat{\mathbf{p}}_3 \cdot \hat{\mathbf{k}}) \right] . \quad (4.43)$$

Recall that the numerator in the first-order solution (2.69) is

$$\text{Num} = \int \frac{d\Omega_{\hat{\mathbf{p}}_1}}{4\pi} \left[\int d\epsilon_{\mathbf{p}_1} \, \delta I_{\mathbf{p}_1}[\psi^{(0)}_{\hat{\mathbf{p}}_i}] \right] \psi^{(0)}_{\hat{\mathbf{p}}_1} \,. \tag{4.44}$$

Inserting the above expression for $\delta I_{\mathbf{p}_1}$ into this formula, we have

$$\begin{aligned} \text{Num} = -\beta N(0) \times 2\psi_2^{(0)} \int d\epsilon_1 \int \frac{d\Omega_{\hat{\mathbf{p}}_1}}{4\pi} \, \psi^{(0)}_{\hat{\mathbf{p}}_1} \int \frac{d^3q}{(2\pi)^3} \int \frac{d\Omega_{\hat{\mathbf{p}}_3}}{4\pi} \, W(2\epsilon_1, \mathbf{q}) \\ \times \left[P_2(\hat{\mathbf{p}}_1 \cdot \hat{\mathbf{k}}) - P_2(\hat{\mathbf{p}}_3 \cdot \hat{\mathbf{k}}) \right] \times \frac{1}{2} n_{\text{F}}(\epsilon_1)^2 \big(1 - n_{\text{F}}(\epsilon_1)\big)^2 . \end{aligned} \tag{4.45}$$

The angular variables $\hat{\mathbf{q}}$, $\hat{\mathbf{p}}_3$, and $\hat{\mathbf{p}}_1$ can then be integrated out easily.[3] We obtain

$$\text{Num} = -\beta \frac{27\pi}{N(0)} \frac{(\psi_2^{(0)})^2}{5} \int d\epsilon_1 \int \frac{q^2 dq}{2\pi^2} \left[\frac{7}{75} |t_0|^2 + 2 \times \frac{1}{4} \frac{32}{75} |t_1|^2 \right] n_{\text{F}}(\epsilon_1)^2 \big(1 - n_{\text{F}}(\epsilon_1)\big)^2 , \tag{4.46}$$

where

$$t_m \equiv \frac{1}{\vartheta + \xi_{1m}^2 q^2 - i \frac{\pi 2\epsilon_1}{8T}} , \quad m = 0, \pm 1 \tag{4.47}$$

is the singular part of the Cooper instability.

The remaining integrals $\int d\epsilon_1$ and $\int dq$ cannot be done analytically, and we also need some renormalization procedure to account for the short-wavelength contribution. Samalam and Serene considered a approximation with a cutoff procedure [10]. We repeat their analysis with our result above for comparison. We also consider another approach which does not require a cutoff.

[3] Note that the spherical harmonics in the expression (4.33) is defined with respect to the direction $\hat{\mathbf{q}}$, while the expansion of $\psi^{(0)}_{\hat{\mathbf{p}}}$ is defined relative to the propagation direction of zero sound, $\hat{\mathbf{k}}$.

4.3.1. Samalam and Serene's analysis

Samalam and Serene considered the approximation [10]

$$\beta n_{\rm F}(\epsilon_1)(1-n_{\rm F}(\epsilon_1)) \approx -n'_{\rm F}(0)\,. \tag{4.48}$$

This approximation, together with a cutoff in the q integral, leads to an analytic expression. The ϵ_1 integral is given by

$$\int_{-\infty}^{\infty} d\epsilon_1\ |t_m|^2 = \frac{4T}{\pi}\frac{\pi}{\vartheta+\xi_{1m}^2 q^2}\,. \tag{4.49}$$

The q integral has ultraviolet divergence in 3D space. To regularize the integral, we add a cutoff on the upper limit, which has an physical meaning that the pair fluctuations is only meaningful on the length scale larger than the coherence length ξ_0, and thus it is meaningless to include the modes with wave number $q \gtrsim \xi_0^{-1}$. Therefore, the q integral is given by

$$\begin{aligned}\int dq \frac{q^2}{\vartheta+\xi_{1m}^2 q^2} &= \frac{1}{\xi_{1m}^3}\int_0^{x_{\rm c}} dx \frac{x^2}{\vartheta+x^2} \\ &= \frac{1}{\xi_{1m}^3}\left[x_{\rm c} - \sqrt{\vartheta}\arctan\frac{x_{\rm c}}{\sqrt{\vartheta}}\right],\end{aligned} \tag{4.50}$$

where the dimensionless variable is $x \equiv q\xi_{1m}$, and a cutoff $x_{\rm c}$ has been added. Recall that $\xi_0^2 \equiv \xi_\parallel^2 = \frac{9}{5}\xi_0^2$ and $\xi_{\pm1}^2 \equiv \xi_\perp^2 = \frac{3}{5}\xi_0^2$, where ξ_0 is the coherence length for s-wave pairing. We obtain the numerator

$$\text{Num} = -\frac{\pi}{5}\frac{1}{m^* p_{\rm F}}\frac{27}{4}(\psi_2^{(0)})^2\left[\frac{7}{75}\sqrt{\frac{5}{9}}^3 + 2\frac{8}{75}\sqrt{\frac{5}{3}}^3\right]\times\frac{1}{\xi_0^3}(x_{\rm c}-\sqrt{\vartheta}\arctan\frac{x_{\rm c}}{\sqrt{\vartheta}})\,, \tag{4.51}$$

where we have used $N(0) = m^* p_{\rm F}/2\pi^2$. Note that although the parameter $x_{\rm c}$ depends on the length scale ξ_{1m}, only one parameter $x_{\rm c}$ goes into the final expression. This is because ξ_{10} and $\xi_{1,\pm1}$ are of the same order, and the cutoff should not be considered to be a hard upper limit. Instead, it is just a parameter to exclude unphysical modes.

The denominator of (2.69) is

$$\text{Denom} = \sum_{l\geq 0} \frac{(\nu_l^{(0)})^2}{2l+1}\Big(1 + \frac{F_l^{\rm s}}{2l+1}\Big)\,, \tag{4.52}$$

which can be reduced to a simple expression since we assume $\nu_l = 0$ for $l > 2$. From the equation for zero sound (2.41), we can obtain

$$\nu_1 = \frac{3s_0}{1 + F_1^{\rm s}/3}\nu_0 \tag{4.53}$$

and

$$\nu_2 = \frac{15}{2}\frac{s_0^2 - s_1^2}{(1 + F_1^{\rm s}/3)(1 + F_2^{\rm s}/5)}\nu_0\,. \tag{4.54}$$

See the article by Baym and Pethick for more details [22]. The hydrodynamic sound velocity is given by [2]

$$s_1^2 = \frac{1}{3}(1 + F_0^{\rm s})(1 + F_1^{\rm s}/3)\,, \tag{4.55}$$

where $s_1 \equiv c_1/v_{\rm F}$, and as discussed in (2.42),

$$s_0^2 - s_1^2 \approx \frac{4}{5}\frac{1 + F_2^{\rm s}/5}{1 + F_0^{\rm s}}s_1^2 = \frac{4}{15}\Big(1 + \frac{F_1^{\rm s}}{3}\Big)\Big(1 + \frac{F_2^{\rm s}}{5}\Big)\,. \tag{4.56}$$

Thus we have $\nu_2 = 2\nu_0$. Inserting these relations into the denominator, we have

$$\begin{aligned}\text{Denom} &= \nu_0^2(1+F_0^s) + \nu_0^2\frac{3s_0^2}{1+F_1^s/3} + \frac{4}{5}\nu_0^2\Big(1+\frac{F_2^s}{5}\Big) \\ &= \frac{3\nu_0^2}{1+F_1^s/3}\left(\frac{1}{3}(1+F_0^s)\Big(1+\frac{F_1^s}{3}\Big) + s_0^2 + \frac{4}{15}\Big(1+\frac{F_1^s}{3}\Big)\Big(1+\frac{F_2^s}{5}\Big)\right) \\ &= 3\nu_0^2\frac{m}{m^*}2s_0^2,\end{aligned} \tag{4.57}$$

where we have used the relation $m^*/m = 1 + F_1^s/3$ [2].

Inserting the numerator (4.51) and denominator (4.57) into the formula of zero sound attenuation (2.69), we obtain

$$\delta\alpha_0 = \frac{\pi}{5}\frac{1}{c_0 s_0^2}\frac{1}{m p_F}\Big(1+\frac{F_2^s}{5}\Big)^2 \times \frac{2.24}{\xi_0^3} \times \left[x_c - \sqrt{\vartheta}\arctan(x_c/\sqrt{\vartheta})\right], \tag{4.58}$$

where we have used the relation (2.66) and $\nu_2 = 2\nu_0$, and the number 2.24 is from $\frac{9}{2}\left[\frac{7}{75}\sqrt{\frac{5}{9}}^3 + 2\times\frac{8}{75}\sqrt{\frac{5}{3}}^3\right] \approx 2.24$. Except that the constant in our expression is 2.24 while the corresponding constant in Samalam and Serene is 3.08 [10], the result is the same, even though the transition probability we used is different. In this sense, Emery's heuristic transition probability does capture some physics.

4.3.2. Analysis without cutoff

The above calculation is based on the approximation (4.48). Note that without the approximation, the occupation factor $n_F(\epsilon_1)(1-n_F(\epsilon_1))$ has exponential decay when $|\beta\epsilon_1| \gg 1$, which makes the whole integral convergent, without the need to introduce

a cutoff. The q integral in (4.46) can be done first by

$$\int_0^\infty q^2 dq |t_m|^2 = \frac{1}{\xi_{1m}^3} \frac{i\pi}{4} \frac{\sqrt{\vartheta - ix} - \sqrt{\vartheta + ix}}{x}, \tag{4.59}$$

where $x \equiv 2\pi\epsilon_1/8T$. The occupation factor can be written as

$$\begin{aligned} n_{\mathrm{F}}(\epsilon_1)(1 - n_{\mathrm{F}}(\epsilon_1)) &= \cosh^{-2}(\epsilon_1/2T)/4 \\ &= \mathrm{sech}^2(2x/\pi)/4. \end{aligned} \tag{4.60}$$

The final result is the similar to the previous result, with the factor $x_{\mathrm{c}} - \sqrt{\vartheta}\arctan\frac{x_{\mathrm{c}}}{\sqrt{\vartheta}}$ replaced by a function

$$\Theta(\vartheta) \equiv \frac{1}{2}\int_{-\infty}^{\infty} dx \, \frac{\mathfrak{Im}\sqrt{\vartheta + i|x|}}{|x|} \times \mathrm{sech}^4(2x/\pi)\,. \tag{4.61}$$

This integral cannot be done analytically, but it is convergent for any ϑ. The function Θ, derived from the integral (4.59), includes contribution from short wavelengths, while fluctuations are relevant only in the long-wavelength region $q \lesssim 1/\xi_0$, as discussed earlier. The short-wavelength contribution is insensitive to the reduced temperature ϑ, and thus the unphysical short-wavelength contribution can be compensated by a constant counter term, which is independent of ϑ. Thus the expression becomes

$$\delta\alpha_0 = \frac{\pi}{5}\frac{1}{c_0 s_0^2}\frac{1}{m p_{\mathrm{F}}}\left(1 + \frac{F_2^{\mathrm{s}}}{5}\right)^2 \frac{2.24}{\xi_0^3}\Theta(\vartheta) + \delta\alpha_\infty\,. \tag{4.62}$$

The constant $\delta\alpha_\infty$, which represents the short-wavelength contribution, can be obtained by fitting experimental values of α. As shown in Figure 4.3, both expressions (4.58) and (4.62) can fit the experimental data. For the expression (4.58) with

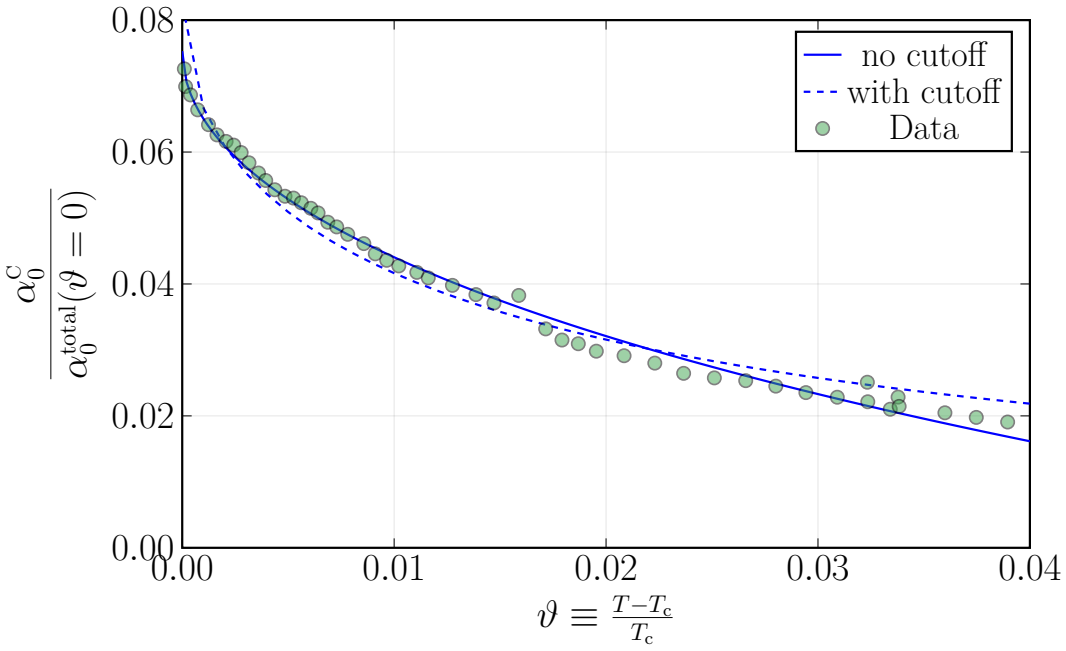


Figure 4.3. The blue curve is obtained from the equation (4.58), with x_c and F_2^s as the fitting parameters. The red curve is for (4.62), with $\delta\alpha_\infty$ and F_2^s as the fitting parameters. See the main text for more information. The data points are taken from Paulson and Wheatley's article [8].

the cutoff, the least square fit gives $x_c \approx 0.236$ and $F_2^s \approx 3.25$, while for the cutoff-free expression (4.62), the fit gives $\delta\alpha_\infty \approx -0.215$ and $F_2^s \approx 1.39$. The cutoff-free theory fits the data better than the theory with the cutoff, especially in the region near T_c. Moreover, the value $F_2^s = 1.39$ obtained from the cutoff-free theory is close to current experimental results.[4]

[4]We do not have a precise value for F_2^s, but at high pressures the value $F_2^s \sim 1$ is possible. See the summary plot given by Halperin's group in http://spindry.phys.northwestern.edu/3HeCalculator/F2s_plot.html, as well as the book by Dobbs [23].

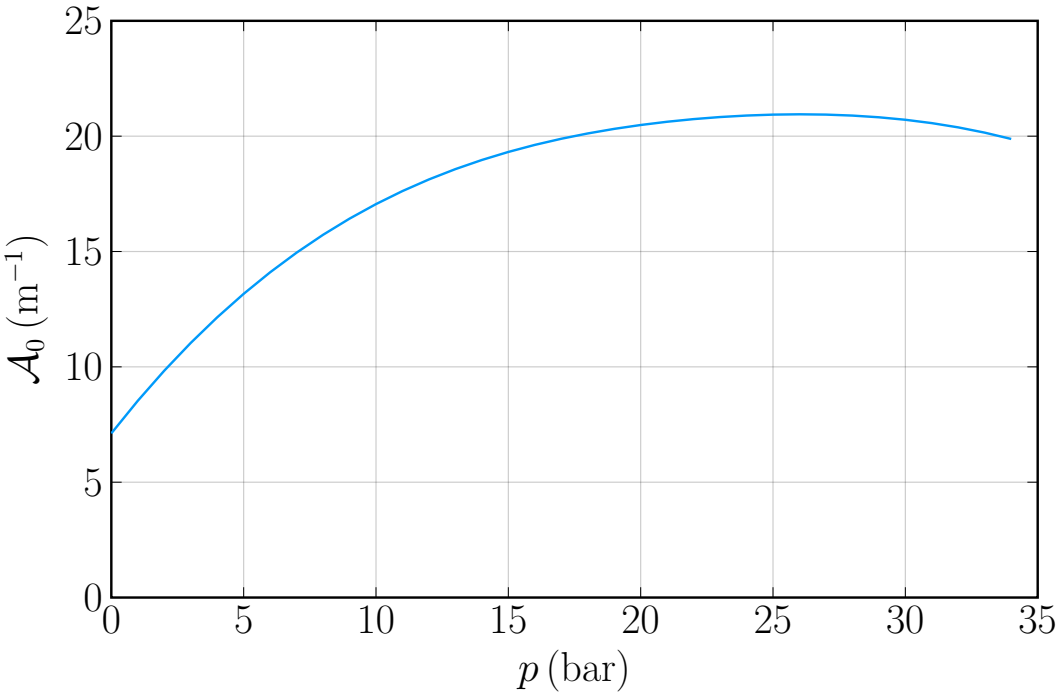


Figure 4.4. This figure gives pressure dependence of the magnitude of the excess attenuation, described by the prefactor (4.63). The pressure ranges from 0 to 34 bar. See the text.

The magnitude of the excess attenuation is controlled by the prefactor in the expression (4.58) (or (4.62)),

$$\mathcal{A}_0 \equiv \frac{\pi}{5}\frac{1}{c_0 s_0^2}\frac{1}{m p_{\mathrm{F}}}\Big(1+\frac{F_2^{\mathrm{s}}}{5}\Big)^2\frac{2.24}{\varsigma_0^3}\,. \tag{4.63}$$

This quantity is independent of the reduced temperature ϑ, and its pressure dependence is given in Figure 4.4. Over the pressure range from 0 to 34 bars, the zero sound attenuation at $T = T_{\mathrm{c}}$ ranges from $150\,\mathrm{m}^{-1}$ to $250\,\mathrm{m}^{-1}$, which are obtained from data fitting and extrapolation [43]. Thus the excess attenuation is a few percent of the total

attenuation. The Figure 4.4 shows that at high pressures the excess attenuation tends to be larger, which is consistent with the report by Paulson and Wheatley [8].

4.4. Corrections to Fermi velocity and Landau parameters

The Cooper pair fluctuations not only generate a collision integral, but also modify the parameters of the Fermi liquid. To see the change of parameters, we focus on the left-hand side of equation (3.45), and neglect the right-hand side. Thus we have

$$(\partial_t + \mathbf{v}_\mathbf{p} \cdot \nabla_\mathbf{R}) n_\mathbf{p} - \nabla_\mathbf{R} \mathfrak{Re}\Sigma^{11}\big|_{\epsilon=\epsilon_\mathbf{p}} \cdot \nabla_\mathbf{p} n_\mathbf{p} + \nabla_\mathbf{p} \mathfrak{Re}\Sigma^{11}\big|_{\epsilon=\epsilon_\mathbf{p}} \cdot \nabla_\mathbf{R} n_\mathbf{p} - \partial_\epsilon \mathfrak{Re}\Sigma^{11}\big|_{\epsilon=\epsilon_\mathbf{p}} \partial_t n_\mathbf{p} = 0\,. \quad (4.64)$$

The real part of the self-energy Σ^{11} determines the Fermi velocity (or equivalently, the effective mass), and the Landau parameters. As can be shown by the quasiclassical method [35], the dominating contribution to these parameters comes from the molecular field, i.e., the mean-field contribution. When the temperature is near T_c, the contribution from the diagram in Figure 4.1, which contains the effect of Cooper pair fluctuations, becomes strong. In this section we calculate the corrections from the fluctuation self-energy by considering the equation (4.64). We first separate the equation into two parts. The first part comes from the molecular field, giving the Landau's kinetic equation with the standard parameters. The second part is given by

the fluctuation self-energy. Thus the equation can be written as

$$\big(1-\partial_\epsilon \mathfrak{Re}\Sigma^{11}_{\rm fluc}\big|_{\epsilon=\epsilon_{\mathbf{p}}}\big)\partial_t n_{\mathbf{p}} + \big(\mathbf{v}_{\mathbf{p}} + \nabla_{\mathbf{p}}\mathfrak{Re}\Sigma^{11}_{\rm fluc}\big|_{\epsilon=\epsilon_{\mathbf{p}}}\big)\cdot\nabla_{\mathbf{R}} n_{\mathbf{p}} - \nabla_{\mathbf{R}}\big(\delta\epsilon_{\mathbf{p}} + \mathfrak{Re}\Sigma^{11}_{\rm fluc}\big|_{\epsilon=\epsilon_{\mathbf{p}}}\big)\cdot\nabla_{\mathbf{p}} n_{\mathbf{p}} = 0\,, \quad (4.65)$$

where the velocity $v_{\rm F} \equiv \mathbf{p}/m^* = \partial_{\mathbf{p}}\epsilon^0_{\mathbf{p}}$ and the energy from the nonequilibrium distribution is given by

$$\delta\epsilon_{\mathbf{p}} = \frac{1}{V}\sum_{\mathbf{p}'\sigma'} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}\delta n_{\mathbf{p}'\sigma'}\,. \quad (4.66)$$

Note that the coefficient of the time derivative term is also modified. Assume the nonequilibrium is not strong and the kinetic equation can be linearized. To the leading order, we have

$$\big(1-\partial_\epsilon \mathfrak{Re}\Sigma^{11}_{\rm fluc}\big|^{\rm eq}_{\epsilon=\epsilon_{\mathbf{p}}}\big)\partial_t \delta n_{\mathbf{p}} + \big(\mathbf{v}_{\mathbf{p}} + \nabla_{\mathbf{p}}\mathfrak{Re}\Sigma^{11}_{\rm fluc}\big|^{\rm eq}_{\epsilon=\epsilon_{\mathbf{p}}}\big)\cdot\nabla_{\mathbf{R}}\delta n_{\mathbf{p}} - \nabla_{\mathbf{R}}\big(\delta\epsilon_{\mathbf{p}} + \mathfrak{Re}\delta\Sigma^{11}_{\rm fluc}\big|_{\epsilon=\epsilon_{\mathbf{p}}}\big)\cdot\nabla_{\mathbf{p}} n^{\rm eq}_{\mathbf{p}} = 0\,, \quad (4.67)$$

Note that the for the first two terms, the self-energy $\Sigma^{11}_{\rm fluc}$ is evaluated with the equilibrium distribution, while for the last term, the small deviation $\delta\Sigma^{11}_{\rm fluc}$ due to the nonequilibrium distribution δn is considered. We drop the subscript "fluc" in the following.

Recall that $\mathfrak{Re}\Sigma^{11} = (\Sigma^{11} - \Sigma^{22})/2$. Using

$$\Sigma^{11}(p) = \sum_q \Gamma^{11}_{\alpha\beta,\gamma\delta}(p,p;q) G^{11}_{\beta\delta}(q-p) \tag{4.68}$$

$$\Sigma^{22}(p) = \sum_q \Gamma^{22}_{\alpha\beta,\gamma\delta}(p,p;q) G^{22}_{\beta\delta}(q-p) \tag{4.69}$$

and

$$\Gamma^{11} = \sum_\lambda \hat{p}_i \frac{-L^{\mathrm{GL}}_\lambda - (GG)^{(21)}_\lambda}{|L^{\mathrm{GL}}_\lambda|^2} P^\lambda_{ij} \hat{p}'_j \tag{4.70}$$

$$\Gamma^{22} = \sum_\lambda \hat{p}_i \frac{L^{\mathrm{GL}}_\lambda - (GG)^{(12)}_\lambda}{|L^{\mathrm{GL}}_\lambda|^2} P^\lambda_{ij} \hat{p}'_j \tag{4.71}$$

and $G^{11*}(p) = -G^{22}(p)$, we obtain

$$\begin{aligned}\frac{\Sigma^{11} - \Sigma^{22}}{2} = \frac{1}{2}\int_Q \sum_\lambda \left[\hat{p}_i \frac{-(GG)^{21}_\lambda - (GG)^{12}_\lambda}{|L^{\mathrm{GL}}_\lambda|^2} P^\lambda_{ij} \hat{p}_j\right] \mathfrak{Re} G^{11}(q-p) \\ + \frac{1}{2}\int_Q \sum_\lambda \left[\hat{p}_i \frac{(L^{\mathrm{GL}}_\lambda)^* - L^{\mathrm{GL}}_\lambda}{|L^{\mathrm{GL}}_\lambda|^2} P^\lambda_{ij} \hat{p}_j\right] i\mathfrak{Im} G^{11}(q-p)\,. \end{aligned} \tag{4.72}$$

We will evaluate this quantity in equilibrium for the time derivative and the Fermi velocity, and in nonequilibrium for the Landau parameters.[5]

4.4.1. Fermi velocity and the time derivative

The self-energy term for the Fermi velocity and the time derivative in (4.67) is calculated in equilibrium. The relevant expressions can be found in Section 4.1. Recall that

[5] Recall that the Landau interaction modifies the quasiparticle energy by $\delta\epsilon \sim F\delta n$, which is proportional to the nonequilibrium distribution δn.

in equilibrium

$$(GG)^{12} = -\pi\frac{N(0)}{3}\Big[\Big(\frac{1}{4} - \frac{\omega}{8T}\Big)\delta_{ik} - \frac{3\pi^2}{4\times 5\times 7\zeta(3)}\xi_0^2 q^2(\delta_{ik} + 2\hat{q}_i\hat{q}_k)\Big]\,, \tag{4.73}$$

and using $(GG)^{12}(q) = (GG)^{21}(-q)$, we have

$$\begin{aligned}(GG)^{12}_{ij} + (GG)^{21}_{ij} &= -\pi\frac{N(0)}{3}\Big[\frac{1}{2}\delta_{ij} - \frac{1}{2\times 16\times 5}\frac{v_{\rm F}^2 q^2}{T^2}(\delta_{ij} + 2\hat{q}_i\hat{q}_j)\Big] \\ &= -\pi\frac{N(0)}{3}\Big[\frac{1}{2}(P^{\parallel}_{ij} + P^{\perp}_{ij}) - \frac{3\pi^2}{2\times 5\times 7\zeta(3)}\xi_0^2 q^2(3P^{\parallel}_{ij} + P^{\perp}_{ij})\Big]\,.\end{aligned} \tag{4.74}$$

The above expression can be separated into the two components $\lambda = \parallel, \perp$. For the first term in the expression (4.72) we have

$$(GG)^{12}_{\lambda} + (GG)^{21}_{\lambda} = -\pi\frac{N(0)}{6}\Big[1 - \frac{\pi^2}{7\zeta(3)}\xi_\lambda^2 q^2\Big] \tag{4.75}$$

with $\xi^2_{\parallel} = \frac{9}{5}\xi_0^2$ and $\xi^2_{\perp} = \frac{3}{5}\xi_0^2$. For the second term in (4.72), we have

$$-L^{\rm GL}_{\lambda} + (L^{\rm GL}_{\lambda})^* = -2i\frac{N(0)}{3}(\vartheta + \xi_\lambda^2 q^2)\,, \tag{4.76}$$

and the denominator is

$$|L^{\rm GL}_{\lambda}|^2 = \frac{N(0)^2}{9}\Big[(\vartheta + \xi_\lambda^2 q^2)^2 + \big(\frac{\pi\omega}{8T}\big)^2\Big]\,, \tag{4.77}$$

which contains the singular part. Using the relation $G^{11} = G^{\rm R} + G^{12}$, the time-ordered Green function can be written as

$$G^{11}(q-p) = {\rm P}\frac{1}{\omega - \epsilon - \epsilon_{\mathbf{q}-\mathbf{p}}} + i\pi\delta(\omega - \epsilon - \epsilon_{\mathbf{q}-\mathbf{p}})[2n(\omega-\epsilon) - 1]\,. \tag{4.78}$$

From the equation (4.67), the correction is $\nabla_{\mathbf{p}} \mathfrak{Re}\Sigma^{11}_{\text{fluc}}\big|^{\text{eq}}_{\epsilon=\epsilon_{\mathbf{p}}}$. We can write the derivative as $\mathbf{v}_{\mathbf{p}}\partial_{\epsilon_{\mathbf{p}}}$. Plugging these expressions into (4.72) for $\mathfrak{Re}\Sigma^{11}_{\text{fluc}}$ and taking the derivative give the result

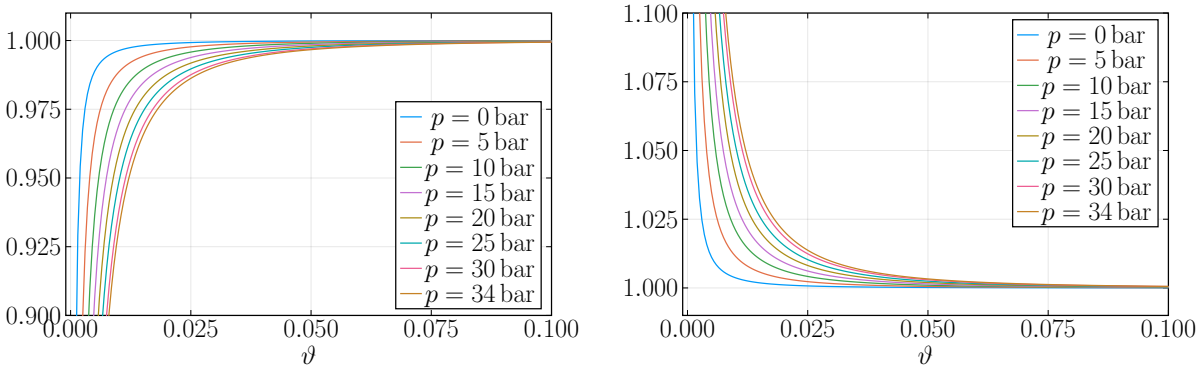
$$\begin{aligned}\frac{\delta v_{\text{F}}}{v_{\text{F}}} = -\frac{4.7174}{2^{10}}\frac{1}{N(0)T_{\text{c}}\xi_0^3} \times \Big[&\frac{\arctan\frac{x_{\text{c}}}{\sqrt{\vartheta}}}{\sqrt{\vartheta}^3} + \frac{x_{\text{c}}^3 - \vartheta x_{\text{c}}}{\vartheta x_{\text{c}}^4 + 2\vartheta^2 x_{\text{c}}^2 + \vartheta^3} \\ &- \frac{\pi^2}{7\zeta(3)}\Big(\frac{3\arctan\frac{x_{\text{c}}}{\sqrt{\vartheta}}}{\sqrt{\vartheta}} - \frac{5x_{\text{c}}^3 + 3\vartheta x_{\text{c}}}{x_{\text{c}}^4 + 2\vartheta x_{\text{c}}^2 + \vartheta^2}\Big)\Big] \\ -5.975 \times 10^{-2} \times \frac{1}{N(0)T_{\text{c}}\xi_0^3}&\Big[x_{\text{c}} - \sqrt{\vartheta}\arctan\frac{x_{\text{c}}}{\sqrt{\vartheta}}\Big],\end{aligned} \tag{4.79}$$

where we have taken $\epsilon_{\mathbf{p}} = 0$. This correction is always negative, so the fluctuations reduce the Fermi velocity.

The correction to the constant in front of the time derivative (see (4.67)) can be calculated in a similar way. The result is

$$\begin{aligned}\delta = \frac{4.7174}{2^{10}}\frac{1}{N(0)T_{\text{c}}\xi_0^3} \times \Big[&\frac{\arctan\frac{x_{\text{c}}}{\sqrt{\vartheta}}}{\sqrt{\vartheta}^3} + \frac{x_{\text{c}}^3 - \vartheta x_{\text{c}}}{\vartheta x_{\text{c}}^4 + 2\vartheta^2 x_{\text{c}}^2 + \vartheta^3} \\ &- \frac{\pi^2}{7\zeta(3)}\Big(\frac{3\arctan\frac{x_{\text{c}}}{\sqrt{\vartheta}}}{\sqrt{\vartheta}} - \frac{5x_{\text{c}}^3 + 3\vartheta x_{\text{c}}}{x_{\text{c}}^4 + 2\vartheta x_{\text{c}}^2 + \vartheta^2}\Big)\Big]\end{aligned} \tag{4.80}$$

and the time derivative becomes $(1+\delta)\partial_t$. The result of the Fermi velocity and the time constant is shown in Figure 4.5, using the cutoff $x_{\text{c}} = 0.236$, which is a reasonable estimate obtained from the Paulson and Wheatley's data [8]; see the result from Section 4.3.



(a) (b)

Figure 4.5. Corrections to the Fermi velocity and the constant in front of the time derivative. Panel (a) gives $1 + \delta v_F / v_F$, using the formula (4.79) with different pressures, and Panel (b) gives $1 + \delta$, with the correction δ given by (4.80). These curves are calculated with the cutoff $x_c = 0.236$.

4.4.2. Landau parameters

The corrections to Landau parameters depend on the nonequilibrium distribution. From the expressions in (4.8)–(4.11) and (4.13), we see that the real part of Σ^{11} (4.72) depends on the distribution function in a complicated way. To simplify the calculation, we keep the denominator the same as its equilibrium value, $L^{GL} \sim (\vartheta + \xi_\lambda^2 q^2 - i\pi\omega/8T)^{-1}$, which gives the singular contribution as $\vartheta \to 0$. This is the same as what we do in the calculation of the collision integral, and is justified for linear response. Moreover, we assume that the nonequilibrium affects the system mainly through the explicit dependence of the distribution function (i.e. through the G^{12} and G^{21} lines in the diagrams), and the implicit dependence involved in the quasiparticle energy $\epsilon_{\mathbf{p}}$ has little influence on the ladder diagrams.

The self-energy has dependence on the distribution function through different Green functions in the expression (4.72). As can be seen in the last diagram in Figure 4.1, the dependence on the distribution can be from the three internal lines. In the following, we use the symbols $p_2 \equiv q - p$, $p_3 \equiv p'$, $p_4 \equiv q - p'$, and $p_1 \equiv p$. We will see the energy shift due to nonequilibrium distribution comes from the distribution of the particle $2, 3$ and 4. From the energy shift, we can obtain the correction to the Landau parameters.

Consider the first term in the right-hand side of (4.72). The real part of the time-order Green function (4.78) has no explicit dependence on the distribution function, and thus we only need to consider

$$(GG)^{21}_{ij} + (GG)^{12}_{ij} = -2\pi \int_{\mathbf{p}'} \delta(\omega - \epsilon_{\mathbf{p}'} - \epsilon_{\mathbf{q}-\mathbf{p}'}) \times [n_{\mathbf{p}'} n_{\mathbf{q}-\mathbf{p}'} + (1 - n_{\mathbf{p}'})(1 - n_{\mathbf{q}-\mathbf{p}'})] \hat{p}'_i \hat{p}'_j \quad (4.81)$$

from (4.9) and (4.10). Let $n_{\mathbf{p}'} = n^{\mathrm{eq}}(\epsilon_{\mathbf{p}'}) + \delta n_{\mathbf{p}'}$. Recall that we want the first-order variation $\delta\Sigma$. Consider

$$-\delta\big((GG)^{21} + (GG)^{12}\big)_{ij} = 2\pi \int_{\mathbf{p}'} \delta(\omega - \epsilon_3 - \epsilon_4) \times \big[-\delta n_3 - \delta n_4 + 2n_3 \delta n_4 + 2n_4 \delta n_3 \big] \hat{p}'_i \hat{p}'_j \,, \quad (4.82)$$

where the subscripts 3 and 4 represent $\mathbf{p}'$ and $\mathbf{q}-\mathbf{p}'$, and the superscript "eq" has been dropped. After integrating the energy ω, we have

$$\int \frac{d\omega}{2\pi}\frac{1}{2}\left(\frac{-\delta((GG)^{21}_{ij}+(GG)^{12}_{ij})}{|L^{\rm GL}_\lambda|^2}\right)\mathfrak{Re}G^{11} = \frac{1}{2}\int_{\mathbf{p}'}\frac{(2n_4-1)\delta n_3+(2n_3-1)\delta n_4}{|L^{\rm GL}_\lambda(\omega=\epsilon_3+\epsilon_4,\mathbf{q})|^2}\times\frac{1}{\epsilon_3+\epsilon_4-\epsilon-\epsilon_2}\hat{p}'_i\hat{p}'_j\,, \quad (4.83)$$

where the external energy is ϵ, and the energy $\epsilon_2\equiv\epsilon_{\mathbf{q}-\mathbf{p}}$ is for the internal fermion line in Figure 4.1.

For the second term in the right-hand side of (4.72), the variation δn can happen in both (GG) terms and $\mathfrak{Im}G^{11}$. From (4.13), the variation gives

$$\delta((L^{\rm GL}_{ij})^*-L^{\rm GL}_{ij}) = 2i\int_{\mathbf{p}'}\frac{\delta n_{\mathbf{p}'}+\delta n_{\mathbf{q}-\mathbf{p}'}}{\omega-\epsilon_{\mathbf{p}'}-\epsilon_{\mathbf{q}-\mathbf{p}'}+i0}\hat{p}'_i\hat{p}'_j\,. \quad (4.84)$$

Recall that

$$i\mathfrak{Im}G^{11}(q-p) = i\pi\delta(\omega-\epsilon-\epsilon_2)[2n(q-p)-1]\,. \quad (4.85)$$

The ω integral gives

$$\int \frac{d\omega}{2\pi}\frac{1}{2}\left(\frac{\delta((L^{\rm GL}_{ij})^*-L^{\rm GL}_{ij})}{|L^{\rm GL}_\lambda|^2}\right)i\mathfrak{Im}G^{11} = -\frac{1}{2}\int_{\mathbf{p}'}\frac{(2n_2-1)}{|L^{\rm GL}_\lambda(\omega=\epsilon+\epsilon_2,\mathbf{q})|^2}\times\frac{(\delta n_3+\delta n_4)}{\epsilon+\epsilon_2-\epsilon_3-\epsilon_4}\hat{p}'_i\hat{p}'_j\,. \quad (4.86)$$

For the $\mathfrak{Im}G^{11}$,

$$i\delta\mathfrak{Im}G^{11}(q-p) = i\pi\delta(\omega-\epsilon-\epsilon_2)\times 2\delta n_2\,. \quad (4.87)$$

In equilibrium,

$$-L_\lambda^{\rm GL}+(L_\lambda^{\rm GL})^*=-2i\frac{N(0)}{3}(\vartheta+\xi_\lambda^2 q^2)\,. \tag{4.88}$$

Thus the ω integral gives

$$\int\frac{d\omega}{2\pi}\frac{1}{2}\frac{-L_\lambda^{\rm GL}+(L_\lambda^{\rm GL})^*}{|L_\lambda^{\rm GL}|^2}i\delta\Im\mathrm{m}G^{11}=\frac{1}{2}\frac{\frac{N(0)}{3}(\vartheta+\xi_\lambda^2 q^2)}{|L_\lambda^{\rm GL}(\omega=\epsilon+\epsilon_2,\mathbf{q})|^2}\times 2\delta n_2\times\hat{p}_i' P_{ij}^\lambda\hat{p}_j'\,. \tag{4.89}$$

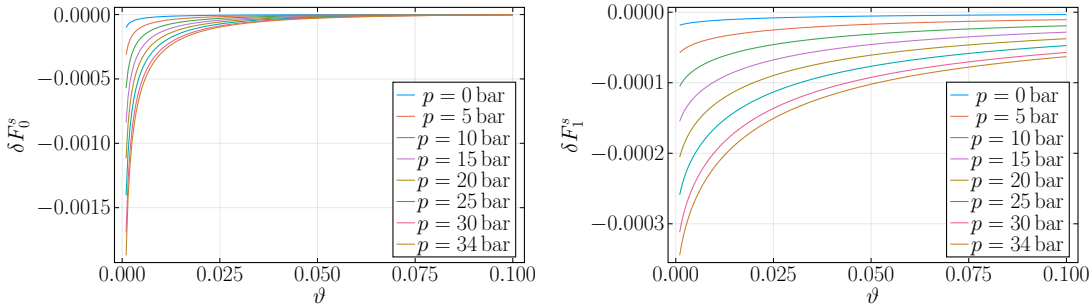
Note that the $\mathbf{p}'$ integral has been done in this case. After the integration over $\mathbf{q}$, the sum of (4.83), (4.86) and (4.89) gives the correction to the Landau parameters. The detail of the calculation can be found in Appendix B. The results are the following.

$$\begin{aligned}\delta F_0^{\rm s}=&-4.617\times10^{-3}\frac{1}{N(0)\xi_0^3 T_{\rm c}}\left[\frac{\arctan\frac{x_{\rm c}}{\sqrt{\vartheta}}}{\sqrt{\vartheta}}-\frac{x_{\rm c}}{x_{\rm c}^2+\vartheta}\right]\\&+3.136\times10^{-2}\frac{1}{N(0)T_{\rm c}\xi_0^3}\left[x_{\rm c}-\frac{3}{2}\sqrt{\vartheta}\arctan\frac{x_{\rm c}}{\sqrt{\vartheta}}+\frac{\vartheta x_{\rm c}}{2(x_{\rm c}^2+\vartheta)}\right]\\&+5.975\times10^{-2}\frac{1}{N(0)\xi_0^3 T_{\rm c}}\left[x_c-\sqrt{\vartheta}\arctan\frac{x_{\rm c}}{\sqrt{\vartheta}}\right]\end{aligned} \tag{4.90}$$

and

$$\begin{aligned}\delta F_1^{\rm s}=&\,1.211\times10^{-2}\frac{1}{N(0)T_{\rm c}\xi_0^3}\left[x_{\rm c}-\frac{3}{2}\sqrt{\vartheta}\arctan\frac{x_{\rm c}}{\sqrt{\vartheta}}+\frac{\vartheta x_{\rm c}}{2(x_{\rm c}^2+\vartheta)}\right]\\&-1.792\times10^{-1}\frac{1}{N(0)\xi_0^3 T_{\rm c}}\left[x_c-\sqrt{\vartheta}\arctan\frac{x_{\rm c}}{\sqrt{\vartheta}}\right].\end{aligned} \tag{4.91}$$

The result is shown in Figure 4.6 at different pressures, using $x_{\rm c}=0.236$. Note that the Landau parameters in the normal state are $F_0^{\rm s}\sim100$ and $F_1^{\rm s}\sim10$, and thus the corrections from our result give $\delta F_i^{\rm s}/F_i^{\rm s}\lesssim10^{-5}$ for both $i=0,1$, which is much smaller than the corrections to Fermi velocity and the time constant (see Figure 4.5).



(a) (b)

Figure 4.6. The figure (a) and (b) show the corrections to the Landau parameters F_0 and F_1 at different pressures. Note that the corrections depend on the reduced temperature. The corrections are larger when the temperature is closer to the transition temperature. The curves are given by Equation (4.90) and (4.91). The cutoff is fixed at $x_c = 0.236$.

As shown in Figure 4.5 and 4.6, most Fermi liquid parameters have infrared divergence at $\vartheta = 0$. The perturbative calculation based on the normal-state fermion propagators is not valid when ϑ is too close to 0, because the correction would be large and strongly modify the normal-state propagator. Nonetheless, if the temperature is not too close to T_c, the corrections are small and the perturbation method is valid.

4.4.3. Relation between the effective mass and the Landau parameter F_1^s

In Landau's Fermi liquid theory [2], there is a relation between the effective mass and the Landau parameter F_1^s for a Fermi liquid with Galilean invariance,

$$\frac{m^*}{m} = 1 + \frac{F_1^s}{3}\,. \tag{4.92}$$

This relation does not hold for the corrections from pair fluctuations. Indeed, the Fermi velocity correction is negative and thus gives a higher effective mass, while the

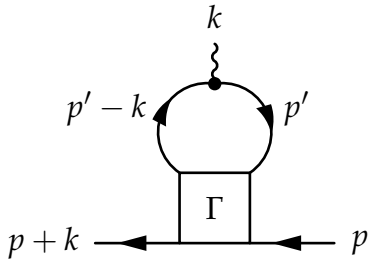


Figure 4.7. The diagrammatic representation of the right-hand side of the Ward identity.

F_1^s correction is negative. The inconsistency means the above relation holds only for the standard Landau interaction. The relation (4.92) can be derived from the Ward identity [31, 44]. The Ward identity from gauge invariance has the form [44]

$$\Sigma(p+k)-\Sigma(p)=i\sum_{p'}\Gamma G(p')G(p'-k)\Big(\omega-\frac{\mathbf{k}\cdot(\mathbf{2p'}-\mathbf{k})}{2m}\Big)\,, \tag{4.93}$$

where Σ is the self-energy for the fermion, and $\Gamma(p,p';k)$ is the two-particle vertex, with the momentum transfer k. Let $\omega=0$ and consider the limit $\mathbf{k}\to 0$. We have

$$\frac{\partial\Sigma(p)}{\partial\mathbf{p}}=-i\sum_{p'}\{\Gamma G^2\}^{\omega=0,k\to 0}\frac{\mathbf{p}'}{m}\,. \tag{4.94}$$

The right-hand side of the Ward identity can be represented in a diagram, shown in Figure 4.7. This identity is exact, without any approximation. We can see left-hand side of the identity is related to the Fermi velocity, and the right-hand side is related to the interaction. To derive the relation (4.92), we need further assumptions.

The key assumption is that the vertex Γ has no energy dependence. This assumption is reasonable under quasiclassical approximation and for the vertex obtained from the particle-hole channel with small momentum transfer. Using this assumption, we only need to sum (or integrate) the product $G(p')G(p'-k)$ over different

frequencies, which gives the factor

$$i\delta(\xi_{\mathbf{p}'})\frac{\mathbf{v}\cdot\mathbf{k}}{\omega-\mathbf{v}\cdot\mathbf{k}}\,. \tag{4.95}$$

The delta function $\delta(\xi_{\mathbf{p}'})$ restricts the fermion momenta to the Fermi surface. The $k\to 0$ limit after $\omega=0$ of the vertex gives the forward scattering amplitude [30], parameterized by A_l. Using the relation

$$1+\frac{F_l}{2l+1}=\frac{1}{1-\frac{A_l}{2l+1}}\,, \tag{4.96}$$

the relation (4.92) can be obtained.

The above assumption is not valid for the pair fluctuations. The vertex resulted from pair fluctuations depends on the total momentum and frequency of the incoming particles, $q\equiv p+p'=(\omega,\mathbf{q})$, and it has the form (see Section 4.1)

$$\Gamma\sim\frac{1}{\vartheta+\xi_0^2q^2-i\frac{\omega}{T}}\,. \tag{4.97}$$

The assumption of dropping the frequency dependence is no longer correct. Indeed, the calculation given in the previous section includes the frequency dependence, and it does give nontrivial result. Furthermore, the vertex from pair fluctuations is calculated in the particle-particle channel, and it only gives significant contribution when the total momentum q of the two incoming particles is small. This means that the two fermion momenta p and p' in the Ward identity (see the diagram in Figure 4.7) are almost opposite, $p'\approx -p$, while for the Landau interaction, the two incoming particles can have arbitrary momenta on the Fermi surface. The corrections obtained

in the previous section is calculated under the restriction of small total momentum q. Since this restriction put strong correlation between p and p', the result cannot be the same as the Landau interaction.

4.5. Velocity drop of zero sound

As formerly discussed, the self-energy contributes not only to the collision integral, but also to the quasiparticle energy. In the last section, we show that the Fermi velocity, the time derivative, and the interaction between quasiparticles in the kinetic equation are changed by the fluctuations. A direct result of these corrections is the velocity change of zero sound. Using the corrections obtained in the previous section, the kinetic equation becomes

$$((1+\delta)\omega - \mathbf{v}_{\hat{\mathbf{p}}}\cdot\mathbf{k})\nu_{\hat{\mathbf{p}}} - (\mathbf{v}_{\hat{\mathbf{p}}}\cdot\mathbf{k})\int\frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi}F^{\mathrm{s}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\nu_{\hat{\mathbf{p}}'} = 0\,, \tag{4.98}$$

where the Fermi velocity and Landau parameters are corrected by pair fluctuations. Note that the coefficient of the time derivative also gets a correction, given by the expression (4.80). We retain only the first three spherical harmonics, and the zero sound velocity is given by [22]

$$\frac{c_0^2}{v_{\mathrm{F}}^2} = \frac{1}{3}(1+F_0^{\mathrm{s}})(1+\frac{F_1^{\mathrm{s}}}{3}) + \frac{4}{15}(1+\frac{F_1^{\mathrm{s}}}{3})(1+\frac{F_2^{\mathrm{s}}}{5})\,. \tag{4.99}$$

Although we do not have a precise value for F_2^{s}, most data indicate $|F_2^{\mathrm{s}}| \lesssim 1$.[6] Since the Landau parameter F_2^{s} is always much smaller than F_0^{s} and we do not have a precise value for it, we ignore the F_2^{s} term in the formula (4.99) for simplicity. Note that the

[6] See the graph in http://spindry.phys.northwestern.edu/3HeCalculator/F2s_plot.html.

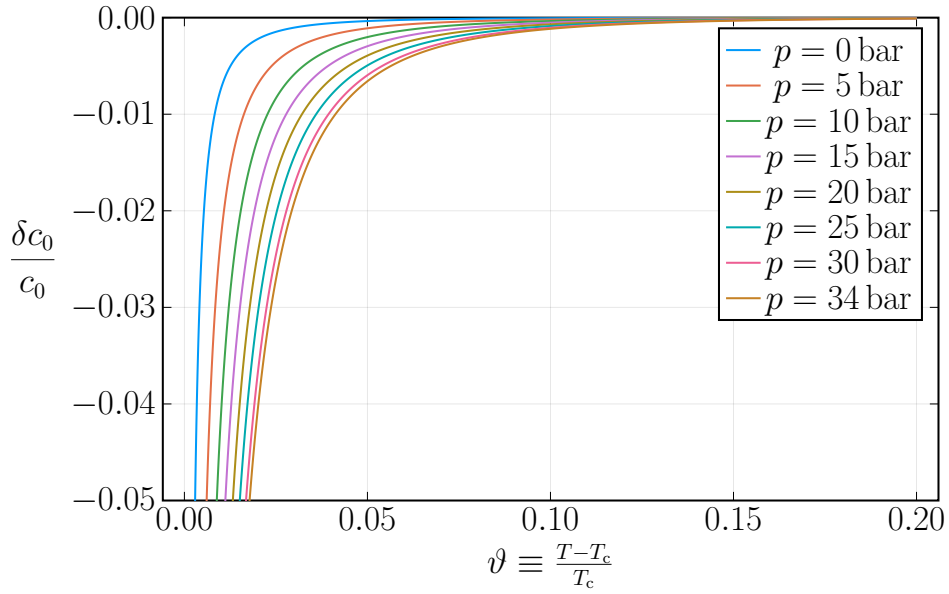


Figure 4.8. The figure shows the deviation of zero sound velocity relative to its normal value, $\delta c_0/c_0$, at different pressures. The curves are calculated with the fixed cutoff $x_c = 0.236$.

above formula is derived from the original kinetic equation, in which the coefficient of the frequency ω is unity, i.e. $\delta = 0$. To correctly include the influence of pair fluctuations, the result needs to be *divided by the factor* $1 + \delta$ *since this factor effectively reduces the frequency*. The zero sound velocity difference between the value from corrected Fermi liquid parameters and the value from the standard parameters is shown in Figure 4.8, normalized by the value obtained from the standard parameters. The curves in this figure are calculated at a low pressure, and using the cutoff $x_c = 0.236$, which is obtained from the best fit in Figure 4.3.

The experiment done by Lee et al. [12] showed that there is a velocity drop as $\vartheta \lesssim 0.2$ by the amount about 30 ppm, as shown in Figure 1.3. The velocity drop appears in our theory, but our theory predicts a much larger decrease. A possible explanation

is that the frequency used in the experiment is higher than the gap energy [12], which means the pair-breaking effect is strong. The high-frequency zero sound used in the experiment may reduce the magnitude of the pair fluctuations in the normal state, leading to a smaller velocity drop.[7] With the assumption of the fixed cutoff, our theory shows that the zero sound velocity drop is larger at higher pressures, consistent with the pressure dependence of the parameter $1/N(0)\xi_0^3 T_c$ (see Figure 4.2). Thus the velocity drop should be more observable at high pressures. We cannot draw a conclusion here, and more studies are needed in order to solve the discrepancy.

4.6. Summary

In this chapter, we consider the self-energy derived from pair fluctuations. We find that the self-energy gives rise to the collision integral which has substantial difference from the one proposed by Emery [5]. Our result is based on microscopic calculation, and thus gives a solid foundation to the fluctuation-induced collision integral. In particular, our transition probability (Eq. (4.33)) does not contain interference terms, consistent with the fact that no phase coherence exists above T_c. We derive zero sound attenuation using our collision integral, and find the result fits the experimental data well. Furthermore, we propose a better renormalization procedure for the zero sound attenuation, which is free of the cutoff, and gives a better value for the Landau parameter F_2^s.

[7] Another unknown factor is the cutoff. The cutoff $x_c = 0.236$ is obtained from the data for zero sound attenuation, measured at pressure $p = 30.87\,\text{bar}$ [8], while the experiment done by Lee et al. was at pressure $p = 0.6\,\text{bar}$. The cutoff for velocity does *not need to be the same as the cutoff for the attenuation, and it can have pressure dependence*. Nevertheless, the cutoff should be still of the same order of magnitude, and thus the influence of the cutoff should be less important.

Based on the same self-energy, we calculate the corrections to the Fermi liquid parameters. These corrections lead to a velocity drop of zero sound when the temperature is near T_c. The velocity drop was observed by Lee et al. [12], but the magnitude of the drop in our theory is much larger than the experimental data. As discussed at the end of Section 4.5, we do not understand the cause of the discrepancy, and more studies are necessary. In particular, experiments at different pressures are important, since the influence of pair fluctuations is strongly pressure-dependent (see Figure 4.2). We believe the pressure dependence is a key to understand the physics of pair fluctuations near T_c.

CHAPTER 5

Bosonic Field Theory for Zero Sound and Pair Fluctuations

In the previous chapters we study the Fermi liquid by fermionic operators. Since both the zero sound and pair fluctuations are bosonic, we want to explore the possibility of expressing the theory in terms of boson fields. This approach provides an alternative formulation of the interaction between zero sound and pair fluctuations. In this chapter we develop a bosonic field theory for zero sound and pair fluctuations using functional integrals. Although the numerical result does not agree with the experimental observation, this theory contains the theoretical structure similar to the theory of paraconductivity in superconductors [20]. Since the pair fluctuations in superconductors have been studied extensively [13–19], our bosonic theory might provide a connection to the well-developed fluctuation theory, and make the theory for zero sound with pair fluctuations more complete.

5.1. Functional integral for the Fermi liquid

We start from a Hamiltonian which includes interactions in the particle-hole channel and particle-particle channel. The particle-hole channel represents the Landau interaction, which should describe the zero sound, while the particle-particle channel represents the Cooper pairing, which is p-wave for ^{3}He. The Hamiltonian contains three parts:

$$H = H_0 + H_{\rm L} + H_{\rm C}\,, \tag{5.1}$$

with

$$H_0 = \int d^3x\ \psi^\dagger_\sigma(\mathbf{x})\Big[-\frac{\nabla^2}{2m} - \mu\Big]\psi_\sigma(\mathbf{x})\,, \tag{5.2}$$

$$H_{\rm L} = \frac{1}{2V}\sum_{\mathbf{p},\mathbf{p}',\mathbf{k}} \psi^\dagger_{\mathbf{p}\sigma}\psi_{\mathbf{p}+\mathbf{k}\sigma} f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}\psi^\dagger_{\mathbf{p}'\sigma'}\psi_{\mathbf{p}'-\mathbf{k}\sigma'}\,, \tag{5.3}$$

and

$$H_{\rm C} = -\frac{3\lambda}{2}\int d^3x\Big[\psi^\dagger(\mathbf{x})g_\alpha\frac{\overleftrightarrow{\nabla}_i}{2ip_{\rm F}}\psi^\dagger(\mathbf{x})\Big]\Big[\psi(\mathbf{x})g^\dagger_\alpha\frac{\overleftrightarrow{\nabla}_i}{2ip_{\rm F}}\psi(\mathbf{x})\Big]\,. \tag{5.4}$$

The fields ψ and $\psi^\dagger$ are fermion operators. We have used the summation convention for the spin indices. From the non-interacting Hamiltonian H_0, the fermion field have dimension $[\psi(x)] = 1/\mathsf{V}^{\frac{1}{2}}$.[1] The Landau interaction $H_{\rm L}$ is given by the interaction in the particle-hole channel, where the *momentum transfer is small* compared to the Fermi momentum, $k \ll p_{\rm F}$. Thus the summation $\sum_{\mathbf{k}}$ is restricted to a thin shell around the Fermi surface. Note the Landau interaction $H_{\rm L}$ is expressed in terms of the Fourier components of the fields

$$\psi(\mathbf{x}) = \sum_{\mathbf{p}}\psi_{\mathbf{p}}\frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{V}}, \quad \psi^\dagger(\mathbf{x}) = \sum_{\mathbf{p}}\psi^\dagger_{\mathbf{p}}\frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{V}} \tag{5.5}$$

and the Fourier components are dimensionless, $[\psi_{\mathbf{p}}] = 1$. The coupling function has dimension $[f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'}] = \mathsf{EV}$. This form is consistent with the convention used by Baym and Pethick [22]. For the pairing interaction $H_{\rm C}$, we define $g_\alpha \equiv i\sigma_\alpha\sigma_y$ and $\overleftrightarrow{\nabla} \equiv \overrightarrow{\nabla} - \overleftarrow{\nabla}$. The ψ and $\psi^\dagger$ are two-component spinors, multiplied by the Pauli matrices. This form gives the p-wave and spin-triplet pairing. The coupling constant also has dimension $[\lambda] = \mathsf{EV}$.

[1] We use the symbol V to denote the dimension of volume, and the symbol E to denote the dimension of energy.

For simplicity, we consider the equilibrium field theory. The thermodynamics of the system can be described by the partition function $Z = \mathrm{Tr}\, e^{-\beta H}$, which can be studied by the imaginary-time formalism. As mentioned at the beginning of this chapter, we want to have a bosonic field theory to describe the zero sound and pair fluctuations. To this end, we use the functional integral representation of the partition function [45]

$$Z = \int_{\psi(\beta)=-\psi(0)} D[\bar{\psi}\psi] \exp(-S[\bar{\psi},\psi])\,, \tag{5.6}$$

with the Euclidean action

$$S[\bar{\psi},\psi] = \int_0^\beta d\tau \Big[\sum_{\mathbf{p}} \bar{\psi}_{\mathbf{p}}(\tau)\partial_\tau \psi_{\mathbf{p}}(\tau) + H(\bar{\psi},\psi)(\tau) \Big]\,, \tag{5.7}$$

where the fields $\psi(\tau)$ and $\bar{\psi}(\tau)$ are Grassmann numbers, and depend on the imaginary time τ. The action can be written as $S = S_0 + S_{\mathrm{L}} + S_{\mathrm{C}}$ with

$$S_0 = \int_0^\beta d\tau \sum_{\mathbf{p}} \bar{\psi}_{\mathbf{p}}(\tau) \left[\partial_\tau + \xi^0_{\mathbf{p}}\right] \psi_{\mathbf{p}}(\tau)\,, \tag{5.8}$$

$$S_{\mathrm{L}} = \frac{1}{2V} \int_0^\beta d\tau \sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} \bar{\psi}_{\mathbf{p}\sigma}(\tau)\psi_{\mathbf{p}+\mathbf{q}\sigma}(\tau) f_{\mathbf{p}\sigma,\mathbf{p}'\sigma'} \bar{\psi}_{\mathbf{p}'\sigma'}(\tau)\psi_{\mathbf{p}'-\mathbf{q}\sigma'}(\tau)\,, \tag{5.9}$$

and

$$S_{\mathrm{C}} = -\frac{3\lambda}{2} \int_0^\beta d\tau \int d^3x \Big[\bar{\psi}(\mathbf{x},\tau) g_\alpha \frac{\overleftrightarrow{\nabla}_i}{2ip_{\mathrm{F}}} \bar{\psi}(\mathbf{x},\tau)\Big] \Big[\psi(\mathbf{x},\tau) g^\dagger_\alpha \frac{\overleftrightarrow{\nabla}_i}{2ip_{\mathrm{F}}} \psi(\mathbf{x},\tau)\Big]\,. \tag{5.10}$$

In the following, we will integrate out the fermion fields ψ and $\psi^\dagger$, in favor of some boson fields.

We remark that the above action can be obtained from the low-energy effective action (3.1). As restricted to the Fermi surface, the interaction Γ^Λ can be divided into two distinct classes [38]: The normal scattering in which two incoming particles have an angle $\angle(\mathbf{p}_1, \mathbf{p}_2) \neq \pi$, and the Cooper scattering in which the two particles have opposite momenta. The Landau interaction S_L comes from the normal scattering restricted to the forward direction, i.e. the momentum transfer $\mathbf{k} \approx 0$. Scatterings with finite momentum transfer give normal collisions, which we do not want to study here. The Cooper interaction S_C comes from the scattering with opposite momenta, and we restrict it to the p-wave channel only. In summary, the action S is the low-energy effective action for the relevant physics.

5.2. Bosonic action for zero sound and Cooper pairs

The above action is a functional of fields ψ and $\bar{\psi}$, representing the fermions. Since our goal is to study the zero sound and Cooper pair fluctuations, we want to express the action in terms of relevant degrees of freedom. In the following, we use the Hubbard-Stratonovich transformation to express the original action in terms of relevant boson fields. This procedure integrates out the fermion degrees of freedom in the particle-hole and particle-particle channel, and at the end we will see the collisionless kinetic equation and the GL equation emerge naturally.

For the pairing interaction S_C, we introduce an complex field $A_{\alpha i}$ which satisfies

$$\int D[\bar{A}A] \exp\Big[-\frac{1}{2} \int d\tau \int d^3x \; \bar{A}_{\alpha i}(\mathbf{x}, \tau) \frac{1}{3\lambda} A_{\alpha i}(\mathbf{x}, \tau) \Big] = 1 \,, \tag{5.11}$$

with a properly normalized measure $D[\bar{A}A]$. Note that $[A(\mathbf{x},\tau)] = \mathsf{E}$. By shifting the dummy variable $A_{\alpha i}$, we can get

$$\int D[\bar{A}A] \exp\Big[-\frac{1}{2}\int d\tau \int d^3x (\bar{A}_{\alpha i} + 3\lambda \bar{\psi} g_\alpha \frac{\overleftrightarrow{\nabla}_i}{2ip_{\mathrm{F}}}\bar{\psi}) \frac{1}{3\lambda}(A_{\alpha i} + 3\lambda \psi g_\alpha^\dagger \frac{\overleftrightarrow{\nabla}_i}{2ip_{\mathrm{F}}}\psi)\Big] = 1\,. \tag{5.12}$$

Inserting this identity into the (5.6), the quartic term (5.10) is canceled and we have

$$S_{\mathrm{C}}[\bar{A}A,\bar{\psi}\psi] = \frac{1}{2}\int d\tau \int d^3x \Big[\frac{1}{3\lambda}\bar{A}_{\alpha i}A_{\alpha i} + \bar{\psi} g_\alpha \frac{\overleftrightarrow{\nabla}_i}{2ip_{\mathrm{F}}}\bar{\psi}A_{\alpha i} + \bar{A}_{\alpha i}\psi g_\alpha^\dagger \frac{\overleftrightarrow{\nabla}_i}{2ip_{\mathrm{F}}}\psi\Big]\,. \tag{5.13}$$

The fermion field ψ and $\bar{\psi}$ are quadratic in all terms. For later use, we express this action in the momentum space. Using $\psi(\mathbf{x},\tau) = \sum_{\mathbf{p}} \frac{e^{i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{V}}\psi_{\mathbf{p}}(\tau)$, $\bar{\psi}(\mathbf{x},\tau) = \sum_{\mathbf{p}} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{\sqrt{V}}\bar{\psi}_{\mathbf{p}}(\tau)$ and $A(\mathbf{x},\tau) = \sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}}A(\mathbf{q},\tau)$, we have

$$\begin{aligned} S_{\mathrm{C}}[\bar{A}A,\bar{\psi}\psi] = {} & \frac{1}{6\lambda}\int_0^\beta d\tau\, V \sum_{\mathbf{q}} \bar{A}(\mathbf{q},\tau)A(\mathbf{q},\tau) \\ & + \frac{1}{2}\int_0^\beta d\tau \sum_{\mathbf{p},\mathbf{p}'} \Big[\bar{\psi}_{\mathbf{p}}(\tau) g_\alpha \frac{(\mathbf{p}-\mathbf{p}')_i}{2p_{\mathrm{F}}}\bar{\psi}_{\mathbf{p}'}(\tau)A_{\alpha i}(\mathbf{p}+\mathbf{p}',\tau) \\ & \qquad + \bar{A}_{\alpha i}(\mathbf{p}+\mathbf{p}',\tau)\psi_{\mathbf{p}}(\tau) g_\alpha^\dagger \frac{-(\mathbf{p}-\mathbf{p}')_i}{2p_{\mathrm{F}}}\psi_{\mathbf{p}'}(\tau)\Big]\,. \end{aligned} \tag{5.14}$$

For the Landau interaction (5.9), consider a new field $n_{\mathbf{p}}(\mathbf{q},\tau)$ satisfying

$$\begin{aligned} \int D[n] \exp\Big[-\frac{1}{2}\int_0^\beta d\tau \frac{1}{V}\sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} & [n_{\mathbf{p}}(\mathbf{q},\tau) + i\bar{\psi}_{\mathbf{p}}(\tau)\psi_{\mathbf{p}+\mathbf{q}}(\tau)] \\ & \times f^{\mathrm{s}}_{\mathbf{p},\mathbf{p}'}[n_{\mathbf{p}'}(-\mathbf{q},\tau) + i\bar{\psi}_{\mathbf{p}'}(\tau)\psi_{\mathbf{p}'-\mathbf{q}}(\tau)]\Big] = 1\,. \end{aligned} \tag{5.15}$$

After inserting this identity into (5.6), we have

$$S_{\mathrm{L}}[n,\bar{\psi}\psi] = \frac{1}{2V}\int_0^\beta d\tau \sum_{\mathbf{p},\mathbf{p}',\mathbf{q}} n_{\mathbf{p}}(\mathbf{q},\tau) f^{\mathrm{s}}_{\mathbf{p},\mathbf{p}'} n_{\mathbf{p}'}(-\mathbf{q},\tau) \\ + i\bar{\psi}_{\mathbf{p}}(\tau)\psi_{\mathbf{p}+\mathbf{q}}(\tau) f^{\mathrm{s}}_{\mathbf{p},\mathbf{p}'} n_{\mathbf{p}'}(-\mathbf{q},\tau) + n_{\mathbf{p}}(\mathbf{q},\tau) f^{\mathrm{s}}_{\mathbf{p},\mathbf{p}'} i\bar{\psi}_{\mathbf{p}'}(\tau)\psi_{\mathbf{p}'-\mathbf{q}}(\tau)\,. \quad (5.16)$$

To simplify the expression, we introduce the notation

$$\phi_{\mathbf{p}}(\mathbf{q},\tau) \equiv \frac{1}{V}\sum_{\mathbf{p}'} f^{\mathrm{s}}_{\mathbf{p},\mathbf{p}'} n_{\mathbf{p}'}(\mathbf{q},\tau)\,, \quad (5.17)$$

and have

$$S_{\mathrm{L}}[n,\bar{\psi}\psi] = \frac{1}{2}\int_0^\beta d\tau \sum_{\mathbf{p},\mathbf{q}} \Big[n_{\mathbf{p}}(\mathbf{q},\tau)\phi_{\mathbf{p}}(-\mathbf{q},\tau) \\ + i\bar{\psi}_{\mathbf{p}}(\tau)\psi_{\mathbf{p}+\mathbf{q}}(\tau)\phi_{\mathbf{p}}(-\mathbf{q},\tau) + \phi_{\mathbf{p}}(\mathbf{q},\tau) i\bar{\psi}_{\mathbf{p}}(\tau)\psi_{\mathbf{p}-\mathbf{q}}(\tau) \Big]\,. \quad (5.18)$$

Note the dimensions are $[n_{\mathbf{p}}(\mathbf{q},\tau)] = 1$ and $[\phi_{\mathbf{p}}(\mathbf{q},\tau)] = \mathsf{E}$.

The action $S = S_0 + S_{\mathrm{C}} + S_{\mathrm{L}}$ is quadratic in the fermion fields now, in favor of the auxiliary fields A and n, and thus we can integrate out the fermion fields exactly. Since the action contains combinations the particle-particle pair $\psi\psi$ in additional to the particle-hole pair $\bar{\psi}\psi$, we need the Nambu spinor to write them in a single matrix. Moreover, the anti-commutativity of the Grassmann numbers requires the anti-symmetry of the involved matrix. We give the details in the appendix. For

the particle-hole structure, we define

$$\widehat{\mathcal{G}}_0^{-1}(p,p') \equiv \frac{1}{T}\begin{pmatrix} ip_n - \xi_{\mathbf{p}} & 0 \\ 0 & -ip_n + \xi_{\mathbf{p}} \end{pmatrix} \delta_{\sigma\sigma'}\delta_{pp'}\,, \tag{5.19}$$

$$\widehat{\phi}(p,p') \equiv \begin{pmatrix} \phi_{\mathbf{p}}(\mathbf{p}-\mathbf{p}', p_n - p_n') & 0 \\ 0 & -\phi_{\mathbf{p}'}(\mathbf{p}'-\mathbf{p}, p_n' - p_n) \end{pmatrix} \delta_{\sigma\sigma'}\,, \tag{5.20}$$

and

$$\widehat{A}(p,p') \equiv \begin{pmatrix} 0 & g_\alpha \frac{(\mathbf{p}-\mathbf{p}')_i}{2p_{\mathrm{F}}} A_{\alpha i}(\mathbf{p}+\mathbf{p}', p_n + p_n') \\ g_\alpha^\dagger \frac{(\mathbf{p}'-\mathbf{p})_i}{2p_{\mathrm{F}}} \bar{A}_{\alpha i}(\mathbf{p}+\mathbf{p}', p_n + p_n') & 0 \end{pmatrix}, \tag{5.21}$$

with

$$\phi_{\mathbf{p}}(\mathbf{p}-\mathbf{p}', \tau) = T\sum_{k_m} e^{-ik_m\tau}\phi_{\mathbf{p}}(\mathbf{p}-\mathbf{p}', k_m)\,, \tag{5.22}$$

$$A_{\alpha i}(\mathbf{x},\tau) = T\sum_{q_m}\sum_{\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}} e^{-iq_m\tau} A_{\alpha i}(\mathbf{q}, q_m)\,, \tag{5.23}$$

and

$$\bar{A}_{\alpha i}(\mathbf{x},\tau) = T\sum_{q_m}\sum_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}} e^{+iq_m\tau} \bar{A}_{\alpha i}(\mathbf{q}, q_m)\,. \tag{5.24}$$

Note that the functions $\widehat{\mathcal{G}}_0$, $\widehat{\phi}$, and $\widehat{A}$ are dimensionless. The result can be written as

$$\begin{aligned} S = &-\frac{1}{2}\,\mathrm{Tr}\ln(-\widehat{\mathcal{G}}_0^{-1} + i\widehat{\phi} + \widehat{A}) \\ &+ \frac{1}{2}\int_0^\beta d\tau \sum_{\mathbf{p},\mathbf{q}} n_{\mathbf{p}}(\mathbf{q},\tau)\phi_{\mathbf{p}}(-\mathbf{q},\tau) + \frac{1}{6\lambda}\int_0^\beta d\tau V \sum_{\mathbf{q}} \bar{A}_{\alpha i}(\mathbf{q},\tau) A_{\alpha i}(\mathbf{q},\tau)\,, \end{aligned} \tag{5.25}$$

which contains the fields $n_{\mathbf{p}}$ ($\phi_{\mathbf{p}}$ is linear on $n_{\mathbf{p}}$, see the definition (5.17)) and $A_{\alpha i}$.

5.3. Expansion around the equilibrium state

The bosonic action (5.25) describes the thermodynamics in terms of the distribution function $n_{\mathbf{p}}$ and the order parameter $A_{\alpha i}$. If we neglect the fluctuations of the fields $n_{\mathbf{p}}$ and $A_{\alpha i}$, the equilibrium distribution can be determined by minimizing the action (5.25). Indeed, if we assume the order parameter $A_{\alpha i} = 0$, and take the variation with respect to $\phi_{\mathbf{p}}$, we have

$$\frac{\delta S}{\delta \phi_{\mathbf{p}}} \sim -\operatorname{Tr} \frac{i}{-ip_n + \xi_{\mathbf{p}} + i\phi_{\mathbf{p}}} + n_{\mathbf{p}} \,. \tag{5.26}$$

In equilibrium, the variation must vanish and we have

$$in^{\mathrm{eq}} \sim \operatorname{Tr} \frac{1}{ip_n - \xi_{\mathbf{p}} - i\phi_{\mathbf{p}}^{\mathrm{eq}}} \sim \frac{1}{e^{\beta(\xi_{\mathbf{p}} + i\phi_{\mathbf{p}}^{\mathrm{eq}})} + 1} \,, \tag{5.27}$$

which indeed gives the equilibrium distribution in Fermi liquids (2.30). The presence of the imaginary unit i is a common feature in the imaginary time formalism (see, for example, [46, 47]).

We will expand the bosonic action around this equilibrium state. The order parameter vanishes in equilibrium for $T > T_{\mathrm{c}}$, so the field $A_{\alpha i}$ in (5.25) has already described the fluctuations. Let $n = n^{\mathrm{eq}} + \delta n$. The equilibrium value ϕ^{eq} can be absorbed into the single-particle Green function $\mathcal{G}_0$. We assume the deviation δn and the pair fluctuations $A_{\alpha i}$ are small, so we can expand the Tr ln to leading orders. We first decompose the logarithm as follows:

$$\operatorname{Tr} \ln(-\widehat{\mathcal{G}}_0^{-1} + i\delta\widehat{\phi} + \widehat{A}) = \operatorname{Tr} \ln(-\widehat{\mathcal{G}}_0^{-1}) + \operatorname{Tr} \ln(\widehat{1} - \widehat{\mathcal{G}}_0(i\delta\widehat{\phi} + \widehat{A})) \,. \tag{5.28}$$

Note that if there is no nonequilibrium δn and the pair fluctuations A, the above expression becomes $\mathrm{Tr}\ln(-\widehat{\mathcal{G}}_0^{-1})$, which gives the equilibrium properties. We will neglect this part in the following expressions.

Using the expansion $\log(1-x) = -\sum_{n=1}^{\infty} x^n/n$, we have

$$-\frac{1}{2}\mathrm{Tr}\ln(\widehat{1}-\widehat{\mathcal{G}}_0(i\delta\widehat{\phi}+\widehat{A})) = \frac{1}{2}\mathrm{Tr}\sum_{n=1}^{\infty}\frac{1}{n}[\widehat{\mathcal{G}}_0(i\delta\widehat{\phi}+\widehat{A})]^n\,. \tag{5.29}$$

Recall that the trace Tr contains summations over spin, particle-hole, and momentum indices. Using this expansion with the last two quadratic terms in the bosonic action (5.25), we obtain

$$S = \frac{1}{2}\mathrm{Tr}\sum_{n=2}^{\infty}\frac{1}{n}[\widehat{\mathcal{G}}_0(i\delta\widehat{\phi}+\widehat{A})]^n + \frac{1}{2}\int_0^\beta d\tau\sum_{\mathbf{p},\mathbf{q}}\delta n_{\mathbf{p}}(\mathbf{q},\tau)\delta\phi_{\mathbf{p}}(-\mathbf{q},\tau) + \frac{1}{6\lambda}\int_0^\beta d\tau V\sum_{\mathbf{q}}\bar{A}(\mathbf{q},\tau)A(\mathbf{q},\tau)\,. \tag{5.30}$$

Note that we have dropped all linear terms in δn and A, since the expansion is around the equilibrium. The quadratic terms in this expansion give rise to the linearized collisionless kinetic equation and the TDGL equation to the second order.[2] Hence the non-interacting part of the bosonic action correctly describe the zero sound and the Cooper pair fluctuations independently. The higher-order terms give the interactions between these fluctuations. We focus on the interaction between the zero sound and pair fluctuations, and neglect other contributions.

[2]Note that there is no cross term like $A\delta n$ because the trace over particle-hole space vanishes.

5.3.1. Collisionless kinetic equation

The quadratic action containing second-order δn or $\delta\phi$ is

$$S_0^{\phi} = \frac{1}{2}\,\mathrm{Tr}\,\frac{1}{2}(\widehat{\mathcal{G}}_0 i\delta\widehat{\phi})^2 + \frac{1}{2}\int_0^{\beta} d\tau \sum_{\mathbf{p},\mathbf{q}} \delta n_{\mathbf{p}}(\mathbf{q},\tau)\delta\phi_{\mathbf{p}}(-\mathbf{q},\tau)\,. \tag{5.31}$$

Evaluating the trace of spin and particle-hole indices, the above expression becomes

$$S_0^{\phi} = -T^2 \sum_k \sum_p \mathcal{G}_0(p)\mathcal{G}_0(p+k)\delta\phi_{\mathbf{p}}(k)\delta\phi_{\mathbf{p}}(-k) + \frac{1}{2}T\sum_k \sum_{\mathbf{p}} \delta n_{\mathbf{p}}(k)\delta\phi_{\mathbf{p}}(-k)\,, \tag{5.32}$$

where $\mathcal{G}_0(p) = 1/(ip_n - \xi_{\mathbf{p}})$. Evaluating the summation over Matsubara frequency ip_n, we obtain

$$S_0^{\phi} = T\sum_k \left\{ \sum_{\mathbf{p}} n'_{\mathrm{F}}(\xi_{\mathbf{p}}) \frac{\mathbf{v}_{\mathbf{p}}\cdot\mathbf{k}}{ik_m - \mathbf{v}_{\mathbf{p}}\cdot\mathbf{k}} \delta\phi_{\mathbf{p}}(k)\delta\phi_{\mathbf{p}}(-k) + \frac{1}{2}\sum_{\mathbf{p}} \delta n_{\mathbf{p}}(k)\delta\phi_{\mathbf{p}}(-k) \right\}. \tag{5.33}$$

Recall that $\delta\phi_{\mathbf{p}} = \frac{1}{V}\sum_{\mathbf{p}} f_{\mathbf{p},\mathbf{p}'}\delta n_{\mathbf{p}'}$. Because the matrix $f_{\mathbf{p},\mathbf{p}'}$ is invertible, $\delta\phi_{\mathbf{p}}$ contains the same information as $\delta n_{\mathbf{p}}$. We can get the equation of motion by taking the variation with respect to $\delta\phi_{\mathbf{p}}(-k)$, which gives

$$\delta n_{\mathbf{p}}(k) + 2n'_{\mathrm{F}}(\xi_{\mathbf{p}}) \frac{\mathbf{v}_{\mathbf{p}}\cdot\mathbf{k}}{ik_m - \mathbf{v}_{\mathbf{p}}\cdot\mathbf{k}} \delta\phi_{\mathbf{p}}(k) = 0\,. \tag{5.34}$$

This is just the linearized kinetic equation, describing zero sound. Therefore, the nonequilibrium distribution δn indeed follows the dynamics of zero sound.

5.3.2. Second-order TDGL equation

The quadratic terms in $A_{\alpha i}$ give rise to the time-dependent Ginzburg-Landau equation to the second order. As long as the temperature is not too close to the critical point,

the fluctuations can be well described by this gaussian form.[3] The quadratic part is

$$S_0^A \equiv \frac{1}{2}\,\mathrm{Tr}\,\frac{1}{2}(\widehat{\mathcal{G}}_0\widehat{A})^2 + \frac{1}{6\lambda}\int_0^\beta d\tau V \sum_{\mathbf{q}} \bar{A}(\mathbf{q},\tau)A(\mathbf{q},\tau)\,. \tag{5.35}$$

The trace term gives

$$\frac{1}{2}\,\mathrm{Tr}\,\frac{1}{2}(\widehat{\mathcal{G}}_0\widehat{A})^2 = \frac{1}{2}\sum_{p,q}\sum_{\sigma}\mathcal{G}_0(p)g_\alpha\hat{p}_i A_{\alpha i}(q)[-\mathcal{G}_0(q-p)]g^\dagger_\beta\hat{p}_j\bar{A}_{\beta j}(q)\,, \tag{5.36}$$

where we have summed over the particle-hole indices. Using $\sum_\sigma g_\alpha g^\dagger_\beta = 2\delta_{\alpha\beta}$, we have

$$\frac{1}{2}\,\mathrm{Tr}\,\frac{1}{2}(\widehat{\mathcal{G}}_0\widehat{A})^2 = T^2\sum_{p,q}\mathcal{G}_0(p)\hat{p}_i A_{\alpha i}(q)[-\mathcal{G}_0(q-p)]\hat{p}_j\bar{A}_{\alpha j}(q)\,. \tag{5.37}$$

The remaining p integral can be done with small q expansion, as shown in Appendix A. The result is

$$S_0^A = \frac{N(0)V}{3}T\sum_q \bar{A}_{\alpha i}(q)\left[\vartheta\delta_{ij} + \frac{\pi|q_m|}{8T}\delta_{ij} + \frac{3}{5}\xi_0^2 q^2(\delta_{ij}+2\hat{q}_i\hat{q}_j)\right]A_{\alpha j}(q)\,, \tag{5.38}$$

where $\xi_0^2 \equiv \frac{7\zeta(3)v_F^2}{48\pi^2T^2}$ is the s-wave coherence length. This action S^A is the p-wave GL functional to the second order.

5.3.3. Fluctuation propagator

For the later use, we consider the average $\langle\bar{A}_{\alpha i}(q)A_{\beta j}(q')\rangle$ here, which is the correlation function of pair fluctuations, or the fluctuation propagator. From the action (5.38),

[3] The critical region can be estimated by the Ginzburg criterion. See Coleman [48], for example. For liquid ^{3}He, the critical region has the size of the order of 10^{-12} around T_c, which is not accessible experimentally.

we immediately have

$$\langle \bar{A}_{\alpha i}(q) A_{\beta j}(q') \rangle = \Big(\frac{N(0)VT}{3}\Big)^{-1} \Big[(\vartheta + \frac{\pi |q_m|}{8T}) \delta_{ij} + \frac{3}{5} \xi_0^2 q^2 (\delta_{ij} + 2\hat{q}_i \hat{q}_j) \Big]^{-1} \delta_{q,q'} \delta_{\alpha\beta} \,. \tag{5.39}$$

Recall the definitions introduce in the Section 4.1. We have

$$\langle \bar{A}_{\alpha i}(q) A_{\beta j}(q') \rangle = \sum_{\lambda = \parallel, \perp} \mathcal{G}_A^\lambda(q) P_{ij}^\lambda(\hat{\mathbf{q}}) \delta_{q,q'} \delta_{\alpha\beta} \,, \tag{5.40}$$

where $P^{\parallel,\perp}(\hat{\mathbf{q}})$ is the projection along the direction parallel and perpendicular to $\hat{\mathbf{q}}$, and

$$\mathcal{G}_A^\lambda(q) = \frac{3}{N(0)VT} \Big[\vartheta + \frac{\pi |q_m|}{8T} + \xi_\lambda^2 q^2 \Big]^{-1} \tag{5.41}$$

with

$$\xi_\lambda^2 = \begin{cases} \frac{9}{5}\xi_0^2 & \lambda = \parallel \\ \\ \frac{3}{5}\xi_0^2 & \lambda = \perp \end{cases} . \tag{5.42}$$

5.4. Interactions between zero sound and pair fluctuations

We have shown that the second-order terms give rise to the description of zero sound and pair fluctuations. To see how the pair fluctuations influence zero sound, we need the interactions between them, which come from the higher-order terms in the expansion (5.30). *We assume the fluctuations are weak, and thus consider only the leading-order terms*. We also ignore all self-interacting terms.[4] For $n = 3$, the only

[4]The higher-order terms in $A_{\alpha i}$ are not important, as the temperature of interest is not within the critical region. The zero sound is described well by the linear approximation.

interaction term is

$$\begin{aligned} S^{(3)} &= 2iT^3 \sum \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q-p)\phi_{\mathbf{p}}(k)\hat{p}_i\bar{A}_{\alpha i}(q+k)\hat{p}_j A_{\alpha j}(q) \\ &= 2iT^3 \sum \mathcal{G}(p)\mathcal{G}_0(p+q-q')\mathcal{G}_0(q'-p)\phi_{\mathbf{p}}(q-q')\hat{p}_i\bar{A}_{\alpha i}(q)\hat{p}_j A_{\alpha j}(q')\,, \end{aligned} \tag{5.43}$$

while for $n = 4$, there are two distinct interactions,

$$\begin{aligned} S^{(4,1)} = 2T^4 \sum \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q'-p)\mathcal{G}_0(q-q'+p) \\ \times \phi_{\mathbf{p}}(k)\phi_{\mathbf{p}}(-k+q-q')\hat{p}_i\bar{A}_{\alpha i}(q)\hat{p}_j A_{\alpha j}(q')\,, \end{aligned} \tag{5.44}$$

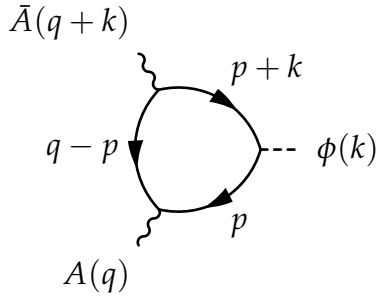
and

$$\begin{aligned} S^{(4,2)} = T^4 \sum \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q'-p)\mathcal{G}_0(q-p-k) \\ \times \phi_{\mathbf{p}}(k)\phi_{-\mathbf{p}}(-k+q-q')\hat{p}_i\bar{A}_{\alpha i}(q)\hat{p}_j A_{\alpha j}(q')\,, \end{aligned} \tag{5.45}$$

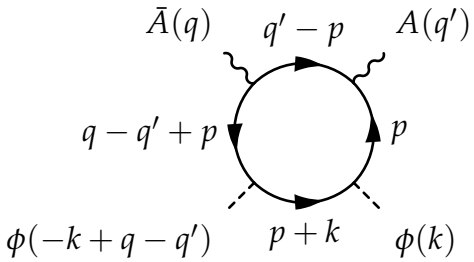
where $\mathcal{G}_0(p) = 1/(ip_n - \xi_{\mathbf{p}})$ is the fermion propagator, and the summation is over all momenta, with the restriction $q, k \ll p$. Together with the terms obtained from the last section, the approximate action is

$$S = S_0^{\phi} + S_0^{A} + S^{(3)} + S^{(4,1)} + S^{(4,2)}\,. \tag{5.46}$$

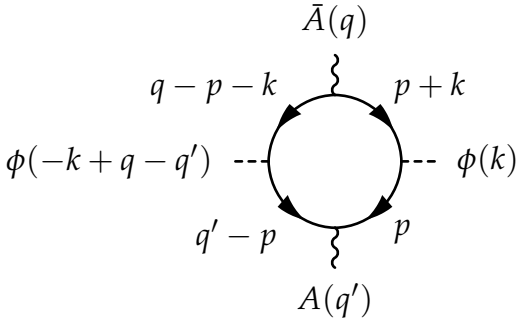
Recall that the partition function is given by $Z = \int \exp(-S)$. To get the effect of pair fluctuations, we consider the functional integration over $A_{\alpha i}$. Using the cumulant



(a) Diagrammatic representation of $S^{(3)}$.



(b) Diagrammatic representation of $S^{(4,1)}$.



(c) Diagrammatic representation of $S^{(4,2)}$.

Figure 5.1. The three interaction terms (5.43), (5.44), and (5.45) can be represented by the above diagrams. The direction of the momentum flow for ϕ is considered to be outward.

expansion, we have

$$\langle e^{-V}\rangle = e^{-\langle V\rangle+\frac{1}{2}(\langle V^2\rangle-\langle V\rangle^2)+\cdots}, \tag{5.47}$$

where $\langle\cdot\rangle$ represents the average over the pair fluctuations, with the weight $\exp(-S_0^A)$. When taking the average over the fluctuations, we consider the zero sound as a weak external perturbation. Thus we consider only the terms quadratic in $\phi_{\mathbf{p}}$, and ignore higher-order terms. The linear term in ϕ is not important, because it gives a driving term in the equation of motion, which does not change the dispersion relation of zero sound. In the following, we consider the contribution from the three interaction terms in the action (5.46). We find the leading order contributions are given by diagrams similar to the ones in the paraconductivity theory.

5.4.1. DOS contribution

We first consider the 4th-order terms, because they are simpler than the 3rd-order term. These terms are quadratic in ϕ in the first-order cumulant expansion. Using the expression (5.44), we have

$$\begin{aligned}\langle S^{(4,1)}\rangle &= 2T^4\sum \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q'-p)\mathcal{G}_0(q-q'+p)\\ &\qquad\qquad \times \phi_{\mathbf{p}}(k)\phi_{\mathbf{p}}(-k+q-q')\hat{p}_i\hat{p}_j\times\langle\bar{A}_{\alpha i}(q)A_{\alpha j}(q')\rangle\\ &= 2T^4\sum \mathcal{G}_0(p)^2\mathcal{G}_0(p+k)\mathcal{G}_0(q-p)\phi_{\mathbf{p}}(k)\phi_{\mathbf{p}}(-k)\hat{p}_i\hat{p}_j\times 3\sum_\lambda \mathcal{G}_A^\lambda(q)P_{ij}^\lambda(\hat{\mathbf{q}}),\end{aligned} \tag{5.48}$$

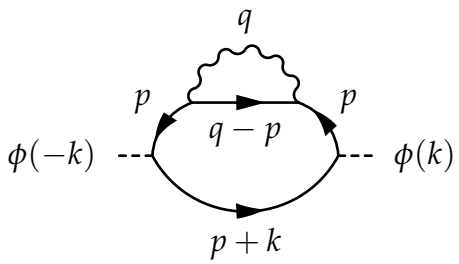


Figure 5.2. The DOS diagram. This diagram can be obtained by connecting the A and $\bar{A}$ fields in the diagram 5.1(b).

where we have used the average (5.40), and summed over the delta symbols. This expression has the same structure as the DOS contribution in the theory of paraconductivity [20], as shown in Figure 5.2, and thus we also denote it by DOS.

Define

$$\lambda_{\mathbf{p}}^{\mathrm{DOS}} \equiv 2T^2 \times T\sum_{ip_n} \mathcal{G}_0(p)^2 \mathcal{G}_0(p+k)\mathcal{G}_0(q-p). \tag{5.49}$$

The expression is reduced to

$$S^{\mathrm{DOS}} \equiv \langle S^{(4,1)} \rangle = T\sum_{k,q}\sum_{\mathbf{p}} \lambda_{\mathbf{p}}^{\mathrm{DOS}} \phi_{\mathbf{p}}(k)\phi_{\mathbf{p}}(-k)\hat{p}_i\hat{p}_j \times 3\sum_{\lambda} \mathcal{G}_A^{\lambda}(q)P_{ij}^{\lambda}(\hat{\mathbf{q}}). \tag{5.50}$$

The singular contribution is from the $q_m = 0$ mode, so we take only this part. We also ignore the dependence of $\mathbf{q}$ in $\lambda_{\mathbf{p}}^{\mathrm{DOS}}$, because it is smooth at $\mathbf{q} = 0$. Thus we have

$$\begin{aligned}
\sum_{q} \mathcal{G}_A^{\lambda}(q)P_{ij}^{\lambda}(\hat{\mathbf{q}}) &= \sum_{\mathbf{q}} \frac{3}{N(0)VT}\left[\vartheta + \xi_{\lambda}^2 q^2\right]^{-1} P_{ij}^{\lambda}(\hat{\mathbf{q}}) \\
&= \frac{3}{N(0)T}\int \frac{d^3q}{(2\pi)^3}\frac{1}{\vartheta + \xi_{\lambda}^2 q^2} P_{ij}^{\lambda}(\hat{\mathbf{q}}) \\
&= \frac{1}{N(0)T\xi_0^3}\frac{1}{2\pi^2}\left[x_{\mathrm{c}} - \sqrt{\vartheta}\,\mathrm{atan}(\frac{x_{\mathrm{c}}}{\sqrt{\vartheta}})\right]\delta_{ij} \times \left(\frac{1}{\sqrt{9/5}^3} + \frac{2}{\sqrt{3/5}^3}\right),
\end{aligned} \tag{5.51}$$

where we have used the same regularization scheme as in the (4.50). See the text below (4.51). Let

$$C(\vartheta) \equiv \frac{3}{2\pi^2}\left[x_c - \sqrt{\vartheta}\,\mathrm{atan}(\frac{x_c}{\sqrt{\vartheta}})\right] \times \left(\frac{1}{\sqrt{9/5}^3} + \frac{2}{\sqrt{3/5}^3}\right). \tag{5.52}$$

We get

$$S^{\mathrm{DOS}} = T\sum_k \sum_{\mathbf{p}} \lambda_{\mathbf{p}}^{\mathrm{DOS}} \phi_{\mathbf{p}}(k)\phi_{\mathbf{p}}(-k) \frac{1}{N(0)T\xi_0^3} C^{\mathrm{DOS}}(\vartheta) \tag{5.53}$$

Note that the dimensionless parameter $1/N(0)T_c\xi_0^3$ is a small number, as shown in Figure 4.2.

5.4.2. MT contribution

The $S^{(4,2)}$ term also has the $\bar{A}A\phi\phi$ structure, and thus the calculation is similar to the DOS contribution. Using the expression (5.45), we have

$$\begin{aligned}\langle S^{(4,2)}\rangle &= T^4 \sum \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q'-p)\mathcal{G}_0(q-p-k)\phi_{\mathbf{p}}(k)\phi_{-\mathbf{p}}(-k+q-q')\hat{p}_i\hat{p}_j \\ &\qquad \times \langle \bar{A}_{\alpha i}(q)A_{\alpha j}(q')\rangle \\ &= T^4 \sum \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q-p)\mathcal{G}_0(q-p-k)\phi_{\mathbf{p}}(k)\phi_{-\mathbf{p}}(-k)\hat{p}_i\hat{p}_j \\ &\qquad \times 3\sum_\lambda \mathcal{G}_A^\lambda(q)P_{ij}^\lambda(\hat{\mathbf{q}}).\end{aligned} \tag{5.54}$$

This expression is similar to the MT contribution in paraconductivity [20], as shown in Figure 5.3. Define

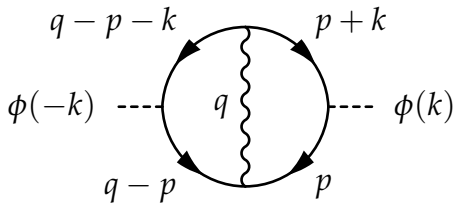


Figure 5.3. The MT diagram. This diagram can be obtained from the diagram 5.1(c).

$$\lambda_{\mathbf{p}}^{\mathrm{MT}} \equiv T^2 \times T \sum_{ip_n} \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q-p)\mathcal{G}_0(q-p-k). \tag{5.55}$$

The above expression is reduced to

$$S^{\mathrm{MT}} \equiv \langle S^{(4,2)} \rangle = T \sum_{k,q} \sum_{\mathbf{p}} \lambda_{\mathbf{p}}^{\mathrm{MT}} \phi_{\mathbf{p}}(k)\phi_{-\mathbf{p}}(-k)\hat{p}_i\hat{p}_j \times 3 \sum_{\lambda} \mathcal{G}_A^{\lambda}(q) P_{ij}^{\lambda}(\hat{\mathbf{q}}). \tag{5.56}$$

Similarly, we consider only the $q_m = 0$ mode, and ignore the $\mathbf{q}$-dependence in $\lambda_{\mathbf{p}}^{\mathrm{MT}}$. Integration over $\mathbf{q}$ gives the factor

$$C(\vartheta) \equiv \frac{3}{2\pi^2}\left[x_{\mathrm{c}} - \sqrt{\vartheta}\,\mathrm{atan}(\frac{x_{\mathrm{c}}}{\sqrt{\vartheta}})\right] \times \left(\frac{1}{\sqrt{9/5^3}} + \frac{2}{\sqrt{3/5^3}}\right), \tag{5.57}$$

which is the same as in the DOS case, and

$$S^{\mathrm{MT}} = T \sum_{k} \sum_{\mathbf{p}} \lambda_{\mathbf{p}}^{\mathrm{MT}} \phi_{\mathbf{p}}(k)\phi_{-\mathbf{p}}(-k) \frac{1}{N(0)T\xi_0^3} C^{\mathrm{MT}}(\vartheta). \tag{5.58}$$

5.4.3. AL contribution

The contribution from $S^{(3)}$ is different from the DOS and MT terms, because $S^{(3)}$ has the structure $\bar{A}A\phi$. In order to get the quadratic term in ϕ, we have to consider the

next order in the cumulant expansion.

$$
\begin{aligned}
[S^{(3)}]^2 = {} & 2iT^3 \sum \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q-p)\phi_{\mathbf{p}}(k)\hat{p}_i \bar{A}_{\alpha i}(q+k)\hat{p}_j A_{\alpha j}(q) \\
& \times 2iT^3 \sum \mathcal{G}_0(p')\mathcal{G}_0(p'+k')\mathcal{G}_0(q'-p')\phi_{\mathbf{p}'}(k')\hat{p}'_l \bar{A}_{\beta l}(q'+k')\hat{p}'_m A_{\beta m}(q').
\end{aligned}
\tag{5.59}
$$

There are two $\bar{A}$ and two A fields, and thus there are two different ways to contract them. One of them leads to a disconnected diagram, and it will be canceled in the cumulant expansion.[5] The only contribution is

$$
\begin{aligned}
\langle [S^{(3)}]^2 \rangle - \langle S^{(3)} \rangle^2 = {} & -2T^3 \sum \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q-p)\phi_{\mathbf{p}}(k)\hat{p}_i\hat{p}_j \\
& \times 2T^3 \sum \mathcal{G}_0(p')\mathcal{G}_0(p'+k')\mathcal{G}_0(q'-p')\phi_{\mathbf{p}'}(k')\hat{p}'_l\hat{p}'_m \\
& \times \langle \bar{A}_{\alpha i}(q+k) A_{\beta m}(q') \rangle \langle A_{\alpha j}(q) \bar{A}_{\beta l}(q'+k') \rangle.
\end{aligned}
\tag{5.60}
$$

Recall that $\langle \bar{A}_{\alpha i}(q) A_{\beta j}(q') \rangle = \sum_\lambda \mathcal{G}_A^\lambda(q) P_{ij}^\lambda(\hat{\mathbf{q}}) \delta_{q,q'} \delta_{\alpha\beta}$. Since the fluctuation propagator is diagonal in momentum variables, we have $q' = q + k$ and $k' = q - q' = -k$ required by the Kronecker deltas. The previous expression becomes

$$
\begin{aligned}
\langle [S^{(3)}]^2 \rangle - \langle S^{(3)} \rangle^2 = {} & -2T^3 \sum \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q-p)\phi_{\mathbf{p}}(k)\hat{p}_i\hat{p}_j \\
& \times 2T^3 \sum \mathcal{G}_0(p')\mathcal{G}_0(p'-k)\mathcal{G}_0(q+k-p')\phi_{\mathbf{p}'}(-k)\hat{p}'_l\hat{p}'_m \\
& \times 3 \sum_\lambda \mathcal{G}_A^\lambda(q+k) P_{im}^\lambda(\widehat{\mathbf{q}+\mathbf{k}}) \sum_{\lambda'} \mathcal{G}_A^{\lambda'}(q) P_{jl}^{\lambda'}(\hat{\mathbf{q}}).
\end{aligned}
\tag{5.61}
$$

This expression can be represented by the diagram in Figure 5.4, which is similar to the AL diagram in the paraconductivity theory.

[5] Recall that the second-order term has the form $\langle V^2 \rangle - \langle V \rangle^2$. The disconnected diagram is canceled by the $\langle V \rangle^2$ term.

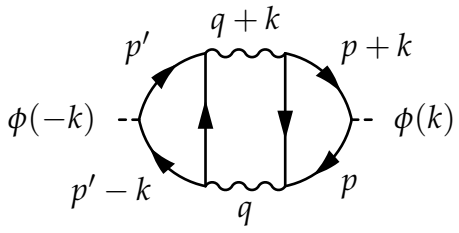


Figure 5.4. The AL diagram, which comes from the second-order contribution of the vertex given in the diagram 5.1(a).

The singular contribution is from the fluctuation propagator $\mathcal{G}_A$, within small regions around the most singular modes $q = 0$ and $q = -k$. As before, for the discrete Matsubara sum $\sum_{iq_m}$, we take only the most singular modes $q_m = 0, -k_m$. For the integral over the momentum $\mathbf{q}$, we pick a cutoff to restrict the integral around $\mathbf{q} = \mathbf{0}, -\mathbf{k}$. Moreover, the fermion propagators $\mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(q-p)$ are smooth at $q = 0, -k$, so we can set q to these values, since the both $\mathbf{q}$ and $\mathbf{k}$ are much smaller than the Fermi momentum. For $q = 0$, the first and second summations over p have the form $\sum \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(-p)$ after appropriate change of variables and approximations. Similarly, for $q = -k$, both have the form $\sum \mathcal{G}_0(p)\mathcal{G}_0(p-k)\mathcal{G}_0(-p)$. Let

$$\lambda_{\mathbf{p}}^{\mathrm{AL}}(k) \equiv 2T \times T \sum_{ip_n} \mathcal{G}_0(p)\mathcal{G}_0(p+k)\mathcal{G}_0(-p). \tag{5.62}$$

Then we have

$$\langle [S^{(3)}]^2 \rangle - \langle S^{(3)} \rangle^2 =$$
$$- T^2 \sum_k \sum_{\mathbf{p}} \sum_{\mathbf{p}'} \Big[\lambda^{\rm AL}_{\mathbf{p}}(k) \phi_{\mathbf{p}}(k) \hat{p}_i \hat{p}_j \times \lambda^{\rm AL}_{\mathbf{p}'}(k) \phi_{\mathbf{p}'}(-k) \hat{p}'_l \hat{p}'_m \times 3 \sum_{\bar{\mathbf{q}}} (q = (0, \bar{\mathbf{q}}))$$
$$+ \lambda^{\rm AL}_{\mathbf{p}}(-k) \phi_{\mathbf{p}}(k) \hat{p}_i \hat{p}_j \times \lambda^{\rm AL}_{\mathbf{p}'}(-k) \phi_{\mathbf{p}'}(-k) \hat{p}'_l \hat{p}'_m \times 3 \sum_{\bar{\mathbf{q}}} (q = (-k_m, -\mathbf{k} + \bar{\mathbf{q}})) \Big], \quad (5.63)$$

where the expressions $(q = (0, \bar{\mathbf{q}}))$ and $(q = (-k_m, -\mathbf{k} + \bar{\mathbf{q}}))$ are the fluctuation propagators $\sum_\lambda \mathcal{G}^\lambda_A(q+k) P^\lambda_{im}(\widehat{\mathbf{q}+\mathbf{k}}) \sum_{\lambda'} \mathcal{G}^{\lambda'}_A(q) P^{\lambda'}_{jl}(\hat{\mathbf{q}})$ around the two most singular modes,

$$q = (0, \bar{\mathbf{q}}) :$$
$$\Big(\frac{3}{N(0)VT} \Big)^2 \frac{1}{\vartheta + \pi |k_m| / (8T) + \xi^2_\lambda (\bar{\mathbf{q}} + \mathbf{k})^2} \times \frac{1}{\vartheta + \xi^2_{\lambda'} \bar{\mathbf{q}}^2} P^\lambda_{im}(\widehat{\bar{\mathbf{q}} + \mathbf{k}}) P^{\lambda'}_{jl}(\hat{\bar{\mathbf{q}}}) \quad (5.64)$$

and

$$q = (-k_m, -\mathbf{k} + \bar{\mathbf{q}}) :$$
$$\Big(\frac{3}{N(0)VT} \Big)^2 \frac{1}{\vartheta + \xi^2_\lambda \bar{\mathbf{q}}^2} \times \frac{1}{\vartheta + \pi |k_m| / (8T) + \xi^2_{\lambda'} (\bar{\mathbf{q}} - \mathbf{k})^2} P^\lambda_{im}(\hat{\bar{\mathbf{q}}}) P^{\lambda'}_{jl}(\widehat{\bar{\mathbf{q}} - \mathbf{k}}). \quad (5.65)$$

The momentum $\bar{\mathbf{q}}$ is the small deviation from the most singular momenta $\mathbf{q} = \mathbf{0}, -\mathbf{k}$.

Since the most singular modes are now given by $\bar{\mathbf{q}} = 0$, we will restricted the integration to a small region around $\bar{\mathbf{q}}$, and also assume $\bar{\mathbf{q}} \pm \mathbf{k} \approx \pm \mathbf{k}$ in the above expressions. For convenience, we drop the bar on top of $\bar{\mathbf{q}}$ in the following. Thus the

summation over the momentum $\mathbf{q}$ becomes

$$\sum_{\mathbf{q}} \left(q = (0, \mathbf{q})\right) : \\ \sum_{\mathbf{q}} \Big(\frac{3}{N(0)VT}\Big)^2 \frac{1}{\vartheta + \pi|k_m|/(8T) + \xi_\lambda^2 \mathbf{k}^2} \times \frac{1}{\vartheta + \xi_{\lambda'}^2 \mathbf{q}^2} P_{im}^{\lambda}(\hat{\mathbf{k}}) P_{jl}^{\lambda'}(\hat{\mathbf{q}}), \quad (5.66)$$

$$\sum_{\mathbf{q}} \left(q = (-k_m, -\mathbf{k} + \mathbf{q})\right) : \\ \sum_{\mathbf{q}} \Big(\frac{3}{N(0)VT}\Big)^2 \frac{1}{\vartheta + \xi_\lambda^2 \mathbf{q}^2} \times \frac{1}{\vartheta + \pi|k_m|/(8T) + \xi_{\lambda'}^2 \mathbf{k}^2} P_{im}^{\lambda}(\hat{\mathbf{q}}) P_{jl}^{\lambda'}(\hat{\mathbf{k}}). \quad (5.67)$$

Let

$$K^{\lambda}(k) \equiv \frac{1}{\vartheta + \pi|k_m|/(8T) + \xi_\lambda^2 \mathbf{k}^2} \approx \frac{1}{\pi|k_m|/(8T) + \xi_\lambda^2 \mathbf{k}^2}, \quad (5.68)$$

which has weak ϑ-dependence. Converting the summation $\sum_{\mathbf{q}} = V \int \frac{d^3q}{(2\pi)^3}$ and integrating the momentum $\mathbf{q}$, we get the same factor

$$C(\vartheta) = \frac{3}{2\pi^2} \Big[x_c - \sqrt{\vartheta}\, \mathrm{atan}(\frac{x_c}{\sqrt{\vartheta}})\Big] \times \Big(\frac{1}{\sqrt{9/5}^3} + \frac{2}{\sqrt{3/5}^3}\Big) \quad (5.69)$$

as before. Putting the results back to the expression (5.63), we obtain

$$\langle [S^{(3)}]^2 \rangle - \langle S^{(3)} \rangle^2 = -T^2 \sum_k \sum_{\mathbf{p},\mathbf{p}'} \left[\lambda_{\mathbf{p}}^{\mathrm{AL}}(k) \lambda_{\mathbf{p}'}^{\mathrm{AL}}(k) + \lambda_{\mathbf{p}}^{\mathrm{AL}}(-k) \lambda_{\mathbf{p}'}^{\mathrm{AL}}(-k)\right] \phi_{\mathbf{p}}(k) \phi_{\mathbf{p}'}(-k) \\ \times \Big(\frac{3}{N(0)VT}\Big)^2 \frac{V}{\xi_0^3} C(\vartheta) K^{\lambda'}(k) \frac{1}{3} \hat{p}_i P_{ij}^{\lambda'}(\hat{\mathbf{k}}) \hat{p}'_j \hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'. \quad (5.70)$$

Recall that the second order cumulant expansion (5.47) has an additional $-\frac{1}{2}$ factor. We define

$$S^{\mathrm{AL}} = -\frac{1}{2}\left(\langle [S^{(3)}]^2\rangle - \langle S^{(3)}\rangle^2\right). \tag{5.71}$$

5.5. Zero sound equation with corrections

In the previous section, we showed that the leading order corrections from pair fluctuations are given in S^{AL}, S^{DOS}, and S^{MT}. Together with the non-interacting action for zero sound (see Section 5.3.1), we have the effective action

$$S^{\phi} = S_0^{\phi} + S^{\mathrm{AL}} + S^{\mathrm{DOS}} + S^{\mathrm{MT}}, \tag{5.72}$$

where all terms are quadratic in ϕ. As done in Section 5.3.1, we take the variation with respect to the field $\phi_{\mathbf{p}}(-k)$, and obtain a linear equation for δn. This linear equation describes the original zero sound with the corrections from S^{AL}, S^{DOS}, and S^{MT}. We can then use the equation to study zero sound near the superfluid phase transition. Taking the variation, we obtain

$$\delta n_{\mathbf{p}}(k) + 2n_{\mathrm{F}}'(\xi_{\mathbf{p}})\frac{\mathbf{v}_{\mathbf{p}}\cdot\mathbf{k}}{ik_m - \mathbf{v}_{\mathbf{p}}\cdot\mathbf{k}}\phi_{\mathbf{p}}(k) + (\mathrm{AL}) + (\mathrm{DOS}) + (\mathrm{MT}) = 0, \tag{5.73}$$

where (AL), (DOS), and (MT) are contributions from taking the variation of the expressions (5.71), (5.53), and (5.58).

Using the ansatz $\delta n_{\mathbf{p}}(k) = -n_{\mathrm{F}}'(\xi_{\mathbf{p}})\nu_{\hat{\mathbf{p}}}(k)$, we have

$$\phi_{\mathbf{p}}(k) \equiv \frac{1}{V}\sum_{\mathbf{p}'} f^{\mathrm{s}}_{\mathbf{p},\mathbf{p}'}\delta n_{\mathbf{p}'}(k) \approx N(0)\int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi} f^{\mathrm{s}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\nu_{\hat{\mathbf{p}}}(k), \tag{5.74}$$

and thus

$$-n'_{\mathrm{F}}(\xi_{\mathbf{p}})\nu_{\hat{\mathbf{p}}}(k)+2n'_{\mathrm{F}}(\xi_{\mathbf{p}})\frac{\mathbf{v}\cdot\mathbf{k}}{ik_m-\mathbf{v}\cdot\mathbf{k}}N(0)\int_{\hat{\mathbf{p}}}f^{\mathrm{s}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}')\nu_{\hat{\mathbf{p}}}(k)$$
$$+(\mathrm{AL})+(\mathrm{DOS})+(\mathrm{MT})=0\,. \quad (5.75)$$

Since the function $-n_{\mathrm{F}}(\xi_{\mathbf{p}})$ is peaked at the Fermi energy, we integrate over the $\xi_{\mathbf{p}}$ variable and thus all momenta are restricted to the Fermi surface. After some standard expansion, the equation becomes

$$\frac{\nu_l}{2l+1}+\sum_{l'}\Omega_{ll'}(s)F^{\mathrm{s}}_{l'}\frac{\nu_{l'}}{2l'+1}+(\mathrm{AL})_l+(\mathrm{DOS})_l+(\mathrm{MT})_l=0\,, \quad (5.76)$$

where the term $(\mathrm{AL})_l$ and the other similar terms are obtained by integrating out $\xi_{\mathbf{p}}$ and applying $\int_{\hat{\mathbf{p}}}P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})$ to get the l-th component. The Ω function is defined by

$$\Omega_{ll'}(s)\equiv\int_{-1}^{1}\frac{du}{2}P_l(u)\frac{u}{u-s}P_{l'}(u)\,. \quad (5.77)$$

See Chapter 2 for more details.

The correction (AL) from (5.71) is

$$(\mathrm{AL})=+T\sum_{\mathbf{p}'}\left[\lambda^{\mathrm{AL}}_{\mathbf{p}}(k)\lambda^{\mathrm{AL}}_{\mathbf{p}'}(k)+\lambda^{\mathrm{AL}}_{\mathbf{p}}(-k)\lambda^{\mathrm{AL}}_{\mathbf{p}'}(-k)\right]\phi_{\mathbf{p}'}(k)$$
$$\times\left(\frac{3}{N(0)VT}\right)^2\frac{V}{\xi_0^3}C(\vartheta)K^{\lambda'}(k)\frac{1}{3}\hat{p}_iP^{\lambda'}_{ij}(\hat{\mathbf{k}})\hat{p}'_j\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}'. \quad (5.78)$$

The factor λ^{AL}, which comes from the fermion loop, gives

$$\lambda^{\mathrm{AL}}_{\mathbf{p}}(k)=2T\frac{-n'_{\mathrm{F}}(\xi_{\mathbf{p}})ik_m}{(ik_m-\mathbf{v}_{\mathbf{p}}\cdot\mathbf{k})^2} \quad (5.79)$$

in the $q_m = 0$ and $\mathbf{q} \to 0$ limit. Therefore

$$(\mathrm{AL}) = +T\sum_{\mathbf{p}'} 4T^2 \frac{-n_{\mathrm{F}}'(\xi_{\mathbf{p}})ik_m}{(ik_m - \mathbf{v}\cdot\mathbf{k})^2} \times \frac{-n_{\mathrm{F}}'(\xi_{\mathbf{p}'})ik_m}{(ik_m - \mathbf{v}'\cdot\mathbf{k})^2} \times 2N(0)\int_{\hat{\mathbf{p}}''} f^{\mathrm{s}}(\hat{\mathbf{p}}'\cdot\hat{\mathbf{p}}'')\nu_{\hat{\mathbf{p}}''}(k) \times \Big(\frac{3}{N(0)VT}\Big)^2 \frac{V}{\xi_0^3} C(\vartheta)K^{\nu'}(k)\frac{1}{3}\hat{p}_i P_{ij}^{\nu'}(\hat{\mathbf{k}})\hat{p}_j'\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}' \,. \quad (5.80)$$

Note that $\lambda_{\mathbf{p}}^{\mathrm{AL}}(k)\lambda_{\mathbf{p}'}^{\mathrm{AL}}(k)$ and $\lambda_{\mathbf{p}}^{\mathrm{AL}}(-k)\lambda_{\mathbf{p}'}^{\mathrm{AL}}(-k)$ are the same and lead to the factor 2. Integrating $-n_{\mathrm{F}}'(\xi_{\mathbf{p}})$ out and applying $\int_{\hat{\mathbf{p}}} P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})$, we obtain

$$(\mathrm{AL})_l = +4s^4\frac{T^2}{\omega^2}\Omega_{l,ik}^{\mathrm{AL}}(s)\sum_{l'}\Omega_{l',jm}^{\mathrm{AL}}(s)F_{l'}^{\mathrm{s}}\frac{\nu_{l'}}{2l'+1} \times \frac{3}{N(0)\xi_0^3 T}C(\vartheta)K^{\lambda'}(k)P_{ij}^{\lambda'}(\hat{\mathbf{k}})\delta_{km}\,, \quad (5.81)$$

where

$$\Omega_{l,ik}^{\mathrm{AL}}(s) \equiv \int_{\hat{\mathbf{p}}} P_l(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})\frac{1}{(s-\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^2}\hat{p}_i\hat{p}_k\,. \quad (5.82)$$

The contributions from S^{DOS} and S^{MT} are simpler. After taking the variation, we have

$$(\mathrm{DOS}) = \Big[\lambda_{\mathbf{p}}^{\mathrm{DOS}}(k) + \lambda_{\mathbf{p}}^{\mathrm{DOS}}(-k)\Big]\phi_{\mathbf{p}}(k)\frac{1}{N(0)T\xi_0^3}C(\vartheta) \quad (5.83)$$

and

$$(\mathrm{MT}) = \Big[\lambda_{-\mathbf{p}}^{\mathrm{MT}}(k) + \lambda_{-\mathbf{p}}^{\mathrm{MT}}(-k)\Big]\phi_{-\mathbf{p}}(k)\frac{1}{N(0)T\xi_0^3}C(\vartheta)\,. \quad (5.84)$$

The fermion loops give

$$\lambda_{\mathbf{p}}^{\mathrm{DOS}}(k) = 2T^2\frac{n_{\mathrm{F}}'(\xi_{\mathbf{p}})ik_m}{(ik_m - \mathbf{v}\cdot\mathbf{k})^3} \quad (5.85)$$

and

$$\lambda_{\mathbf{p}}^{\mathrm{MT}}(k) = 2T^2 \frac{n_{\mathrm{F}}'(\xi_{\mathbf{p}}) i k_m^2}{(ik_m - \mathbf{v}\cdot\mathbf{k})^2 (ik_m + \mathbf{v}\cdot\mathbf{k})^2} \,. \tag{5.86}$$

After some algebra, we obtain for the DOS part

$$(\mathrm{DOS})_l = -2s^3 \frac{T^2}{\omega^2} \frac{1}{N(0) T \xi_0^3} C(\vartheta) \sum_{l'} \Omega_{ll'}^{\mathrm{DOS}}(s) \frac{F_{l'}^{\mathrm{s}}}{2l'+1} \nu_{l'} \,, \tag{5.87}$$

where

$$\Omega_{ll'}^{\mathrm{DOS}}(s) = \int_{-1}^{1} \frac{du}{2} P_l(u) \frac{1}{(s-u)^3} P_{l'}(u) \,, \tag{5.88}$$

and for the MT part

$$(\mathrm{MT})_l = -2s^4 \frac{T^2}{\omega^2} \frac{1}{N(0) T \xi_0^3} C(\vartheta) \sum_{l'} \Omega_{ll'}^{\mathrm{MT}}(s) \frac{(-1)^{l'} F_{l'}^{\mathrm{s}}}{2l'+1} \nu_{l'} \,, \tag{5.89}$$

where

$$\Omega_{ll'}^{\mathrm{MT}}(s) = \int_{-1}^{1} \frac{du}{2} P_l(u) \frac{1}{(s-u)^2 (s+u)^2} P_{l'}(u) \,. \tag{5.90}$$

Recall that the dimensionless variable $s \equiv \omega / v_{\mathrm{F}} k$ for zero sound without pair fluctuations is about 5–10, depending on the pressure. The Ω functions for the three corrections scale as

$$\Omega^{\mathrm{AL}}(s) \sim \frac{1}{s^2}, \quad \Omega^{\mathrm{DOS}}(s) \sim \frac{1}{s^3}, \quad \Omega^{\mathrm{MT}}(s) \sim \frac{1}{s^4} \,, \tag{5.91}$$

when the variable s is large. Using this approximation, we immediately see that the corrections $(\mathrm{AL})_l$, $(\mathrm{DOS})_l$, and $(\mathrm{MT})_l$ are of order s^0, while the normal term from particle-hole bubble scales as $1/s$ (see the second term in the equation (5.76)). Although the corrections contain a small parameter $1/N(0)T_{\mathrm{c}}\xi_0^3$, numerical calculations

show that they are not small compared to the normal particle-hole bubble contribution. The corrections thus strongly distort the zero sound equation, which is inconsistent with the experiments [8, 12].

5.6. Summary

In this chapter, we construct a bosonic theory for zero sound and pair fluctuations using the Hubbard-Stratonovich transformation. By considering the coupling between the bosonics fields, we obtain the zero sound equation with corrections from pair fluctuations. The corrections all have the factor

$$C(\vartheta) \sim \left[x_c - \sqrt{\vartheta}\,\mathrm{atan}\left(\frac{x_c}{\sqrt{\vartheta}}\right)\right], \tag{5.92}$$

which also appears in the result obtained in Chapter 4. The corrections are, however, not small in magnitude, and thus strongly change the zero sound equation. In addition, the corrections have strong frequency dependence. This result is different from the result obtained using kinetic equation (see Chapter 4), and also inconsistent with the experimental observation, where the deviation from the normal behavior is small [8, 12]. The inconsistency implies that the theory proposed in this chapter is incomplete. On the other hand, our approach does generate the three standard diagrams considered in paraconductivity [20] and the factor (5.92). In summary, this approach might capture some physics of the pair fluctuations, but further investigation is needed in order to fully understand the problem and solve the discrepancy.

CHAPTER 6

Conclusion

Fermi liquid theory is a cornerstone of condensed matter physics, and zero sound is a unique feature of a neutral Fermi liquid. We study how Cooper pairing, another important mechanism in physics, can influence zero sound propagation. We consider two very different approaches to construct the theory. The first approach is based on the kinetic equation, described in Chapter 3 and 4. Since the standard description of zero sound is based on Landau's kinetic equation [26], this approach is quite natural and reliable. We include the self-energy induced by the pairing interaction in the standard Boltzmann-Landau equation. This self-energy gives the pair fluctuation correction to the standard Fermi liquid behavior, including the quasiparticle energy and the collision integral. We have shown in Chapter 4 that the calculated zero sound attenuation based on this theory is in close agreement with Paulson and Wheatley's experiment [8]. We also calculate the fluctuation correction to zero sound velocity. Although the magnitude of the velocity correction is inconsistent with the high-frequency experiment by Lee et al. [12], the theory shows that the pair fluctuations do suppress the zero sound velocity. Additional experiments at lower frequencies would be important to test the theory.

The second approach is formulated in terms of functional integrals. We notice that the zero sound can be described by a boson field introduced by the Hubbard-Stratonovich transformation. After integrating out the fermion fields, the mean-field

approximation gives the collisionless zero sound equation. In the presence of pairing interaction, we can use an additional Hubbard-Stratonovich transformation to introduce a field for the pair fluctuations. The zero sound field and the pair fluctuation field interact, and we obtain diagrams similar to the AL, DOS, and MT diagrams in the theory of paraconductivity after averaging over the fluctuations. The $\sqrt{\vartheta}$ behavior also appears, as in the first approach. Unfortunately, the magnitude of the corrections is large and we do not know the cause and how to solve this problem.

In summary, we have studied zero sound and pair fluctuations using two different approaches. The first one follows the standard kinetic equation viewpoint, and generates results in close agreement with experiments on the attenuation. In our second approach, we tried a more novel method to formulate the problem. The result is not satisfying, but the structure of the theory shares some similarities with the conventional theory of paraconductivity. This work is the first try to study zero sound with pair fluctuations from the *viewpoint of conventional paraconductivity theory*. More theoretical and experimental studies are needed to reconcile the inconsistency between the theoretical prediction and the experimental observation, as well as the connection between the two approaches. The first approach is *ready to be applied to other transport properties*. Furthermore, we can consider the presence of external perturbations, for example, impurities and magnetic field. Pair fluctuations in liquid ^{3}He have not been extensively studied. We hope this work can motivate more research on this topic.

APPENDIX A

Fluctuation Propagator in Equilibrium

The pair fluctuations are described by the effective vertex in the Cooper channel, or the fluctuation propagator, as shown in Section 4.1. In this appendix we evaluate the key factor (4.13) in equilibrium. In particular, we show how to introduce the transition temperature T_c into the theory by renormalization. Using the equilibrium distribution $f(\xi) = \frac{1}{e^{\beta\xi}+1}$ and the quasiparticle energy $\xi_{\mathbf{q}-\mathbf{p}} \approx \xi_{\mathbf{p}} - \mathbf{v}\cdot\mathbf{q}$, the integral in (4.13) becomes

$$\int_{\mathbf{p}} \frac{f(\xi_{\mathbf{p}}) + f(\xi_{\mathbf{p}} - \mathbf{v}\cdot\mathbf{q}) - 1}{\omega + \mathbf{v}\cdot\mathbf{q} - 2\xi_{\mathbf{p}} + i0}\hat{p}_i\hat{p}_k\,. \tag{A.1}$$

Changing the variable $\xi_{\mathbf{p}} \to \xi_{\mathbf{p}} + (\omega + \mathbf{v}\cdot\mathbf{q})/2$ and using the approximation $N(\xi_{\mathbf{p}}) \approx N(0)$, we have

$$N(0)\int d\xi_{\mathbf{p}} \int_{\hat{\mathbf{p}}} \frac{f(\xi_{\mathbf{p}} + \frac{\omega+\mathbf{v}\cdot\mathbf{q}}{2}) + f(\xi_{\mathbf{p}} + \frac{\omega-\mathbf{v}\cdot\mathbf{q}}{2}) - 1}{-2\xi_{\mathbf{p}} + i0}\hat{p}_i\hat{p}_k\,. \tag{A.2}$$

Expanding the numerator and keeping only the leading terms in ω and q, the factor (4.13) becomes

$$L_{ik}^{\mathrm{GL}} = \frac{1}{3iV} - iN(0)\int d\xi_{\mathbf{p}} \int_{\hat{\mathbf{p}}} \frac{2f(\xi_{\mathbf{p}}) - 1 + f'(\xi_{\mathbf{p}})\omega + \frac{1}{2}f''(\xi_{\mathbf{p}})\frac{(\mathbf{v}\cdot\mathbf{q})^2}{2}}{-2\xi_{\mathbf{p}} + i0}\hat{p}_i\hat{p}_k\,. \tag{A.3}$$

We calculate the integrals for $f'(\xi_{\mathbf{p}})$ and $f''(\xi_{\mathbf{p}})$ first, since they are convergent integrals. Using $1/(x+i0) = \mathrm{P.V.}\frac{1}{x} - i\pi\delta(x)$, the integral with $f'(\xi_{\mathbf{p}})$ can be obtained

immediately,

$$\int d\xi_{\mathbf{p}} \frac{f'(\xi_{\mathbf{p}})}{-2\xi_{\mathbf{p}} + i0}\omega = i\pi \frac{1}{8}\frac{\omega}{T}\,, \tag{A.4}$$

where the even function $f'(z) = -f(z)(1-f(z))/T$ has been used. The integral with $f''(\xi_{\mathbf{p}})$ can be done with the identity

$$\int d\xi \frac{f''(\xi)}{-2\xi + i0} = -\frac{7\zeta(3)}{(2\pi T)^2}\,. \tag{A.5}$$

The angular part gives

$$\int \frac{d\Omega_{\hat{\mathbf{p}}}}{4\pi}(\mathbf{v}\cdot\mathbf{q})^2 \hat{p}_i \hat{p}_k = \frac{1}{15} v_{\mathrm{F}}^2 q^2 (\delta_{ik} + 2\hat{q}_i \hat{q}_k)\,. \tag{A.6}$$

The remaining algebra is simple and using the definition of projection $P_{\parallel,\perp}$, we can get the coherence length $\xi_{\parallel,\perp}$, which is given below Equation (4.14).

The subtle part comes from the integral with $(2f(\xi_{\mathbf{p}})-1)/\xi_{\mathbf{p}}$, which has ultraviolet divergence. This problem can fixed by renormalization, absorbing the infinity into a measurable quantity. We know the fluctuation propagator has a pole at temperature T_{c} for the $q = 0$ mode, so $L_{ik}^{\mathrm{GL}}(q = 0; T = T_{\mathrm{c}}) = 0$. This requires

$$\frac{1}{3V} + \frac{1}{3}N(0)\int d\xi_{\mathbf{p}} \frac{2f(\xi_{\mathbf{p}})-1}{-2\xi_{\mathbf{p}} + i0}\bigg|_{T_{\mathrm{c}}} = 0\,, \tag{A.7}$$

where the factor $\frac{1}{3}$ in front of $N(0)$ comes from the angular integral. Note that the integral in this condition is divergent, which means a ultraviolet cutoff is needed.[1] The cutoff (related to the divergence) and the unknown interaction V are absorbed into the measurable quantity T_{c}. This condition allows us to eliminate the divergence

[1] This is called regularization in field theory. There are other ways to regularize the integral, but a cutoff is enough in this case.

in the integral. Inserting the condition into (A.3), we obtain

$$\int d\xi_{\mathbf{p}} \frac{\left[2f(\xi_{\mathbf{p}})-1\right]\Big|_{T_c}^{T}}{-2\xi_{\mathbf{p}}} \approx -\int d\xi_{\mathbf{p}} \frac{1}{\xi_{\mathbf{p}}} \frac{\partial f}{\partial \xi_{\mathbf{p}}} \frac{\xi_{\mathbf{p}}}{T}\bigg|_{T_c} (T-T_c) = -\frac{T-T_c}{T_c} \equiv -\theta\,. \tag{A.8}$$

Thus the transition temperature T_c appears in our theory, in place of the unknown interaction V.

APPENDIX B

Calculation of Landau Parameter Correction

In this appendix we derive the expressions (4.90) and (4.91) from (4.83), and (4.86), and (4.89). The expression (4.83) is symmetric under exchange of $\mathbf{p}'$ and $\mathbf{q}-\mathbf{p}'$, so the expression can be reduced to

$$\int_{\mathbf{q}}\int_{\mathbf{p}'} \frac{(2n_4-1)\delta n_3}{|L_\lambda^{\mathrm{GL}}(\omega=\epsilon_3+\epsilon_4,\mathbf{q})|^2} \times \frac{1}{\epsilon_3+\epsilon_4-\epsilon_1-\epsilon_2}\hat{p}'_i\hat{p}'_j\,. \tag{B.1}$$

Moreover, we approximate

$$L_\lambda^{\mathrm{GL}}(\omega=\epsilon_3+\epsilon_4,\mathbf{q}) \approx L_\lambda^{\mathrm{GL}}(\omega=0,\mathbf{q})\,. \tag{B.2}$$

We first integrate the energy $\epsilon_{\mathbf{p}'}$ and the momentum $\mathbf{q}$ (separated as $\hat{\mathbf{q}}$ and q). The integral has a pole and we need to take the principal value.

$$\begin{aligned}\frac{1}{\epsilon_3+\epsilon_4-\epsilon_1-\epsilon_2} &= \frac{1}{\epsilon_{\mathbf{p}'}+\epsilon_{\mathbf{q}-\mathbf{p}'}-\epsilon_{\mathbf{p}}-\epsilon_{\mathbf{q}-\mathbf{p}}} \\ &\approx \frac{1}{2\epsilon_{\mathbf{p}'}-\mathbf{v}'\cdot\mathbf{q}-2\epsilon_{\mathbf{p}}+\mathbf{v}\cdot\mathbf{q}}\,.\end{aligned} \tag{B.3}$$

Change the variable

$$\epsilon_{\mathbf{p}'} \to \epsilon_{\mathbf{p}'}+\epsilon_{\mathbf{p}}-(\mathbf{v}-\mathbf{v}')\cdot\mathbf{q}/2\,, \tag{B.4}$$

so the denominator becomes $2\epsilon_{\mathbf{p}'}$. We assume

$$\delta n_{\mathbf{p}'} = -\frac{\partial n_{\rm eq}}{\partial \epsilon_{\mathbf{p}'}} \nu_{\hat{\mathbf{p}}'}$$

and $\delta n_{\mathbf{p}'+\mathbf{q}} \approx -n'(\epsilon_{\mathbf{p}'+\mathbf{q}})\nu_{\hat{\mathbf{p}}'}$ so long as $q \ll p_{\rm F}$. We neglect the distribution function $\nu_{\mathbf{p}'}$ in the following for simplicity. The numerator can be expanded around $\mathbf{q} = 0$, and we have

$$\int_{\hat{\mathbf{q}}} \int d\epsilon_3 \frac{1}{2\epsilon_3} \Big[-(2n(\epsilon_3)-1)n'(\epsilon_3) - \frac{\mathbf{q}\mathbf{q}}{8} \big[(2f-1)f'''(\mathbf{v}-\mathbf{v}')(\mathbf{v}-\mathbf{v}') + 4n'n''(\mathbf{v}+\mathbf{v}')(\mathbf{v}-\mathbf{v}') + 2n'n''(\mathbf{v}+\mathbf{v}')(\mathbf{v}+\mathbf{v}') \big] \Big] \hat{p}'_i \hat{p}'_j , \quad \text{(B.5)}$$

where all n's and their derivatives are equilibrium distributions with the variable ϵ_3, and the first (second) vector $\mathbf{q}$ is multiplied by the first (second) velocity vector. Note that we have neglected the terms linear in $\mathbf{q}$ since they vanish after the $\hat{\mathbf{q}}$ integration. Recall that we have a $1/|L_\lambda^{\rm GL}|^2$ factor. Since the two components $\lambda = \parallel, \perp$ have different coherence lengths, we need to calculate them separately, by decomposing the integral (B.5) using P_{ij}^λ. The rest is the radial q integral, which can be done with a cutoff $q_{\rm c}$. The final expression is a $\hat{\mathbf{p}}'$ integral. The Landau interaction energy can be written as

$$\delta\epsilon_{\mathbf{p}} = \int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} \sum_l F_l^{\rm s} P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') \nu_{\hat{\mathbf{p}}'} \, . \quad \text{(B.6)}$$

Comparing the result with the above expression, we obtain the first two terms in (4.90) and the first term in (4.91).

The integral (4.86) does not contribute to the leading order. The expansion gives

$$\int_{\mathbf{q}} \int d\epsilon_3 \frac{1}{2\epsilon_3} \Big[-f'(\epsilon_3) - f''(\epsilon_3) \big[-(\mathbf{v}-\mathbf{v}')\cdot\mathbf{q}/2 \big] - \frac{1}{2} f'''(\epsilon_3) \big[-(\mathbf{v}-\mathbf{v}')\cdot\mathbf{q}/2 \big]^2 \Big] , \tag{B.7}$$

where each term is either odd in ϵ_3 or odd in $\mathbf{q}$, so the whole integral vanishes.

The expression (4.89) is

$$\frac{1}{2} \frac{\frac{N(0)}{3}(\vartheta + \xi_\lambda^2 q^2)}{|L_\lambda^{\mathrm{GL}}(\omega = \epsilon + \epsilon_2, \mathbf{q})|^2} \times 2\delta f_2 \times \hat{p}_i' P_{ij}^\lambda \hat{p}_j' . \tag{B.8}$$

Note that there is no $\mathbf{p}'$ integration for this term, which has been done in obtaining the expression $N(0)(\vartheta + \xi_0^2 q^2)$. Obviously, no $\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}'$ term can appear from this term. Integration over $\hat{\mathbf{q}}$ gives

$$\frac{1}{2} \frac{\frac{N(0)}{3}(\vartheta + \xi_\lambda^2 q^2)}{|L_\lambda^{\mathrm{GL}}(\omega = \epsilon + \epsilon_2, \mathbf{q})|^2} \times 2\delta f_2 \times \frac{n_\lambda}{3} , \tag{B.9}$$

with $n_\parallel = 1$ and $n_\lambda = 2$.

Ignoring the $n_\lambda/3$ factor temporarily, the above expression (B.9) equals

$$\frac{\frac{N(0)}{3}(\vartheta + \xi_\lambda^2 q^2)}{|L_\lambda^{\mathrm{GL}}(\omega = \epsilon + \epsilon_2, \mathbf{q})|^2} \big[-f'(\epsilon_{\mathbf{q}-\mathbf{p}}) \big] \nu_{-\hat{\mathbf{p}}} . \tag{B.10}$$

Using the identity

$$\int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} \Big[4\pi \sum_{lm} Y_{lm}(\hat{\mathbf{p}}) Y_{lm}^*(\hat{\mathbf{p}}') \Big] g(\hat{\mathbf{p}}') = g(\hat{\mathbf{p}}) \tag{B.11}$$

for an arbitrary function g defined on a sphere, the above expression can be written as

$$\int \frac{d\Omega_{\hat{\mathbf{p}}'}}{4\pi} \frac{\frac{N(0)}{3}(\vartheta + \xi_\lambda^2 q^2)}{|L_\lambda^{\mathrm{GL}}(\omega = \epsilon + \epsilon_2, \mathbf{q})|^2} \big[-f'(\epsilon_{\mathbf{q}-\mathbf{p}}) \big] \times \Big[4\pi \sum_{lm} Y_{lm}(\hat{\mathbf{p}}) Y_{lm}^*(\hat{\mathbf{p}}') \Big] \nu_{-\hat{\mathbf{p}}'} \,. \tag{B.12}$$

The addition theorem gives $(2l+1)P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') = 4\pi \sum_m Y_{lm}(\hat{\mathbf{p}}) Y_{lm}^*(\hat{\mathbf{p}}')$. We have

$$\int_{\hat{\mathbf{p}}'} \sum_l \frac{\frac{N(0)}{3}(\vartheta + \xi_\lambda^2 q^2)}{|L_\lambda^{\mathrm{GL}}(\omega = \epsilon + \epsilon_2, \mathbf{q})|^2} \big[-f'(\epsilon_{\mathbf{q}-\mathbf{p}}) \big] \times (2l+1)(-1)^l P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') \nu_{\hat{\mathbf{p}}'} \,, \tag{B.13}$$

where the factor $(-1)^l$ comes from changing $-\hat{\mathbf{p}}' \to \hat{\mathbf{p}}'$.

Approximate $f'(\epsilon_{\mathbf{q}-\mathbf{p}}) \approx f'(\epsilon_{\mathbf{p}})$ since q only reduces the singularity. As before, we approximate $\omega = \epsilon + \epsilon_2 \approx 0$ in the denominator. The q integral gives

$$\int \frac{q^2 dq}{2\pi^2} \frac{1}{\frac{N(0)}{3}(\vartheta + \xi_\lambda^2 q^2)} = \frac{1}{2\pi^2 \frac{N(0)}{3} \xi_\lambda^3} \Big[x_c - \sqrt{\vartheta} \arctan\big(\frac{x_c}{\sqrt{\vartheta}}\big) \Big] \,. \tag{B.14}$$

Together with $-f'(\epsilon_{\mathbf{p}}) = -f'(0) \approx 1/4T$ in equilibrium, we have

$$\int_{\hat{\mathbf{p}}'} \sum_l \frac{3}{8\pi^2 N(0) \xi_\lambda^3 T} \big[x_c - \sqrt{\vartheta} \arctan\big(\frac{x_c}{\sqrt{\vartheta}}\big) \big] \times (2l+1)(-1)^l P_l(\hat{\mathbf{p}} \cdot \hat{\mathbf{p}}') \nu_{\hat{\mathbf{p}}'} \,. \tag{B.15}$$

Using $\frac{1}{\xi_\parallel^3} + \frac{2}{\xi_\perp^3} \approx \frac{4.7174}{\xi_0^3}$, the final result gives the last term in (4.90) and the last term in (4.91).

APPENDIX C

Derivation of the Bosonic Action

In this appendix, we derive the bosonic action (5.25). The derivation contains two steps. We first show that a Grassmann gaussian integral is given by the determinant of the involved matrix. Then we apply this result to the action quadratic in fermion fields to derive the bosonic action.

C.1. Gaussian integral of Grassmann numbers

In this appendix we prove the formula

$$\int d\theta_1 \cdots d\theta_N \, e^{-\frac{1}{2}\theta_i M_{ij} \theta_j} = \det(M)^{\frac{1}{2}}, \tag{C.1}$$

where θ_i's are Grassmann numbers, and M is a matrix with normal numbers. Grassmann numbers satisfy $\int d\theta \ \theta = 1$ and $\int d\theta \ 1 = 0$. Therefore the relevant terms in the expansion of the exponential must be a product of the form $\theta_1 \cdots \theta_N$. Otherwise the integration $\int d\theta_1 \cdots d\theta_N$ gives zero. This means that the number N must be even, because the exponent is quadratic in θ_i. The only term giving contribution is the $\frac{N}{2}$-order term

$$\frac{1}{\frac{N}{2}!} \Big[-\frac{1}{2} \sum_{i,j} \theta_i M_{ij} \theta_j \Big]^{\frac{N}{2}}. \tag{C.2}$$

Note that the matrix M can be separated into symmetric and anti-symmetric parts. Only the anti-symmetric part contributes to $\theta_i M_{ij} \theta_j$, since Grassmann numbers are

anti-commutative. Hence we *assume the matrix M is anti-symmetric*. Since there are $N/2$ brackets and we choose a different one in each bracket to get all θ_i's, these terms give

$$\frac{(-1/2)^{N/2}}{\frac{N}{2}!}\theta_1\theta_2\cdots\theta_N\sum_{p\in S_N}\mathrm{sgn}(p)M_{p(1)p(2)}M_{p(3)p(4)}\cdots M_{p(N-1)p(N)}\,, \tag{C.3}$$

where S_N is the symmetric group of order N. The rest is the integral

$$\int d\theta_1\cdots d\theta_N\;\theta_1\cdots\theta_N\,. \tag{C.4}$$

Now recall that a pair of Grassmann numbers "moves" as normal numbers, for example $(\theta_1\theta_2)(\theta_3\theta_4)=(\theta_3\theta_4)(\theta_1\theta_2)$. Thus we have

$$\theta_1\theta_2\cdots\theta_{N-1}\theta_N=(\theta_{N-1}\theta_N)\cdots(\theta_1\theta_2)=(-1)^{\frac{N}{2}}(\theta_N\theta_{N-1})\cdots(\theta_2\theta_1)\,. \tag{C.5}$$

The minus sign in (C.3) is canceled and the integral gives

$$\int d\psi_1\cdots d\psi_N e^{-\frac{1}{2}\theta_i M_{ij}\theta_j}=\frac{1}{2^{\frac{N}{2}}(\frac{N}{2})!}\sum_{p\in S_N}\mathrm{sgn}(p)M_{p(1)p(2)}M_{p(3)p(4)}\cdots M_{p(N-1)p(N)}\,. \tag{C.6}$$

The right-hand side of the above equation is called Pfaffian for an anti-symmetric matrix, which has the property $\mathrm{Pf}(M)=\det(M)^{\frac{1}{2}}$. Therefore, we have

$$\int d\theta_1\cdots d\theta_N e^{-\frac{1}{2}\theta_i M_{ij}\theta_j}=\det(M)^{\frac{1}{2}} \tag{C.7}$$

for an anti-symmetric matrix with even N.

C.2. Anti-symmetry in Nambu space

We want to use the formula (C.7) to integrate out the fermion fields. To this end, we have to show that the action can be written in the form of $\theta_i M_{ij} \theta$, and the matrix M is anti-symmetric.

Define the Nambu spinor

$$\Psi_{\mathbf{p}} \equiv \begin{pmatrix} \psi_{\mathbf{p}\uparrow} \\ \psi_{\mathbf{p}\downarrow} \\ \bar{\psi}_{\mathbf{p}\uparrow} \\ \bar{\psi}_{\mathbf{p}\downarrow} \end{pmatrix} \tag{C.8}$$

and

$$\bar{\Psi}_{\mathbf{p}} \equiv \begin{pmatrix} \bar{\psi}_{\mathbf{p}\uparrow} & \bar{\psi}_{\mathbf{p}\downarrow} & \psi_{\mathbf{p}\uparrow} & \psi_{\mathbf{p}\downarrow} \end{pmatrix}. \tag{C.9}$$

Recall that the ψ-related part in the action S is

$$S^{\psi} = S_0 + S_{\mathrm{L}}^{\psi} + S_{\mathrm{C}}^{\psi} \tag{C.10}$$

with

$$\begin{aligned} S_0 &= \int_0^{\beta} d\tau \sum_{\mathbf{p}} \bar{\psi}_{\mathbf{p}}(\tau) \left[\partial_\tau + \xi_{\mathbf{p}}^0\right] \psi_{\mathbf{p}}(\tau) \\ &= \frac{1}{2} \int_0^{\beta} d\tau \sum_{\mathbf{p}} \left[\bar{\psi}_{\mathbf{p}}(\tau) \left[\partial_\tau + \xi_{\mathbf{p}}^0\right] \psi_{\mathbf{p}}(\tau) + \psi_{\mathbf{p}}(\tau) \left[\partial_\tau - \xi_{\mathbf{p}}^0\right] \bar{\psi}_{\mathbf{p}}(\tau)\right] \\ &= \frac{1}{2} \int_0^{\beta} d\tau \sum_{\mathbf{p}} \left[\bar{\psi}_{\mathbf{p}}(\tau) \left[\partial_\tau + \xi_{\mathbf{p}}^0\right] \psi_{\mathbf{p}}(\tau) + \psi_{-\mathbf{p}}(\tau) \left[\partial_\tau - \xi_{-\mathbf{p}}^0\right] \bar{\psi}_{-\mathbf{p}}(\tau)\right], \end{aligned} \tag{C.11}$$

$$S_{\mathrm{L}}^{\psi} = \frac{1}{2} \int d\tau \sum_{\mathbf{p},\mathbf{p}'} \bar{\psi}_{\mathbf{p}}(\tau) i\phi_{\mathbf{p}}(\mathbf{p}-\mathbf{p}',\tau) \psi_{\mathbf{p}'}(\tau) - \psi_{\mathbf{p}}(\tau) i\phi_{\mathbf{p}'}(\mathbf{p}'-\mathbf{p},\tau) \bar{\psi}_{\mathbf{p}'}, \tag{C.12}$$

and

$$S_{\mathrm{C}}^{\psi}=\frac{1}{2}\int d\tau\sum_{\mathbf{p},\mathbf{p}'}\Big[\bar{\psi}_{\mathbf{p}}(\tau)g_{\alpha}\frac{(\mathbf{p}-\mathbf{p}')_i}{2p_{\mathrm{F}}}A_{\alpha i}(\mathbf{p}+\mathbf{p}',\tau)\bar{\psi}_{\mathbf{p}'}(\tau) + \psi_{\mathbf{p}}(\tau)g_{\alpha}^{\dagger}\frac{-(\mathbf{p}-\mathbf{p}')_i}{2p_{\mathrm{F}}}A_{\alpha i}(\mathbf{p}+\mathbf{p}',\tau)\psi_{\mathbf{p}'}(\tau)\Big]. \quad \text{(C.13)}$$

In terms of the spinors (C.8) and (C.9), we have

$$S^{\psi}=\frac{1}{2}\int_0^{\beta}d\tau\sum_{\mathbf{p}}\bar{\Psi}_{\mathbf{p}}(\tau)[-\widehat{\mathcal{G}}_0^{-1}(\mathbf{p},\tau)\delta_{\mathbf{p},\mathbf{p}'}+i\widehat{\phi}+\widehat{A}]\Psi_{\mathbf{p}'}(\tau) \quad \text{(C.14)}$$

with

$$-\widehat{\mathcal{G}}_0^{-1}(\mathbf{p},\tau)=\begin{pmatrix}(\partial_\tau+\xi_{\mathbf{p}})\delta_{\sigma\sigma'} & 0\\ 0 & (\partial_\tau-\xi_{\mathbf{p}})\delta_{\sigma\sigma'}\end{pmatrix}, \quad \text{(C.15)}$$

$$\widehat{\phi}=\begin{pmatrix}\phi_{\mathbf{p}}(\mathbf{p}-\mathbf{p}',\tau) & 0\\ 0 & -\phi_{\mathbf{p}'}(\mathbf{p}'-\mathbf{p},\tau)\end{pmatrix}, \quad \text{(C.16)}$$

$$\widehat{A}=\begin{pmatrix}0 & g_{\alpha}\frac{(\mathbf{p}-\mathbf{p}')_i}{2p_{\mathrm{F}}}A_{\alpha i}(\mathbf{p}+\mathbf{p}',\tau)\\ g_{\alpha}^{\dagger}\frac{(\mathbf{p}'-\mathbf{p})_i}{2p_{\mathrm{F}}}\bar{A}_{\alpha i}(\mathbf{p}+\mathbf{p}',\tau) & 0\end{pmatrix}. \quad \text{(C.17)}$$

The full action is

$$S=S^{\psi}+\frac{1}{2}\int_0^{\beta}d\tau\sum_{\mathbf{p},\mathbf{q}}n_{\mathbf{p}}(\mathbf{q},\tau)\phi_{\mathbf{p}}(-\mathbf{q},\tau)+\frac{1}{6\lambda}\int_0^{\beta}d\tau V\sum_{\mathbf{q}}\bar{A}(\mathbf{q},\tau)A(\mathbf{q},\tau). \quad \text{(C.18)}$$

For the anti-symmetry and the convenience of calculation, we transform the expression (C.14) to the frequency domain. Define

$$\psi(\tau)=T\sum_n e^{-ip_n\tau}\psi_n \quad \text{and} \quad \bar{\psi}=T\sum_n e^{+ip_n\tau}\bar{\psi}_n. \quad \text{(C.19)}$$

Using these definitions, the quadratic term becomes

$$S^{\psi} = \sum_{n,n'} (T\bar{\Psi}_p) \Big[-\widehat{\mathcal{G}}_0^{-1}(p,p') + i\widehat{\phi}(p,p') + \widehat{A}(p,p') \Big] (T\Psi_{p'}) \,, \tag{C.20}$$

where the matrices $\widehat{\mathcal{G}}_0$, $\widehat{\phi}$, and $\widehat{A}$ have been defined in (5.19)–(5.21). We intentionally keep the combinations $T\bar{\Psi}$ and $T\Psi$, because they are the dimensionless variables used in functional integration.

In order to use the Gaussian integral formula (C.7), we write $\bar{\Psi}_{\mathbf{p}} = \Psi_{\mathbf{p}}^{\mathsf{T}} \widehat{\tau}_1$ with $\widehat{\tau}_1 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$. Therefore the matrix in the middle of (C.20) becomes $\widehat{\tau}_1(-\widehat{\mathcal{G}}_0 + i\widehat{\phi} + \widehat{A})$. Since the matrix $\widehat{\tau}_1$ exchanges the first and second rows in the particle-hole space, it's easy to see from the expressions (5.19)–(5.21) that the matrix $\widehat{\tau}_1(-\widehat{\mathcal{G}}_0 + i\widehat{\phi} + \widehat{A})$ is anti-symmetric under $p \leftrightarrow p'$, $\sigma \leftrightarrow \sigma'$, and particle $\leftrightarrow$ hole. (Note that g_α and $g_\alpha^\dagger$ are symmetric in the spin space.)

The transformation from the time domain to the frequency domain, i.e. $\psi(\tau) \to T\psi_n$, is unitary, as shown by the Fourier expansion in (C.19). Thus the Jacobian for the functional integral is identity. Using the formula (C.7), we have

$$\begin{aligned} \int D[\bar{\psi}\psi] e^{-S^{\psi}} &= \det[-\widehat{\mathcal{G}}_0^{-1} + i\widehat{\phi} + \widehat{A}]^{\frac{1}{2}} \\ &= \exp\Big[\frac{1}{2} \operatorname{Tr} \ln(-\widehat{\mathcal{G}}_0^{-1} + i\widehat{\phi} + \widehat{A})\Big], \end{aligned} \tag{C.21}$$

where the last equality follows from the identity $\det(e^M) = e^{\operatorname{Tr}(M)}$ for any matrix M. Together with other terms independent of ψ and $\bar{\psi}$, we obtain the bosonic action (5.25).

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