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Motivic Contractibility of the Space of Rational Maps

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ABSTRACT

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The moduli stack of G-bundles on a smooth complete curve C over a field, $\mathrm{Bun}_G(C)$, is an immensely rich geometric object and is of central importance to the Geometric Langlands program. This thesis represents a contribution towards a motivic, in the sense of Voevodsky and Morel-Voevodsky, understanding of this stack.

Following the strategy of Gaitsgory and Gaitsgory-Lurie we view the Beilinson-Drinfeld Grassmanian, $\mathrm{Gr}_G(C)$ as a more tractable, homological approximation to $\mathrm{Bun}_G(C)$. In the main theorem of this thesis we prove, using two different approaches, that the motive of the fiber of the approximation map $\mathrm{Gr}_G(C) \to \mathrm{Bun}_G(C)$ is, in a number of different and precise ways, motivically contractible. This fiber is the space of rational maps, as introduced by Gaitsgory. One approach is to work in the context of E-modules where E is a motivic $\mathcal{E}_\infty$-ring spectra and show that there is a motivic equivalence between the space of rational maps and a version of the Ran space. Via various realization functors, we obtain the contractibility theorems of Gaitsgory and Gaitsgory-Lurie. A second, novel approach is to study the unstable motivic homotopy type using a theorem of Suslin and a model of the space of rational maps as introduced by Barlev.

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"*One must have fear when one writes* ∞."

It feels like such a distant past, an entirely different universe almost, since Luca Scala started me down this path. I thank him for that, and for sharing his fear. I still fear ∞, yet it seems most apropos that this thesis is riddled with higher categories.

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CHAPTER 1

Introduction

1.1. Motivation

Let C be a smooth, complete curve over the field of q elements $\mathbf{F}_q$. Suppose $G \to C$ is a connected group scheme over C such that the generic fiber of G is semsimple and simply connected. Then, Gaitsgory-Lurie [**32**] provided a mechanism by which one can calculate the "number of G-bundles" on C defined over $\mathbf{F}_q$. More precisely, they gave a formula for the stacky point-count:

$$|\mathrm{Bun}_G(C)(\mathbf{F}_q)| := \sum_{(\pi_0 \mathrm{Bun}_G(C))(\mathbf{F}_q)} \frac{1}{|\operatorname{Aut}(\mathcal{P})|}; \tag{1.1}$$

we invite the reader to [**33**, Theorem 1.4.4.1] for how the count looks like. This point count verifies the analogue of Weil's conjecture on Tamagawa numbers associated to the group G in the function fields situation.

To explain this mechanism, we recall the *Beilinson-Drinfeld* or the *factorization Grassmanian* (see [**9**], [**10**] for where this object was introduced; [**32**] also provides an excellent exposition. For a shorter, but lucid introduction see [**79**]), denoted by $\mathrm{Gr}_G(C)$. This is a prestack (simply a functor from affine schemes to ∞-groupoids) whose T-points classify triples

$$((f_1, \cdots, f_i) : T \to C, \mathcal{P}, \gamma) \tag{1.2}$$

where f_j's are T-points of C, $\mathcal{P}$ is a G-bundle on the relative curve C_T and γ is a trivialization of $\mathcal{P}$ away from the union of the graph of the f_j's. There is a map

$$\text{approx} : \mathrm{Gr}_G(C) \to \mathrm{Bun}_G(C) \tag{1.3}$$

which attempts to approximate $\mathrm{Bun}_G(C)$. This map is given by forgetting the auxiliary data and remembering only $\mathrm{Bun}_G(C)$.

The object $\mathrm{Gr}_G(C)$ is, at least morally, a simpler object than $\mathrm{Bun}_G(C)$ (see the beginning [**32**, Section 3] for a more elaborate discussion). If we fix the data of the T-points $(f_1, \cdots, f_i) : T \to C$, then $\mathrm{Gr}_G(C)$ are copies of the affine Grassmanian over these points. These affine Grassmanians are *loop groups* and have the "homotopy type" of $\mathbf{\Omega}_{\mathbf{G}_m}G$ (see [**6**] on how to make this formal in motivic homotopy theory), i.e., the (ind-)scheme of maps from the punctured affine line to the group. This is an inherently simpler object than $\mathrm{Bun}_G(C)$ which is the stack of maps from C into the stack BG.

The strategy in computing (1.1) in [**32**] is to work with the Beilinson-Drinfeld Grassmanian, rather than directly with $\mathrm{Bun}_G(C)$, after proving that the map (1.3) has contractible fibers in the sense that its étale-homology is acyclic. This is the statement of nonabelian Poincaré duality as formulated in [**32**, Theorem 3.2.12]. An earlier version of this theorem, proved over Spec $\mathbf{C}$, can be found in [**31**, Section 5].

1.2. What are we trying to do?

Suppose that X is an algebraic variety over a field k. Then its point count is but a shadow of its "motive." Pursuing this line of thought, we can ask if the count in (1.1) is a shadow of the motive of the stack $\mathrm{Bun}_G(\mathrm{C})$.

An adequate theory of motives of varieties over a field exists after Voevodsky's work (see [**72**] for the basic properties of this category, and [**51**] for a textbook account), and a six functor formalism has matured after the pioneering work of Ayoub [**3**] and subsequently by Cisinski-Déglise [**16**], leading to the relative theory of motives (see [**4**] for a comprehensive summary of the state of the art and the flavor of the subject) and, more generally, of *motivic categories*. In the introduction, we use this term loosely as the value on Spec k of a functor $\mathrm{Sch}_k'^{\mathrm{op}} \to \mathrm{TriCat}$ where Sch_k' is some category of k-schemes (e.g. smooth, or quasi-projective etc.) and TriCat is the 2-category of triangulated categories. This functor is required to satisfiy Grothendieck's six functor formalism in the sense of [**16**, Section A.5]. Examples include Voevodsky's category of motives DM, Morel-Voevodsky's stable homotopy category SH, the category of modules over various motivic ring spectra such as the algebraic cobordism spectrum MGL, the algebraic K-theory spectrum KGL, and their various étale counterparts including the classical derived category of étale sheaves.

In any event, a working theory of motivic categories of stacks is currently in progress — the theory of motives or, rather, of motivic homotopy theory of quotient stacks is available after the work of Hoyois on equivariant motivic homotopy theory [**42**]. One validation for this formulation of the motive/motivic homotopy type of stacks is the program of Levine [**47**] to enhance and reproduce calculations in enumerative geometry using this machinery. Unfortunately, no such theory is available for general algebraic stacks.

Nonetheless, one can still make sense of stacks as objects in motivic categories over fields as first formalized in [**14**] in the language of model categories. Thinking of a stack as a presheaf of groupoids on affine k-schemes, one restricts it to smooth affine ones and then extend it to Nisnevich sheaves of spaces; see §2.1. The motivic homotopy type is the resulting $\mathbf{A}^1$-localization. The problem with this approach is that there is *a priori* no relation between the atlas of the stack and the stack itself since there is no étale descent in the motivic homotopy category[1] . However, this approach can sometimes still be useful as there are approximations of these stacks by motives/motivic homotopy types of algebraic varieties — the first examples are the Morel-Voevodsky and Totaro [**55**, Chapter 3] models for classifying stacks of groups. More recently, the notion of *exhaustive stacks* introduced by Hoskins and Pepin-Lehalleur in [**40**] extends this idea to a more general class of stacks, including $\mathrm{Bun}_G(C)$.

[1]But see [**14**] and the appendix of [**40**] for what happens upon rationalization.

While our interest is ultimately in $\mathrm{Bun}_G(C)$, this paper attempts to understand the motive/motivic homotopy type of the fiber of the approximation map (1.3). What we mean by fiber is the *étale local fiber* — the fiber of the map approx after étale sheafification — which, as explained in [**32**, Section 3.3], is the *space of rational maps*. We note that the identification of this fiber requires a nontrivial input: the Drinfeld-Simpson theorem [**23**] on Borel subgroup-reductions of G-bundles.

Roughly speaking, a T-point of the space of rational maps between schemes X and Y (we can actually make sense of for very general targets) $\mathrm{RatMaps}(X,Y)$ classifies maps from a dense open of X complementary to the graph of a finite nonempty collection of T-points of X to Y. We prove

Theorem 1.2.1. [See Theorem 5.1.2] *Let k be a field and suppose that* E *is a motivic ring spectrum in* $\mathrm{SH}(k)$*. Denote by* $\mathrm{M}_{\mathrm{E}}(X)$ *the free* E*-module (see §2.2.0.2) on a motivic space X. Let C be a curve over a field k, and let Y be a connected, separated scheme which has a Zariski cover $\{U_\alpha\}$ where each U_α is a dense open subset of $\mathbf{A}^{n_\alpha}$. Then the canonical map of* E*-modules:*

$$\mathrm{M}_{\mathrm{E}}(\mathrm{L}_{\mathrm{mot}}\mathrm{L}_{\mathrm{h}}\mathrm{RatMaps}(C,Y)) \to \mathrm{E} \tag{1.4}$$

is an equivalence.

Here, L_{h} denote sheafification with respect to the h-topology. In other words, if we call $\mathrm{M}_{\mathrm{E}}(X)$ the E-*motive* of X, then the E-motive of the (h-sheafification of the) space of rational maps is equivalent to the E-motive of the point, i.e., its E-motive is contractible. In the case of the trivial bundle and $D = \emptyset$ we see that this refines [**32**, Lemma 3.6.1] after étale realization, and over Spec $\mathbf{C}$ this also refines the main theorem of [**31**]. We also note C neither needs to be smooth or complete, but this generality was already noted in [**31**, Remark 2.8.3] as the contractibility statement only depends on a dense open subscheme of the curve.

1.2.0.1. Approach 1: generic maps and a theorem of Suslin. One approach to Theorem 1.2.1 is to work in the unstable motivic homotopy category $\mathrm{H}(k)$. The strategy here is to prove that the map $\mathrm{L_h RatMaps}(C, Y) \to \operatorname{Spec} k$ is a trivial Kan fibration after application of $\mathrm{L}_{\mathbf{A}^1}$, using the Suslin construction (reviewed in §B.0.1.1) as a model. After checking appropriate closed descent condition, the condition of being a trivial Kan fibration becomes a problem about extending rational maps. The difficulty stems from the fact that, by rational maps we mean those whose loci of singularity are defined by graphs of T-points of C where T is a test affine scheme.

The problem of extending rational maps while keeping the loci of singularity to be of this form is difficult. We will instead work with a more flexible model considered by Barlev in his thesis [**8**]. This model is called the space of *generic maps* and is denoted by $\mathrm{GenMaps}(X, Y)$ where X is a scheme and Y is a very general target. In this model, the loci of singularity are only required to have open complement being *universally dense over the base*. If $X \to T$ is a map, this means an open subset $U \subset X$ such that for any closed point $t \in T$ the fiber U_t is a dense open subset of X_t. Complements of graphs are examples, but there are generally more such open subsets. These loci of singularity are more controllable — in fact a theorem of Suslin (see Theorem 3.3.4; this is referred to in the motivic cohomology literature as Suslin's "generic equidimensionality theorem") tells us that, up to $\mathbf{A}^1$-homotopy, these loci can be arranged to have the "expected" fiber dimensions over the base. We employ this result to show that the map $\mathrm{GenMaps}(C, Y) \to \operatorname{Spec} k$ is a trivial Kan fibration (in an appropriate category of prestacks) whenever Y is an open subset of affine space (see Proposition 3.3.6).

Another advantage of generic maps is that $\mathrm{GenMaps}(X, -)$ forms a Zariski cosheaf, and thus we can extend our contractibility results to more general kinds of target. This approach was already taken in [**8**] to obtain extensions of the main contractibility result of [**31**].

1.2.0.2. Approach 2: rational maps versus the Ran **space.** We will deduce Theorem 1.2.1 in a more restricted setting — we have to impose some restrictions on C and on

E — but in a different way. This approach mimics more closely the original approach of Gaitsgory in [**31**]. In this approach we utilize the Ran *space of* C, which classifies finite non-empty subsets of $C(T)$ for any test affine scheme T. To proceed, we break down the contractibility of $\mathrm{RatMaps}(C, Y)$ into two steps

(1) A motivic equivalence between $\mathrm{RatMaps}(C, Y)$ and $\mathrm{Ran}(C)$ in $\mathrm{Mod}_{\mathrm{E}}$, and
(2) the motivic contractibility of $\mathrm{Ran}(C)$ after h-sheafification.

Step (1) formalizes the following geometric idea: if $Y \subset \mathbf{A}^n$ is an open dense susbcheme, then $\mathrm{RatMaps}(C, Y) \subset \mathrm{RatMaps}(C, \mathbf{A}^n)$ has a complement which is of infinite codimension hence are motivically equivalent. This motivic equivalence is assured whenever E is an oriented and connective (in the sense of Morel's homotopy t-structure) motivic $\mathcal{E}_\infty$-ring spectra — for example the algebraic cobordism and motivic cohomology spectra. We discuss this in §2.2 and prove slight generalizations of [**40**, Section 2]. The space $\mathrm{RatMaps}(C, \mathbf{A}^n)$ is then easily seen to be a vector bundle over $\mathrm{Ran}(C)$.

Part (2) of this approach introduces a variant of the Ran space which is subject to future investigation. Recall that one of the crucial properties of the Ran space, proved by Beilinson and Drinfeld, is that it is contractible in the sense that its étale/singular homology is acyclic (see [**31**, Appendix] for a proof). We enhance this to the level of motivic homotopy types

Theorem 1.2.2. *Let X be a connected quasiprojective k-scheme, then the map* $\mathrm{L_h Ran}(X) \to \mathrm{Spec}\ k$ *is an equivalence in* $\mathrm{H}(k)$.

The h-sheafification that appears in Theorem 1.2.2 is crucial — note that even if X is a connected scheme, its motivic homotopy type, and hence its Ran space, might not be (elliptic curves form a basic class of examples). The proof of Theorem 1.2.2 goes through an explicit model for the h-sheafification of the Ran space and uses a theorem of Morel which verifies connectedness in motivic homotopy theory. Instead of classifying just finite subsets of C, a T-point of this h-sheafified version of the Ran space classifies closed subsets $Z \subset X \times T$ where the map $Z \to T$ is finite and surjective. We call this the *cycle*-Ran space (see Definition 4.2.4) and we prove that it is an explicit model for the h-sheafification of the Ran space (see Proposition 4.2.5). In a future work, we aim to provide a recognition principle for n-fold $\mathbf{P}^1$-loop spaces using this version of the Ran space in an analogous fashion to the result in topology [**49**, Chapter 5] due to Lurie; the case of infinite $\mathbf{P}^1$-loop spaces was only recently settled in [**26**].

We emphasize again that approach 2 is strictly weaker than approach 1, but its value is in clarifying certain motivic aspects of [**32**] and [**31**] and also in introducing this new version of the Ran space which may be of independent interest (certainly, at least to the author).

1.3. Outline

This thesis consists of five chapters, including the introduction. In Chapter 2, we introduce the basic objects of our study — the E-motives of prestacks where E is a motivic $\mathcal{E}_\infty$-ring spectra. We simply regard a prestack as a presheaf of spaces over Sm_S and perform the usual motivic localization, as we explain in §2.1. In §2.2 we proceed to take its "free E-module" or, as we call it, its E-motive. Putting conditions on E lends us certain computational leverage — the main result of this chapter is Lemma 2.2.9 which computes the E-motives of prestacks presented as a colimit of smooth k-schemes where each term is a compatibly an open immersion in a larger smooth k-scheme. This computation was observed in [**40**] but we perform it in a slightly more general setting. The kind of E where this lemma is valid is quite general — they are essentially oriented motivic spectra over a field where we have inverted the base characteristic of the base field.

In Chapter 3, we study the homotopy type of the space of generic maps as introduced by Barlev in [**8**], beginning with the indexing category of generic maps — the category of domains are reviewed in §3.1. We study the properties of this space which are important from the point of view of motivic homotopy theory in §3.2 such as closed gluing. The main theorem of this chapter occurs in §3.3 as Theorem 3.3.8 which shows that the motivic homotopy type of the space of generic maps from a curve over a field k into a quasi-affine scheme is contractible as an unstable motivic homotopy type. The argument uses a theorem of Suslin (Theorem 3.3.4) to "move" the loci of a singularity of a generic map to a better position, up to $\mathbf{A}^1$-homotopy. This theorem is actually more general than the proof of the contractibility of the E-motive of the space of rational maps which we give in the next chapter.

In Chapter 4, we study the E-motives of the space of rational maps as understood by Gaitsgory in [**31**]. The space of rational maps naturally lives over the Ran space of the source. It turns out that one can run an analogue of the argument in [**31**] to show that the E-motive of the the space of rational maps is equivalent to the E-motive on the Ran space whenever Lemma 2.2.9 holds. This is Theorem 4.2.18. However, there is no reason why the E-motive of the usual Ran space should be contractible, i.e., equivalent to the E-motive of the point. We introduce a variant of the Ran space in §4.1 which is an explicit model for its h-sheafification. This model of the Ran space leads to a Riemann-Roch computation, suggested to us by Lurie, which proves in §4.2 that the E-motive of this version of the Ran space is contractible as Corollary 4.2.7. We believe that this result, and the introduction of this version of the Ran space is of independent interest.

In Chapter 5 we explain how to recover a more general version of the result in Chapter 4 from Chapter 3 in §4.2.0.15. Essentially, this follows from [**8**] and we record it as Theorem 5.1.2 which we consider to be the main result of this thesis. From this we recover contractibility results of [**31**] and [**32**] in §5.2 after realization.

There are two appendices in this thesis. In Appendix A which states a version of Gabber's presentation lemma, crucial to the discussions in Chapter 4. The second is Appendix B which reviews constructions of motivic homotopy in general settings, recording down proofs of folk results that we were unable to find in the literature.

1.4. Conventions

We assume the reader is familiar with the basics of motivic homotopy theory using the syntax of ∞-categories; references are [**60**], [**7**, Section 2], [**25**, Section 2]. Otherwise, a review of the constructions — in the generality that the author considers optimal — is provided in Appendix B. Here are our conventions on motivic homotopy theory.

- By presheaves on a small category C we always mean presheaves of spaces/∞-groupoids/Kan complexes. We denote this ∞-category by $\mathrm{P}(\mathrm{C})$. The full subcategory of presheaves that takes coproducts to products is denoted by $\mathrm{P}_{\boldsymbol{\Sigma}}(\mathrm{C})$.
- For a base scheme S, we denote by $\mathrm{H}(S)$ the ∞-category of motivic spaces, that is, homotopy invariant, Nisnevich sheaves on the category of smooth schemes over S, Sm_S.
- We adopt the following conventions about localizations (in the sense of [**48**, Section 5.2.7]): suppose that we have an adjunction $L : \mathrm{C} \to \mathrm{D} : i$ where i is fully faithful. The *reflection* of an object $X \in \mathrm{C}$ is the application of the endofunctor $i \circ L$ on X. Often we will abusively denote by L the endofunctor $i \circ L$; the context will always be clear but note that $i \circ L$ does not necessarily preserve colimits.
- Often we will refer to the endofunctor L as the reflection functor onto the subcategory D. We say that an arrow $f : X \to Y$ is an *L-equivalence* if $L(f)$ is an equivalence — saying this is unambigious as the functor i reflects equivalences.
- We denote by $\mathrm{L}_{\mathrm{Nis}}, \mathrm{L}_{\mathbf{A}^1}, \mathrm{L}_{\mathrm{mot}}$ the reflection of presheaves on Sm_S onto Nisnevich sheaves, homotopy invariant presheaves and their intersection respectively. So the terms $\mathrm{L}_{\mathrm{Nis}}, \mathrm{L}_{\mathbf{A}^1}, \mathrm{L}_{\mathrm{mot}}$-equivalences should be clear to the reader.
- If $p : X \to S$ is a smooth S-scheme, we always mean that X is of *finite type.* An essentially smooth scheme over k is an inverse limit of smooth S-schemes with affine transition maps.
- If F is a presheaf of abelian groups we denote its τ-sheafification by $a_\tau(F)$, τ any topology.
- We denote unstable motivic spheres by $\mathbf{S}^{p,q} := (S^1)^{p-q} \wedge \mathbf{G}_m^q$ for $p \geq q$, and write $\mathbf{1}$ for the motivic sphere spectrum. Motivic suspensions are denoted by $\boldsymbol{\Sigma}^{p,q}$.

- For a motivic spectrum E over a base scheme S we write $\mathrm{E} \in \mathrm{SH}(S)$, or E_S for emphasis.

Some comments on the notation and implementation of ∞-categories in this thesis:

- We freely use the notions of algebras and modules in higher algebra [**49**]; see [**34**, Chapter 1] for a summary.
- We freely use the basic terminology of ∞-topos theory in the sense of [**48**, Chapter 7].
- Spc is the ∞-category of ∞-groupoids (Kan complexes give a concrete model).
- $\mathrm{Maps}(X, Y)$ is the Kan complex of maps between objects $X, Y \in \mathrm{C}$ and, $\underline{\mathrm{map}}(X, Y)$ is the internal mapping object in C.
- $[X, Y] := \pi_0 \mathrm{Maps}(X, Y)$ denotes homotopy classes of maps between X, $Y \in \mathrm{C}$.
- $\mathrm{Pr}^{L,\otimes}$ —the symmetric monoidal ∞-category of presentable ∞-categories and colimit preserving functors — has a full subcategory $\mathrm{Pr}^{L}_{\mathrm{stab}}$ of *stable* presentable ∞-categories.

CHAPTER 2

The motivic homotopy type and the E-motive of prestacks

In this preliminary section, we will begin describing the environment in which our study of the space of rational maps take place. We first make sense of what it means to take the motivic homotopy type of prestacks and, later, its E-motive where E is a motivic $\mathcal{E}_\infty$-ring spectrum. We extract the properties of E necessary to make the computation of the E-motive of the space of rational maps possible and prove that there are many examples of such E's — they are essentially connective oriented theories where we have inverted the base characteristic.

2.1. Prestacks and motivic homotopy theory

Let us begin with some terminology; a review of motivic homotopy theory and relevant terminology is carried out in Appendix B.

The ∞-category of *prestacks* is just $\mathrm{P}(\mathrm{Aff})$, the ∞-category of presheaves of spaces on affine schemes. If we are working over a base S, the ∞-category of *S-prestacks* is $\mathrm{P}(\mathrm{Aff}_S)$. We will also have occasion to consider the fully faithful embedding from the $(2,1)$-category of presheaves of $(1-)$groupoids, $\mathrm{P}_{\mathrm{Gpd}}(\mathrm{Aff}_S)$, to the ∞-category of S-prestacks

$$\mathrm{P}_{\mathrm{Gpd}}(\mathrm{Aff}_S) \overset{i}{\hookrightarrow} \mathrm{P}(\mathrm{Aff}_S). \tag{2.1}$$

In particular we may consider algebraic stacks $\mathrm{Alg}\,\mathrm{Stk}_S$ [1] as objects of $\mathrm{P}(\mathrm{Aff}_S)$ this way and obtain an embedding $\mathrm{Alg}\,\mathrm{Stk}_S \hookrightarrow \mathrm{P}_{\mathrm{Gpd}}(\mathrm{Aff}_S)$.

[1]By this, we mean an fppf sheaf of groupoids $\mathscr{X} : \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Gpd}$ such that the diagonal $\Delta : \mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable by an algebraic space and there exists an algebaic space U, called the *atlas*, equipped with a morphism $U \to \mathscr{X}$ is smooth and surjective. This is the definition given in [**65**, Tag 026N].

2.1.0.1. We have a fully faithful embedding $j : \mathrm{SmAff}_S \hookrightarrow \mathrm{Aff}_S$ which induces an adjunction

$$j_! : \mathrm{P}(\mathrm{SmAff}_S) \rightleftarrows \mathrm{P}(\mathrm{Aff}_S) : j^* \tag{2.2}$$

where $j^*\mathscr{F} := \mathscr{F} \circ j$ and $j_!$ is given by left Kan extension.

Proposition 2.1.1. *The functor $j_!$ is fully faithful.*

Proof. Follows from the fact that $\mathrm{SmAff}_S \hookrightarrow \mathrm{Aff}_S$ is fully faithful and the fact $j_!$ is computed by a left Kan extension, after applying [**48**, Proposition 4.3.3.8]. □

2.1.0.2. The inclusion $j' : \mathrm{SmAff}_S \hookrightarrow \mathrm{Sm}_S$ again induces an adjunction

$$j'_! : \mathrm{P}(\mathrm{SmAff}_S) \rightleftarrows \mathrm{P}(\mathrm{Sm}_S) : j'^* \tag{2.3}$$

Proposition 2.1.2. *The functor $j'_!$ is fully faithful.*

Proof. Same reasoning as in the proof of (2.1.2). □

2.1.0.3. As a result we obtain a functor

$$\mathbf{\Psi} : \mathrm{P}(\mathrm{Aff}_S) \xrightarrow{j^*} \mathrm{P}(\mathrm{SmAff}_S) \xrightarrow{j'_!} \mathrm{P}(\mathrm{Sm}_S) \xrightarrow{\mathrm{L_{mot}}} \mathrm{H}(S). \tag{2.4}$$

The *motivic homotopy type of a prestack* $\mathscr{F} \in \mathrm{P}(\mathrm{Aff}_S)$ is then defined to be $\mathbf{\Psi}(\mathscr{X})$. We remark that since all the functors in sight are left adjoints, the functor $\mathbf{\Psi}$ preserves colimits.

If $\mathscr{X}$ is an algebraic stack, then its *motivic homotopy type* is again the application of the functor $\mathbf{\Psi}$ to $\mathscr{F}$ thought of as an object of $\mathrm{P}_{\mathrm{Gpd}}(\mathrm{Aff}_S)$.

2.1.0.4. For examples of computations (or, at least, an expression in $\mathrm{H}(S)$ in terms of motivic homotopy type of schemes) of $\mathbf{\Psi}$ of quotient stacks, we refer the reader to

[**55**, Section 4.3] or [**43**, Section 2]. For example, over very general bases S, we have an equivalence in $\mathrm{H}(S)$ expressing $\boldsymbol{\Psi}(\mathsf{B}_{\mathrm{fppf}}G)$ as a colimit of motivic homotopy type of schemes

$$\boldsymbol{\Psi}(\mathsf{B}_{\mathrm{fppf}}G) \simeq \operatorname{colim} \mathrm{L}_{\mathrm{fppf}}(U_i/G). \tag{2.5}$$

For example of $G = \mathrm{GL}_n$ then we obtain $\mathrm{L}_{\mathrm{mot}}$ of $\boldsymbol{\Psi}(\mathsf{B}_{\mathrm{fppf}}GL_n)$ as $\mathrm{L}_{\mathrm{mot}}$ of an infinite Grassmanian.

2.1.0.5. We will use the following conventions for $\mathbf{A}^1$-homotopy sheaves. Recall that if $\mathscr{X}$ is an ∞-topos we have an intrinsic notion of *homotopy groups* [**48**, Definition 6.5.1.1]

$$\pi_n^{\mathscr{X}}(X) : \mathscr{X} \to \mathscr{X}^{\heartsuit}_{/X}, \tag{2.6}$$

here $\mathscr{X}^{\heartsuit}$ denotes the underlying topos of the (also known as the discrete category of 0-truncated objects, see [**48**, Definition 5.5.6.1]). Picking a base point, i.e., a morphism from the terminal object $x : 1 \to X$ defines the object

$$x^*\pi_n^{\mathscr{X}}(X) : \mathscr{X} \to \mathscr{X}^{\heartsuit}_{/X} \to \mathscr{X}^{\heartsuit}_{/1} \simeq \mathscr{X}^{\heartsuit}, \tag{2.7}$$

which we write as $\pi_n^{\mathscr{X}}(X, x) := x^*\pi_n^{\mathscr{X}}(X)$ and we call the n-th *homotopy sheaf of X pointed at x*. As usual $\pi_n^{\mathscr{X}}(X, x)$ is a group object for $n \geq 1$ and is an abelian group object for $n \geq 2$.

Concretely, if $\mathscr{X} = \mathrm{Shv}(\mathrm{C})$, the ∞-category of sheaves of spaces on a small category C with respect to a topology τ, then the n-th homotopy sheaf an object $X \in \mathrm{Shv}(\mathrm{C})$ pointed at $x : 1 \to X$ is just the τ-sheafifcation of the presheaf

$$T \in \mathrm{C} \mapsto \pi_0 \operatorname{Maps}_{\mathrm{Shv}(\mathrm{C})_*}(\boldsymbol{\Sigma}^n_+ T, (X, x)). \tag{2.8}$$

2.1.0.6. Suppose that C is a small category equipped with a topology τ. We shall denote the homotopy sheaves of the ∞-topos $\mathrm{Shv}_\tau(\mathrm{C})$ in this case as

$$\pi_n^\tau(X) := \pi_n^{\mathrm{Shv}_\tau(C)}(X). \tag{2.9}$$

Suppose that $X \in \mathrm{P}(\mathrm{Sm}_S)$ and consider the reflection $\mathrm{L}_{\mathrm{mot}}X$ onto $\mathrm{H}(S)$. Then the $\mathbf{A}^1$*-homotopy sheaves of* X *pointed at* $x : 1_S \to X$ is defined to be

$$\pi_n^{\mathbf{A}^1}(X, x) := x^*\pi_n^{\mathrm{Nis}}(\mathrm{L}_{\mathrm{mot}}X) \tag{2.10}$$

which is an object of the underlying topos $\mathrm{Shv}_{\mathrm{Nis}}(S)^\heartsuit$. There is also the *sheaf of* $\mathbf{A}^1$*-conencted components*

$$\pi_0^{\mathbf{A}^1}(X) := \pi_0^{\mathrm{Nis}}(\mathrm{L}_{\mathrm{mot}}X). \tag{2.11}$$

Of course, if $X \in \mathrm{H}(S)$ then $\pi_n^{\mathbf{A}^1}(X, x) = \pi_n^{\mathrm{Nis}}(X, x)$.

2.1.0.7. We adopt terminology in Morel's book [**54**]. For $0 \leq n \leq \infty$, we say that $X \in \mathrm{P}(\mathrm{Sm}_S)$ is $\mathbf{A}^1$*-n-connected* if the object $\mathrm{L}_{\mathrm{mot}}X$, considered as an object of the ∞-topos $\mathrm{Shv}_{\mathrm{Nis}}(S)$, is $n+1$-connective in the sense of [**48**, Definition 6.5.1.10]. Concretely this means that the $\mathbf{A}^1$-homotopy sheaves $\pi_k^{\mathbf{A}^1}(X, x)$ is isomorphic to the terminal object in the discrete topos of Nisnevich sheaf of sets on Sm_S, for all base points and for $k \leq n$.

2.1.0.8. The "Whitehead Theorem" in $\mathbf{A}^1$-homotopy theory is a consequence of the hypercompleteness of the Nisnevich ∞-topos over certain bases. In this paper, we will also have occasion to compute the homotopy groups sectionswise: if $X \in \mathrm{P}(\mathrm{C})$ and $U \in \mathrm{C}$ then $\pi_n(X(U))$ will be the unambigiously the homotopy groups of the space $X(U)$.

Proposition 2.1.3. *Let S be a quasicompact, locally Noetherian scheme of finite Krull dimension. Then*

$$\{\pi_n^{\mathbf{A}^1} : \mathrm{H}(S) \to \mathrm{Shv}_{\mathrm{Nis}}(S)^\heartsuit\}_{n\geq 0} \tag{2.12}$$

form a conservative family of functors.

Proof. By definition the hypercompletion of an ∞-topos is a localization at ∞-connective morphisms [**48**, Section 6.5.2], hence the claim follows by the hypercompleteness of the ∞-topos $\mathrm{Shv}_{\mathrm{Nis}}(S)$ which is a consequence of the hypotheses in play by [**50**, Corollary 3.7.7.3]. $\square$

In particular, we will later have occasion to work with the $\mathbf{A}^1$-homotopy sheaves of a prestack.

2.1.0.9. One final notational remark: if X is a presheaf on a subcategory of schemes C and $Y \in \mathrm{C}$ is affine, whence $Y = \mathrm{Spec}\ A$ we sometimes denote the section of X at Y to be $X(A)$. The context will always make it clear what we are doing.

2.2. E-motives

The environment in which the theorems below will take place is the ∞-category of E-modules $\mathrm{Mod}_{\mathrm{E}}(S) := \mathrm{Mod}_{\mathrm{E}}(\mathrm{SH}(S))$ where $\mathrm{E} \in \mathrm{CAlg}(\mathrm{SH}(S))$, i.e., a motivic $\mathcal{E}_\infty$-ring spectra. This fits within the formalism of *motivic module categories* as explained by the the author and Kolderup in an upcoming revision of [**27**] and reviewed in the Appendix B.

2.2.0.1. We summarize some key features of $\mathrm{Mod}_{\mathrm{E}}(S)$ in the next proposition.

Proposition 2.2.1. *Let S be a quasicompact, quasiseparated base scheme. Then*

(1) *The ∞-category $\mathrm{Mod}_{\mathrm{E}}(S)$ is presentable and $\mathbb{T}$-stable* [2]. *In particular it is a stable ∞-category and has invertible $\mathbb{T}$-suspension functors*

$$\{\Sigma^q_{\mathbb{T}} : \mathrm{Mod}_{\mathrm{E}}(S) \to \mathrm{Mod}_{\mathrm{E}}(S)\}_{q\in\mathbf{Z}}. \tag{2.13}$$

(2) *The ∞-category $\mathrm{Mod}_{\mathrm{E}}(S)$ is presentably symmetric monoidal and comes equipped with a colimit-preserving functor symmetric monoidal functor*

$$\mathrm{M}_{\mathrm{E}} : \mathrm{SH}(S) \to \mathrm{Mod}_{\mathrm{E}}(S), \tag{2.14}$$

witnessing it as an $\mathrm{SH}(S)$-algebra.

[2] See Definition B.0.23 for what this means.

(3) $\mathrm{Mod}_{\mathrm{E}}(S)$ *is generated under sifted colimits by* $\Sigma^q_{\mathbb{T}}\mathrm{M}_{\mathrm{E}}(\Sigma^\infty_{\mathbb{T}}X_+)$ *where* $X \in \mathrm{SmAff}_S$ *and* $q \in \mathbf{Z}$.

(4) *The generators* $\{\Sigma^q_{\mathbb{T}}\mathrm{M}_{\mathrm{E}}(\Sigma^\infty_{\mathbb{T}}X_+)\}$ *are compact.*

(5) *In particular the functors*

$$\{\mathrm{Maps}(\Sigma^q_{\mathbb{T}}\mathrm{M}_{\mathrm{E}}(\Sigma^\infty_{\mathbb{T}}X_+), -) : \mathrm{Mod}_{\mathrm{E}}(S) \to \mathrm{Spt}\}_{X \in \mathrm{AffSm}_S} \tag{2.15}$$

form a conservative family of exact functors which preserves all small sums (and hence all small colimits).

2.2.0.2. Given a base scheme S, we define the functor assigning a motivic space X to $\mathrm{M}_{\mathrm{E}}(\Sigma^\infty_{\mathbb{T}}X_+) \in \mathrm{Mod}_{\mathrm{E}}(S)$ by

$$\mathrm{M}_{\mathrm{E}} : \mathrm{H}(S) \to \mathrm{Mod}_{\mathrm{E}}(S) \tag{2.16}$$

and we refer to the object $\mathrm{M}_{\mathrm{E}}(X)$ as the E-motive of X. We note, being a composite of left adjoints, the functor (2.16) preserves all small colimits. If $X \in \mathrm{P}(\mathrm{Aff}_S)$ is an S-prestack, its E-motive is then $\mathrm{M}_{\mathrm{E}}(\mathbf{\Psi}(X))$. We will find the following notation useful for any $\mathrm{M} \in \mathrm{M}_{\mathrm{E}}(X)$

(1) We write $\Sigma^q_{\mathbb{T}}\mathrm{M}$ as $\mathrm{M}(q)[2q]$ and write the shift functor as $(-)[1]$.

(2) Using the action of $\mathrm{SH}(S)$ on $\mathrm{Mod}_{\mathrm{E}}(S)$ we have object $\Sigma^\infty_{\mathbb{T}}(\mathbf{G}_m, 1)^{\wedge q} \otimes \mathrm{M}$ for any $q \in \mathbf{Z}$. Denote this object by $\mathrm{M}(q)[q]$. Using the usual $\mathrm{L}_{\mathrm{mot}}$-equivalence $(\mathbf{P}^1, \infty) \simeq (\mathbf{G}_m, 1) \wedge S^1$, this agrees with the previous notation.

Lastly, if $\mathbf{1} \in \mathrm{SH}(S)$ is the sphere spectrum in $\mathrm{SH}(S)$, then we write $\mathrm{M}_{\mathrm{E}}(\mathbf{1})$ as $\mathrm{E} \in \mathrm{Mod}_{\mathrm{E}}(S)$

2.2.0.3. Now we need to work with E-motives of prestacks. The calculations of the E-motives of prestacks in this thesis will boil down to understanding prestacks of the form

$$\mathscr{F} = \operatorname*{colim}_{i \in I} X_i \tag{2.17}$$

where X_i are smooth k-schemes and I is a filtered diagram (often just $\mathbb{N}$).

In particular, we will compare prestacks presented as colimits of diagrams $X, Y : I \to \mathrm{Sch}_k$ and related via a transformation $\{j_i : X_i \to Y_i\}$ where j_i is an open immersion whose closed complement Z_i is of codimension c_i which tends to ∞. Such a situation has been studied recently by Hoskins and Pepin-Lehalleur in [**40**]. We will work in a slightly more general setting.

2.2.0.4. To begin we work with the base $S = \mathrm{Spec}\ k$ being a *perfect* field and write $\mathrm{Mod}_{\mathrm{E}}(k)$ instead of the unwieldy $\mathrm{Mod}_{\mathrm{E}}(\mathrm{Spec}\ k)$. We will also write the symmetric monoidal unit of $\mathrm{M}_{\mathrm{E}}(k)$ as E instead of $\mathrm{M}_{\mathrm{E}}(\mathrm{Spec}\ k)$. The point of working over a perfect field is that every finitely generated field extension has a smooth model [**65**, Tag 030I] hence we may perform stratification arguments (which we will see later).

2.2.0.5. We now introduce several conditions on $\mathrm{Mod}_{\mathrm{E}}(k)$ (which we will later verify for certain oriented theories):

(1) For every separated finite type k-scheme we may functorially associate to it its motive $\mathrm{M}_{\mathrm{E}}(X)$[3] and its compactly supported motive $\mathrm{M}^c_{\mathrm{E}}(X)$ in $\mathrm{Mod}_{\mathrm{E}}(k)$. This association comes equipped with a map

$$\mathrm{M}_{\mathrm{E}}(X) \to \mathrm{M}^c_{\mathrm{E}}(X). \tag{2.18}$$

When X is proper, the map (2.18) is an equivalence. In other words, we have functors

$$\mathrm{M}_{\mathrm{E}}, \mathrm{M}^c_{\mathrm{E}} : \mathrm{Sch}^{\mathrm{ft}}_k \to \mathrm{Mod}_{\mathrm{E}}(k), \tag{2.19}$$

and a transformation

$$\mathrm{M}_{\mathrm{E}} \Rightarrow \mathrm{M}^c_{\mathrm{E}}, \tag{2.20}$$

which is an equivalence on the full subcategory of proper k-schemes.

[3] We can, of course, define $\mathrm{M}_{\mathrm{E}}(X)$ as $\mathrm{M}_{\mathrm{E}}(\Psi(X))$. But, in examples of E, the latter object does not *a priori* good behavior that the former object does — a prominent example is the localization triangle that we will use later.

(2) Suppose that $Z \hookrightarrow X$ is a closed immersion of codimension c where *both X and Z are smooth k-schemes* and $U \hookrightarrow X$ is the open complement. We have a *Gysin triangle*, which is the cofiber sequence in $\mathrm{M}_{\mathrm{E}}(k)$

$$\mathrm{M}_{\mathrm{E}}(U) \to \mathrm{M}_{\mathrm{E}}(X) \to \mathrm{M}_{\mathrm{E}}(Z)(c)[2c]. \tag{2.21}$$

We call the above sequence the *Gysin sequence*.

(3) For any smooth k-scheme X of dimension d we have a functorial equivalence

$$\mathrm{M}^{c}_{\mathrm{E}}(X)(-d)[-2d] \simeq \mathrm{M}_{\mathrm{E}}(X)^{\vee}. \tag{2.22}$$

Remark 2.2.2. We are thinking of $\mathrm{M}_{\mathrm{E}}(X)$ as a *homological motive*, so the map (2.20) is dual to the more familiar map going from compactly supported cohomology to cohomology. This latter map is given by $p_!p* \to p_*p^*$.

Definition 2.2.3. We say that $\mathrm{E} \in \mathrm{CAlg}(\mathrm{SH}(k))$ is an *adequately oriented* theory if the assumptions $(1) - (3)$ above hold.

2.2.0.6. In cases of interest, $\mathrm{Mod}_{\mathrm{E}}(k)$ often has strong finiteness properties — at least if the characteristic of k is invertible; see the author's paper with Levine, Spitweck and Østvær [**25**, Section 5.2] for more details.

Definition 2.2.4. We will say that $\mathrm{Mod}_{\mathrm{E}}(k)$ has *compact-rigid generation* if the following condition holds

- for any $X \in \mathrm{SmAff}_k$, the E-motive of X, $\mathrm{M}_{\mathrm{E}}(X)$, can be written as finite colimits and retracts of $\mathrm{M}_{\mathrm{E}}(Y)$ where Y is a smooth projective k-scheme.

In other words, if we write $\mathrm{Mod}_{\mathrm{E}}(k)^{\mathrm{proj}}$ as the subcategory of $\mathrm{Mod}_{\mathrm{E}}(k)$ generated under finite colimits and retracts of E-motives of smooth projective schemes, then the condition asserts that any $\mathrm{M}_{\mathrm{E}}(X)$ where X is smooth affine is actually in $\mathrm{Mod}_{\mathrm{E}}(k)^{\mathrm{proj}}$.

2.2.0.7. The compact-rigid generation of $\mathrm{Mod}_{\mathrm{E}}(k)$ will allow us to find a more convenient set of generators of $\mathrm{Mod}_{\mathrm{E}}(k)$, which will be useful later.

Proposition 2.2.5. *Suppose that* $\mathrm{Mod}_{\mathrm{E}}(k)$ *has compact-rigid generation, then the* ∞*-category* $\mathrm{Mod}_{\mathrm{E}}(k)$ *is generated under filtered colimits by strongly dualizable objects*

Proof. According to [**25**, Proposition 5.6], which is essentially a theorem of Riou [**59**], we have inclusions

$$\mathrm{Mod}_{\mathrm{E}}(k)^{\mathrm{proj}} \subset \mathrm{Mod}_{\mathrm{E}}^{\mathrm{dual}}(k) \subset \mathrm{Mod}_{\mathrm{E}}(k)^{\omega}. \tag{2.23}$$

where $\mathrm{Mod}_{\mathrm{E}}^{\mathrm{dual}}(k)$ denote the full subcategory of $\mathrm{Mod}_{\mathrm{E}}(k)$ spanned by the fully dualizable objects and $\mathrm{Mod}_{\mathrm{E}}(k)^{\omega}$ the full subcategory of $\mathrm{Mod}_{\mathrm{E}}(k)$ spanned by the compact objects. By [**25**, Lemma 5.2] the last inclusion collapses into an equality whenever the unit is compact and this is the case by Proposition 2.2.1.4 (see Proposition B.0.13 for a proof of the general statement). To collapse the first inclusion, we use [**25**, Theorem 5.8] which proves the desired claim for $\mathrm{SH}(k)$. Noting that the functor M_{E} is symmetric monoidal and preserves finite colimits by Proposition 2.2.1.2, we obtain the desired claim

$\square$

2.2.0.8. We need one more condition on E to perform our calculations. This is one instance where we need to work over a perfect field, over which Morel has constructed the *homotopy t-structure* on $\mathrm{SH}(k)$ in [**52**]. To define the nonnegative part, we recall that if $\mathrm{E} \in \mathrm{SH}(k)$ then we have the Nisnevich sheaves $\pi_i(\mathrm{E})_j$ on Sm_k defined by the Nisnevich sheafification of the presheaf

$$U \mapsto \pi_0 \operatorname{Maps}(\Sigma^{\infty}_{\mathbb{T}} U_+[i], \mathrm{E} \wedge (\mathbf{G}_m, 1)^{\wedge j}). \tag{2.24}$$

Remark 2.2.6. We note that this grading convention might be confusing in light of the usual convention that $S^{p,q} := (S^1)^{\wedge p-q} \wedge (\mathbf{G}_m, 1)^q$, [4] but it is motivated by Morel's notion of a *homotopy module* or a "$\mathbf{\Omega}$-$\mathbf{G}_m$-spectrum in strictly homotopy invariant sheaves" (see [**53**]). We explain this notion briefly.

[4]This has been distressing to the author in the past

For a fixed i, the collection $\{\pi_i(E)_j\}_{j\in\mathbf{Z}}$ forms such a structure: the bonding maps in the underlying $\mathbf{G}_m$-spectrum of E defines maps

$$\pi_i(\mathrm{E}) \overset{\cong}{\to} (\pi_i(\mathrm{E})_{i+1})_{-1}. \tag{2.25}$$

Where $(-)_{-1}$ denote the contraction operator as explained in, for example, [**51**, Lecture 23]; this takes a presheaf of abelian groups on $F : \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Ab}$ to $F_{-1} = \mathrm{Hom}(\mathbf{Z}(\mathbf{G}_m), F) : \mathrm{Sm}_k^{\mathrm{op}} \to \mathrm{Ab}$, its "$\mathbf{G}_m$-loop space". These homotopy modules form the heart of the homotopy t-structure on $\mathrm{SH}(k)$ which we recall briefly below.

We define

$$\mathrm{SH}(k)_{\geq 0} := \{\mathrm{E} : \pi_i(\mathrm{E})_j = 0, i < 0, \forall j \in \mathbf{Z}\}. \tag{2.26}$$

This generates a t-structure $(\mathrm{SH}(k)_{\geq 0}, \mathrm{SH}(k)_{\leq 0})$ formally by [**49**, Proposition 1.4.4.11] but Morel's connectivity theorem identifies the non-positive part as

$$\mathrm{SH}(k)_{\leq 0} := \{\mathrm{E} : \pi_i(\mathrm{E})_j = 0, i > 0, \forall j \in \mathbf{Z}\}. \tag{2.27}$$

See [**44**, Section 2.1] for details. We note that the construction of the nonnegative part of the t-structure does not require that we work over a perfect base field, but the identification of the nonnegative part does because it replies on Morel's stable connectivity theorem [**53**].

2.2.0.9. From now on we say that E is *connective* if it is t-connective with respect to the above t-structure.

Proposition 2.2.7. *Let k be a perfect field and suppose that $\mathrm{E} \in \mathrm{SH}(k)$ is connective. Then $\mathrm{E}^{p,q}(X) = 0$ for $p > q + \dim X$. if*

- *if $X \in \mathrm{Sm}_k$, or*
- *If $X \in \mathrm{Sch}_k^{\mathrm{ft}}$ and E has* cdh-*descent.*

Proof. For $\tau = \mathrm{Nis}$ or cdh, and X a smooth scheme in the first case or any finite type scheme in the latter, we have the strongly convergent spectral sequence (due to the hypercompleteness of the large Nis and cdh-topoi and the finiteness of cohomological dimension of smooth finite type schemes in each of these topologies; we use this latter fact again below):

$$\mathrm{H}^s_{\mathrm{Nis}}(X, \pi_{-t}(\mathrm{E})_q) \Rightarrow [X, \mathrm{E}[s+t] \otimes (\mathbf{G}_m)^{\wedge q}](= \mathrm{E}^{s+t+q,q}(X)^5). \tag{2.28}$$

Since the Nisnevich and the cdh cohomological dimension is bounded about by the Krull dimension ([**46**] and [**68**] respectively), we obtain the desired estimate. □

Remark 2.2.8. According to [**15**], if E is a Cartesian section of the Cartesian fibration associated to the functor $\mathrm{SH} : \mathrm{Sch}_S^{\mathrm{op}} \to \mathrm{Cat}_\infty$, for a Noetherian base S, then E has cdh-descent. This includes the examples of all the motivic $\mathcal{E}_\infty$-ring spectra that are of interest — the motivic sphere spectrum, MGL, KGL and Spitzweck's motivic cohomology spectrum which is defined over $\mathbf{Z}$ [**64**].

2.2.0.10. Now suppose that E is connective and is an adequately oriented theory and that $\mathrm{Mod}_{\mathrm{E}}(k)$ has compact-rigid generation. We can prove the following key lemma (compare [**40**, Proposition 2.13])

Lemma 2.2.9. *Let k be a perfect field and suppose that $X_{(-)}, Y_{(-)} : \mathbb{N} \to \mathrm{Sm}_k$ are digrams of smooth schemes and $j : X \Rightarrow Y$ is a transformation where for each $i \in \mathbb{N}$ $j_i : X_i \to Y_i$ is an open immersion. Let Z_i be the reduced complement of j_i with codimension c_i and suppose that $c_i \to \infty$ as $i \to \infty$. Then the map*

$$\mathrm{M}_{\mathrm{E}}(\operatorname{colim} X_i) \to \mathrm{M}_{\mathrm{E}}(\operatorname{colim} Y_i) \tag{2.29}$$

is an equivalence.

Proof. Write $X := \operatorname{colim} X_i, Y := \operatorname{colim} Y_i$. Since the ∞-category $\mathrm{Mod}_{\mathrm{E}}$ is generated by $\mathrm{M}_{\mathrm{E}}(T)(q)[p]$ where $T \in \mathrm{Sm}_k$ and $q, p \in \mathbf{Z}$ (Proposition 2.2.1) we need only check that,

[5] Beware the potentially confusing indexing conventions!

for a fixed $q, p \in \mathbf{Z}$ the map

$$\mathrm{Hom}(\mathrm{M_E}(T)(q)[p], \mathrm{M_E}X) \to \mathrm{Hom}(\mathrm{M_E}(T)(q)[p], \mathrm{M_E}Y) \tag{2.30}$$

is an isomorphism. Since $\mathrm{M_E}T$ is compact, we need only show that

$$\mathrm{Hom}(\mathrm{M_E}(T)(q)[p], \mathrm{M_E}X_i) \to \mathrm{Hom}(\mathrm{M_E}(T)(q)[p], \mathrm{M_E}Y_i) \tag{2.31}$$

is an equivalence for large enough i. For each i we have a cofiber sequence

$$\mathrm{M_E}(X_i) \to \mathrm{M_E}(Y_i) \to C_i. \tag{2.32}$$

One is tempted to say that C_i is exactly a shift of the motive of $\mathrm{M_E}(Z_i)$ but this is not guaranteed to be the case because Z_i is not a smooth k-scheme.

To get around this, let us first fix an $i \in \mathbb{N}$ and examine the closed subscheme Z_i of X_i which is assumed to be of codimension c_i. Since k is perfect, we have a finite stratification of Z_i

$$\emptyset = W^i_{-1} \subset W^i_0 \subset W^i_1 \subset \cdots \subset Z_i \tag{2.33}$$

for which:

(1) For all j, the inclusion $W^i_{j-1} \subset W^i_j$ is a closed immersion of schemes.
(2) The open complement $W^i_j \setminus W^i_{j-1}$ is smooth.

Therefore we have a cofiber sequence

$$\mathrm{M_E}(W^i_j \setminus W_{j-1}) \to \mathrm{M_E}(W^i_j) \to \mathrm{M_E}(W^i_{j-1})(c^i_j)[2c^i_j], \tag{2.34}$$

whence the cofiber C_i can be expressed as successive extensions of shifts and suspensions of smooth k-schemes where, most importantly, the shifts and suspensions are all of the form $(c^i_j)[2c^i_j]$ where $c^i_j \geq c_i$. Since these $c_i \to \infty$ as $i \to \infty$, so does the c^i_j's. Hence we may assume that $Z_i's$ are smooth. In this situation we have the cofiber sequence

$$\mathrm{M_E}(X_i) \to \mathrm{M_E}(Y_i) \to \mathrm{M_E}(Z)(c_i)[2c_i], \tag{2.35}$$

and so we are reduced to proving the following vanishing statement: for a sequence of integers $c_i \to \infty$ we have that

$$\mathrm{Hom}(\mathrm{M_E}(T)(q)[p], \mathrm{M_E}(Z)(c_i)[2c_i]) = 0 \tag{2.36}$$

for $i \gg 0$. Using duality and writing $d_Z := \dim Z$, we get that

$$\begin{aligned}
&\mathrm{Hom}(\mathrm{M_E}(T)(q)[p], \mathrm{M_E}(Z)(c_i)[2c_i]) \\
&\cong \mathrm{Hom}(\mathrm{M_E}(T) \otimes (\mathrm{M_E}(Z))^{\vee}, \mathrm{E}(-q+c_i)[-p+2c_i]) \\
&\cong \mathrm{Hom}(\mathrm{M_E}(T) \otimes \mathrm{M^c_E}(Z)(-d_Z)[-2d_Z], \mathrm{E}(-q+c_i)[-p+2c_i]) \\
&\cong \mathrm{Hom}(\mathrm{M_E}(T) \otimes \mathrm{M^c_E}(Z), \mathrm{E}(-q+c_i+d_Z)[-p+2c_i+2d_Z]).
\end{aligned}$$

We make one last reduction: E has compact-rigid generation so we may assume that Z is smooth and projective. In this case, we are reduced to proving that

$$\mathrm{Hom}(\mathrm{M_E}(T \times Z), \mathrm{E}(-q+c_i+d_Z)[-p+2c_i+2d_Z]) = 0. \tag{2.37}$$

This happens, according to Proposition 2.2.7, as soon as $-p+2c_i+2d_Z-(-q+c_i+d_Z) > d_Z + \dim, T$. The left hand side is equal to $-p+q+c_i+d_Z$ so we will have vanishing as soon as

$$-p+q+c_i > \dim T \tag{2.38}$$

which we can certainly arrange as $c_i \to \infty$.

$\square$

2.2.0.11. Examples of E**.** We will now produce examples of E which are amenable to the above analysis. We first rapidly recall some notions from the six functor formalism following the summary of [**41**, Section 2]. For our purposes we consider Sch_k the category of separated k-schemes of finite type. We have a functor

$$\mathrm{SH} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Pr}^{L,\otimes}_{\mathrm{stab}}; f : X \to Y \mapsto f^* : \mathrm{SH}(Y) \to \mathrm{SH}(X). \tag{2.39}$$

As the notation indicates the functor f^* is symmetric monoidal so that, in particular, $f^*\mathbf{1}_Y \simeq \mathbf{1}_X$. The right adjoint is denoted by $f_* : \mathrm{SH}(X) \to \mathrm{SH}(Y)$. Furthermore there is the exceptional adjunction associated to $f : X \to Y$ whenever f is separated of finite type

$$f_! : \mathrm{SH}(X) \to \mathrm{SH}(Y) : f^!. \tag{2.40}$$

2.2.0.12. When f is smooth, f^* further admits a left adjoint $f_\#$, characterized by the property that whenever $T \to X$ is a smooth then $f_\#(\Sigma^\infty_{\mathbb{T}} T_+) \simeq \Sigma^\infty_{\mathbb{T}} T_+$ where T is regarded as a smooth Y-scheme.

2.2.0.13. There is always a transformation

$$f_! \to f_* \tag{2.41}$$

which is an equivalence whenever f is proper.

2.2.0.14. We freely use the following notation for Thom spectra of vector bundles (we will soon switch to numbers instead of vector bundles in the presence of orientations). Suppose that $p : \mathcal{E} \to X$ is a vector bundle and $s : X \to \mathcal{E}$ is the zero section, then we have the invertible endofunctor (which is naturally a left adjoint)

$$p_\# s_* : \mathrm{SH}(X) \to \mathrm{SH}(X); \mathrm{M} \mapsto p_\# s_* \mathrm{M} := \Sigma^{\mathcal{E}}\mathrm{M}, \tag{2.42}$$

which has an inverse endofunctor (which is naturally a right adjoint)

$$s^! p^* : \mathrm{SH}(X) \to \mathrm{SH}(X); \mathrm{M} \mapsto p_\# s_* \mathrm{M} := \Sigma^{-\mathcal{E}}\mathrm{M}. \tag{2.43}$$

In the event that $f : X \to \operatorname{Spec} k$ is smooth, we have that $f_{\#}p_{\#}s_*\mathbf{1}_X \simeq \Sigma^\infty_{\mathbb{T}} \operatorname{Th}_X(\mathcal{E}) \in \operatorname{SH}(k)$. The purpose of introducing these operations are the *duality equivalences*: if $f : X \to \operatorname{Spec} k$ is smooth then we have canonical equivalences

$$f_! \simeq f_{\#}\Sigma^{-\Omega_f}, f^! \simeq \Sigma^{\Omega_f} f^*. \tag{2.44}$$

where $\mathbf{\Omega}_f$ denote the sheaf of relative differentials or, more accurately, the associated vector bundle, aka, the tangent bundle of X over $\operatorname{Spec} k$.

2.2.0.15. Suppose that $f : X \to \operatorname{Spec} k$ is the structure morphism, then the *homological motive* of X is defined to be

$$\operatorname{M}(X) := f_! f^! \mathbf{1}_k. \tag{2.45}$$

Indeed, whenever X is smooth we get that

$$f_! f^! \mathbf{1}_k \simeq f_{\#}\Sigma^{-\Omega_f}\Sigma^{\Omega_f} f^*\mathbf{1}_k \simeq f_{\#}\mathbf{1}_X = \Sigma^\infty_{\mathbb{T}} X_+, \tag{2.46}$$

recovering the suspension spectrum of X. Indeed the assignment 2.45 easily assembles, from the six functor formalism, into a functor

$$\operatorname{Sch}_k \to \operatorname{SH}(k), X \mapsto \operatorname{M}(X). \tag{2.47}$$

2.2.0.16. On the other hand, we define the *compactly supported motive* of X to be

$$\operatorname{M}^c(X) := f_* f^! \mathbf{1}_k, \tag{2.48}$$

which also assembles into a functor

$$\operatorname{Sch}_k \to \operatorname{SH}(k), X \mapsto \operatorname{M}^c(X). \tag{2.49}$$

The transformation in (2.41) then gives us the transformation (2.20)

$$\operatorname{M}(-) \Rightarrow \operatorname{M}^c(-) \tag{2.50}$$

2.2.0.17. One of the key properties of the six functor formalism is the localization sequence. We have a diagram in Sch_k

$$Z \overset{i}{\hookrightarrow} X \overset{j}{\hookleftarrow} U \tag{2.51}$$

where i is a closed immersion and j is its open complement. Then

Theorem 2.2.10. *[Morel-Voevodsky* [**55**]*, Ayoub* [**3**]*; localization and purity] Given a diagram as in* (2.51) *then we have the following cofiber sequences of endofunctors in* $\mathrm{SH}(X)$

$$j_! j^! \to \mathrm{id} \to i_* i^* \tag{2.52}$$

and

$$i_! i^! \to \mathrm{id} \to j_* j^*. \tag{2.53}$$

Furthermore, if $Z \hookrightarrow X$ is a morphism of smooth k schemes, and $\mathcal{N}_i$ is the normal bundle of the immersion i, the cofiber sequence in (2.53) *reads as*

$$j_! j^! \to \mathrm{id} \to \Sigma^{i^* \mathcal{N}_i}. \tag{2.54}$$

We remark that these localization sequences hold more generally than just for finite type schemes over a field. Now, denote by $f : X \to \operatorname{Spec} k$ the structure map. Then applying the sequence 2.54 above to $f^! \mathbf{1}_k$, then applying $f_!$ to the resulting sequence gives us a cofiber sequence (since $f_!$ is an exact functor) in $\mathrm{SH}(k)$

$$f_! j_! j^! f^! \mathbf{1} \to f_! f^! \mathbf{1}_k \to f_! \Sigma^{i^* \mathcal{N}_i} f^! \mathbf{1}_k, \tag{2.55}$$

which unwinds to a cofiber sequence in $\mathrm{SH}(k)$,

$$\mathrm{M}(U) \to \mathrm{M}(X) \to \mathrm{M}(\mathrm{Th}_Z(\mathcal{N}_i)). \tag{2.56}$$

where $\mathrm{Th}_Z(\mathcal{N}_i)) \in \mathrm{H}(k)$ is the Thom space of the normal bundle of Z. To obtain (2.21), we now need to discuss oriented motivic spectra.

Remark 2.2.11. The check that the sequence (2.56) is the stabilization of the purity equivalence one obtains unstably in [**55**] is difficult but accomplished in [**3**]; a different proof that the author can somewhat understand is found in the equivariant setting in [**42**, Section 5.3].

2.2.0.18. Just as in topology, an *orientation* is, roughly, a coherent choice of trivializations of the thom spaces of vector bundles and virtual bundles. Suppose that T is a Sm_k*-fibered premotivic category with the six functor formalism* in the sense of [**16**, Chapter 1]; see [**18**, Definition A.1.1, Definition A.1.10] for a friendlier list of axioms. We remark that while the original axiom presupposes that we work with triangulated categories, the enhancements of these axioms to the ∞-categorical setting can be found in the thesis of Khan [**45**]; we will work in this enriched setting without further comment but do not need the coherence in any crucial way.

In particular T is a functor

$$\mathrm{T} : \mathrm{Sch}_k^{\mathrm{op}} \to \mathrm{Pr}^{L,\otimes}_{\mathrm{stab}}; f : X \to Y \mapsto f^* : \mathrm{T}(Y) \to \mathrm{T}(X). \tag{2.57}$$

with the same kind of functorialities as SH discussed above — so there are functors $f_!, f^!, f_\#, f_*$ which are appropriately adjoint and we can make sense of the Thom spectrum of a vector bundle $\mathcal{E} \to X$, $\Sigma^{\mathcal{E}}$ in this setting, as discussed in §2.2.0.14. We define the T-motive and the T-compactly supported motive in the same way as in (2.45) and (2.48) and denote them by $\mathrm{M}_{\mathrm{T}}(X)$ and $\mathrm{M}^c_{\mathrm{T}}(X)$ respectively. If $\mathrm{M} \in \mathrm{T}(Y)$ we write $\mathrm{M}(d)[2d]$ to be $\Sigma^{\mathbf{A}^n_Y}\mathrm{M}$; by $\mathbf{A}^1$-invariance this conforms to the conventions of §2.2.0.2.

2.2.0.19. An *orientation*, in the sense of [**16**, Definition 2.4.38], is a natural equivalence satisfiying properties spelled out in *loc. cit.*:

$$t_E : \Sigma^{\mathcal{E}}(-) \overset{\simeq}{\Rightarrow} (-)(d)[2d]. \tag{2.58}$$

We say that T is *oriented* if an orientation exists. Suppose that T is oriented and the localization sequences in Theorem 2.2.10 holds in T (which will be true for all the examples we consider), then the cofiber sequence (2.55) in $\mathrm{T}(k)$ reads as

$$\mathrm{M_T}(U) \to \mathrm{M_T}(X) \to \mathrm{M_T}(Z)(d)[2d], \tag{2.59}$$

giving us the Gysin sequence in (2.21).

2.2.0.20. The last point is about duality. We need this along with the property that $\mathrm{T}(k)$ has compact-rigid generation in the sense of Definition 2.2.4. So we ask that $\mathrm{T}(k)$ satisfies:

- for any $X \in \mathrm{SmAff}_k$, the E-motive of X, $\mathrm{M_T}(X)$, can be written as finite colimits and retracts of $\mathrm{M_T}(Y)$ where Y is a smooth projective k-scheme.

Proposition 2.2.12. *For any smooth k-scheme X of dimension d and* T *is oriented has compact rigid generation and is equipped with a symmetric monoidal transformation of* Sm_k*-fibered premotivic category with the six functor formalism* $\mathrm{SH}(-) \Rightarrow \mathrm{T}(-)$*. Then we have a functorial equivalence:*

$$\mathrm{M^c_T}(X) \simeq \mathrm{M_T}(X)^{\vee}(d)[2d]. \tag{2.60}$$

Proof. Let $p : X \to \operatorname{Spec} k$ be the structure morphism. First, we have functorial equivalences

$$\begin{aligned} \mathrm{M}^c_{\mathrm{T}}(X) &= p_* p^! \mathbf{1}_k \\ &\simeq p_* \mathbf{\Sigma}^{\Omega_p} p^* \mathbf{1}_k \\ &\simeq p_* p^* \mathbf{1}(d)[2d]. \end{aligned}$$

We have used the orientation in the last equivalence. Now it remains to show that $p_* p^*$ is the strong dual to $p_! p^!$. Using the fact that $\mathrm{T}(k)$ has compact-rigid generation and the fact that duality and the strong dual is preserved under colimits in a symmetric monoidal ∞-category, we need only check this for X a smooth projective k-scheme. In this case, we are reduced to [**41**, Section 3] where it is proved in $\mathrm{SH}(X)$ that the strong dual of

$$p_* p^! \mathbf{1}_k \simeq p_! p^! \mathbf{1}_k \tag{2.61}$$

is indeed

$$p_! p^* \mathbf{1}_k \simeq p_* p^! \mathbf{1}_k. \tag{2.62}$$

$\square$

2.2.0.21. Here is the first and most important example: given any base scheme S (such as Spec $\mathbf{Z}$) we have Voevodsky's *algebraic cobordism spectrum* $\mathrm{MGL}_S \in \mathrm{SH}(S)$ (introduced in [**71**]) defined as a colimit

$$\mathrm{MGL}_S = \operatorname{colim} \mathbf{\Sigma}^{-2n}_{\mathbb{T}} \mathbf{\Sigma}^{\infty}_{\mathbb{T}} \mathrm{Th}_{\mathrm{Gr}_n}(\gamma_n) \tag{2.63}$$

where $\gamma_n \to \mathrm{Gr}_n$ is the tautological n-plane over Gr_n, the smooth (ind-)scheme classifying n-planes. Just like in topology MGL is the universal example of an oriented ring spectrum — see [**16**, Definition 12.2.2] in terms of classes in $\mathbf{P}^{\infty}_S$ and the resulting chern classes and [**16**, Example 2.4.40] where it is shown that the resulting category of modules over an oriented ring spectrum is oriented in the sense of §2.2.0.19. Indeed, Vezzosi proved (as phrased in [**16**, Theorem 12.2.10]) that there is a bijection between orientations of a motivic ring spectrum $\mathrm{E} \in \mathrm{SH}(S)$ and maps of ring spectra from MGL to E.

Proposition 2.2.13. *Let k be a perfect field of characteristic p, then $\mathrm{Mod}_{\mathrm{MGL}}(k)[\frac{1}{p}]$ is an adequately oriented ring spectra, which has compact-rigid generation. Furthermore $\mathrm{MGL}_k[\frac{1}{p}]$ is connective. In particular, the conclusion of Lemma 2.2.9 holds for $\mathrm{E} = \mathrm{MGL}$.*

Proof. The compact-rigid generation of $\mathrm{SH}(k)[\frac{1}{p}]$ is verified in [**25**, Theorem 5.9]. This implies the compact-rigid generation of $\mathrm{Mod}_{\mathrm{MGL}}[\frac{1}{p}]$. The discussions above show that $\mathrm{MGL}[\frac{1}{p}]$ is adequately oriented (not that this is not necessarily true before inverting p because of reasons related to compact-rigid generation as seen in the proof of Proposition 2.2.12). The fact that MGL and hence $\mathrm{MGL}[\frac{1}{p}]$ is connective is found in [**44**, Corollary 3.9].

$\square$

From the universality of MGL we get:

Corollary 2.2.14. *Let k be a perfect field of characteristic p, let E be an oriented motivic $\mathcal{E}_\infty$-ring spectrum in $\mathrm{SH}(k)$ which is furthermore connective, then the conclusion of Lemma 2.2.9 holds for E.*

Example 2.2.15. The conclusion of Lemma 2.2.9 then holds for $\mathrm{KGL}_{\geq 0}$, the connective cover of algebraic K-theory, and $\mathrm{M}\mathbf{Z}[\frac{1}{p}]$, the motivic cohomology spectrum (which is the case considered in [**40**]) by [**44**, Lemma 7.3]. Clearly the étale analogue of these spectra also satisfy Lemma 2.2.9.

CHAPTER 3

The space of generic maps

In this section we will study the motivic homotopy type of one model of the space of rational maps which are christened in [**8**] as the space of *generic maps*. This model of the space of rational maps has the advantage that it is can be studied in $\mathrm{H}(k)$, the unstable motivic homotopy ∞-category. Working unstably lends ourselves to some "perturbative methods" (or moving lemmas), and the main geometric input to our study of the space of generic maps is Suslin's generic equidimensionality Theorem (see Theorem 3.3.4). Roughly speaking, this lemma allows us to move cycles $Z \subset X \times \Delta^n$, where X is an affine k-scheme, to one where the map $Z \to \Delta^n$ has equidimensional fibers. We will use this to move the locus of singularity of a rational map.

3.1. The category of domains

To begin let us recall some standard notions in algebraic geometry. Suppose that S is a base scheme and X an S-scheme, and $p : X \to S$ is the canonical morphism.

Definition 3.1.1. An open subscheme $U \subset X$ is *S-universally dense* if for any morphism $T \to S$, the open subset $U \times_S T \subset X \times_S T$ remains dense.

If the base scheme S is clear, we sometimes say that U is universally dense.

3.1.0.1. The space of generic maps from X to Y naturally lives over a certain category parametrizing universally dense open subsets of X. For a while we will be able to work under the generality that X is an arbitrary scheme, in which case the prestacks involved are all simply objects of $\mathrm{P}(\mathrm{Aff})$.

Definition 3.1.2. The category Dom_X has as objects (S, U) where $U \subset X_S$ is universally dense. We call objects in Dom_X a *domain of X.* A morphism in this category, displayed as $(S, U) \to (T, V)$, is a map $q : S \to T$ of affine schemes such that the diagram

$$\begin{tikzcd} V \subset X_T \arrow[r] \arrow[d] & U \subset X_S \arrow[d] \\ S \arrow[r, "q"] & T. \end{tikzcd} \tag{3.1}$$

commutes (but not necessarily Cartesian).

The following is easy to check

Lemma 3.1.3. *The functor $q : \mathrm{Dom}_X \to \mathrm{Aff}$ $(S, U) \mapsto S$ is a Cartesian fibration.*

Proof. Follows from the requirement that $U \subset X_S$ is universally dense so that q-Cartesian lifts exist. $\square$

We will have occasion to work with the straightening $\mathrm{Dom}_X : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Set}$.

3.1.0.2. A universally dense subscheme of X_S is a generalization of the notion of (the complement) of a relative effective Cartier divisor. To motivate this notion, suppose that we have a morphism $f : X \to S$ and $D \subset X$ an effective Cartier divisor [1] and $p : T \to S$ is a morphism of schemes. We would like conditions for the pullback of schemes $D \times_S T$ to be also be an effective Cartier divisor of X_T. This is assured if $\mathscr{O}_{X_S}/\mathscr{L}$ is S-flat (iff the morphism $D \to S$ is flat) [**65**, Tag 056P].

Definition 3.1.4. Suppose that $f : X \to S$ is a morphism of schemes. A *relative effective divisor on X/S* is an effective Cartier divisor $D \subset X$ such that the morphism $D \to S$ is a flat morphism of schemes (iff the coherent sheaf $\mathscr{O}_X/\mathscr{L}$ is S-flat). Given an effective Cartier divisor $D \subset X$ the *divisor complement* of D is the open subscheme of X which is the complement of D (equivalently, the complement of the support of $\mathscr{O}_X/\mathscr{L}$).

[1] Recall that this is equivalent to the data of an invertible sheaf $\mathscr{L}$ equipped with a monomorphism of coherent sheaves $i : \mathscr{L} \hookrightarrow \mathscr{O}_{X_S}$; D is then obtained as the support of the sheaf cokernel $\mathscr{O}_{X_S}/\mathscr{L}$.

3.1.0.3. Indeed, a class of examples of universally dense subschemes are given by complements of these relative effective divisors. First, we observe that S-universally dense open subschemes of X are simply those whose fibers over closed points of S are dense.

Lemma 3.1.5. *A universally dense subscheme can be tested on closed points: if $p : X \to S$ is morphism of schemes, then an open subset $U \subset X$ is universally dense if and only for all closed points $\{s\} \hookrightarrow S$ the scheme $U \times_S \{s\}$ is nonempty.*

Proof. One implication is trivial. Let $p : T \to S$ be a morphism of schemes and suppose that $V \subset X_T$ is a nonempty open subset. If U_T is not dense then $U_T \cap V$ must be empty. Since V is nonempty we may pick a point $v \in V$ which maps to some point $t \in T$ with closure $\bar{t} \in T$ then we get that $p(\bar{t}) \times_S U$ must be empty, which is a contradiction. $\square$

Proposition 3.1.6. *Let $X \to S$ be a morphism of schemes, and consider an effective Cartier divisor defined by a monomorphism $i : \mathscr{L} \to \mathscr{O}_{X_S}$ where $\mathscr{L}$ is an invertible sheaf. Then the sheaf $\mathscr{O}_{X_S}/\mathscr{L}$ is S-flat if and only if the complement of the support of $\mathscr{O}_{X_S}/\mathscr{L}$ is universally dense.*

Proof. This is implied by [**8**, Lemma 3.2.8]. By Lemma 3.1.5, being universally dense is the same as saying that for any closed point $\{s\} \hookrightarrow S$ the pullback $U \times_S \{s\} \neq \emptyset$, while a coherent sheaf $\mathscr{F}$ being S-flat is the same as saying $\mathrm{Tor}^1(\mathscr{F}, \mathcal{I}_s) = 0$ for any $\mathcal{I}_s \subset \mathscr{O}$ a sheaf of ideals on S corresponding to the closed point s.

Hence, $U \times_S \{s\}$ is nonzero if and only if $\mathscr{O}/\mathscr{L}_s$ vanishes if and only if $\mathscr{L}_s \hookrightarrow \mathscr{O}_s$ is an injection of vector bundles on $X \times_S \{s\}$ if and only if $\mathrm{Tor}^1(\mathscr{O}_s/\mathscr{L}_s, \mathcal{I}_s) = 0$ if and only if $\mathscr{O}_s/\mathscr{L}_s$ is flat as a coherent sheaf on X.

$\square$

3.1.0.4. As indicated by Proposition 3.1.6 a way to construct objects in Dom_X is via relative effective divisors on X_S. We see that this can always be arranged Zariski-locally.

Proposition 3.1.7. *Let $f : X \to S$ be a quasiprojective morphism of schemes, and suppose that $U \subset X$ is a universally dense open subscheme of X. Then there exist a*

Zariski-cover of S, $\tilde{S} \to S$ such that the dense open subscheme $U \times_S \tilde{S} \subset X \times_S \tilde{S}$ is the complement of an effective Cartier divisor on $X \times_S \tilde{S}$.

Proof. Since $f : X \to S$ is quasiprojective, there exists an f-relatively ample invertible sheaf on X, $\mathscr{L}$ with sections $s_1, \cdots, s_n$ for which the non-vanishing loci of these sections are open subsets that cover X. Denote by X_{s_i} the open subset correspond to s_i. Hence we may take the intersections $U_{s_i} := U \cap X_{s_i}$ and $U_{s_i} \subset X$ are divisor complements. Define $S_i \subset S$ to be the open subscheme of S which is the image of $U_{s_i} \subset X \to S$; note that $U_{s_i} \subset X \times_S S_i$ is S_i-universally dense. Then $\tilde{S} = \coprod S_i$ is the desired Zariski cover of S. □

3.1.0.5. We can endow Dom_X with Grothendieck topologies. Suppose that τ is a topology on Aff then we consider the Grothendieck topology on Dom_X defined in the following way:

- A collection $\{(S_\alpha, U_\alpha) \to (T, V)\}$ is a τ-sieve if and only if the $\{S_\alpha \to T\}$ is a τ-sieve.

Now consider the functor $q^* : \mathrm{P}(\mathrm{Aff}) \to \mathrm{P}(\mathrm{Dom}_X), Y \mapsto Y \circ q$. Given a topology τ on Aff, the definition of the induced topology on Dom_X ensures that if Y is a τ-sheaf on $\mathrm{P}(\mathrm{Aff})$, then q^*Y is also a τ-sheaf on Dom_X. As a result we have a commutative diagram

$$\begin{array}{ccc} \mathrm{P}_\tau(\mathrm{Aff}) & \xrightarrow{q^*} & \mathrm{P}_\tau(\mathrm{Dom}_X) \\ \downarrow & & \downarrow \\ \mathrm{P}(\mathrm{Aff}) & \xrightarrow{q^*} & \mathrm{P}(\mathrm{Dom}_X) \end{array} \tag{3.2}$$

where the vertical arrows are fully faithful embeddings. Taking the left adjoints (which exists by the adjoint functor theorem), we obtain

Proposition 3.1.8. *The functor* $q^* : \mathrm{P}(\mathrm{Aff}) \to \mathrm{P}(\mathrm{Dom}_X), Y \mapsto Y \circ q$ *admits a left adjoint* $q_! : \mathrm{P}(\mathrm{Dom}_X) \to \mathrm{P}(\mathrm{Aff})$ *such that the diagram*

$$\begin{tikzcd} \mathrm{P}(\mathrm{Dom}_X) \arrow[r, "q_!"] \arrow[d, "\mathrm{L}_\tau"'] & \mathrm{P}(\mathrm{Aff}) \arrow[d, "\mathrm{L}_\tau"] \\ \mathrm{P}_\tau(\mathrm{Dom}_X) \arrow[r, "q_!"] & \mathrm{P}_\tau(\mathrm{Aff}) \end{tikzcd} \tag{3.3}$$

commutes. Furthermore the functors $q_!$ *are both localizations.*

3.1.0.6. According to [**8**] *a generic moduli problem* is an object $Y \in \mathrm{P}(\mathrm{Dom}_X)$. Since the functor $q^* : \mathrm{P}(\mathrm{Aff}) \to \mathrm{P}(\mathrm{Dom}_X)$ is fully faithful [**8**, Page 7], we may also regard $q_! Y$ as prestack and we are safe in not making a distinction between generic moduli problems as an object of $\mathrm{P}(\mathrm{Dom}_X)$ and $\mathrm{P}(\mathrm{Aff})$.

3.2. Space of generic maps

We now define the *space of generic maps* associated to a scheme X and a presheaf $Y \in \mathrm{P}(\mathrm{Sch})$. When Y is a scheme this is written down by hand in [**8**, Example 2.3.3]. First, we have a functor

$$o : \mathrm{Dom}_X \to \mathrm{Sch}, (S, U) \mapsto U, \tag{3.4}$$

picking out the universally dense open subset of X_S. We will need this functor to define maps which are regular only on U.

Construction 3.2.1. Let X be a scheme and let $Y \in \mathrm{P}(\mathrm{Sch})$ be a presheaf. We define the presheaf

$$\mathrm{GenMaps}(X, Y)_{\mathrm{Dom}_X} : \mathrm{Dom}_X^{\mathrm{op}} \to \mathrm{Spc} \tag{3.5}$$

as the straightening of the Cartesian fibration $\mathrm{GenMaps}(X, Y)_{\mathrm{Dom}_X} \to \mathrm{Dom}_X$ defined via the pullback

$$
\begin{tikzcd}
\mathrm{GenMaps}(X,Y)_{\mathrm{Dom}_X} \arrow[r] \arrow[d] & \mathrm{Sch}_{/\!/Y} \arrow[d] \\
\mathrm{Dom}_X \arrow[r, "o"] & \mathrm{Sch}\,.
\end{tikzcd} \tag{3.6}
$$

The *space of generic maps between X and Y* is then defined as

$$\mathrm{GenMaps}(X,Y) := q_!\mathrm{GenMaps}(X,Y)_{\mathrm{Dom}_X} : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Spc}. \tag{3.7}$$

3.2.0.1. We note that $\mathrm{GenMaps}(X,Y)_{\mathrm{Dom}_X}((T,U))$ is the space classifying spans

$$
\begin{tikzcd}
& U \arrow[dl, "j"'] \arrow[dr, "f"] & \\
X \times T & & Y,
\end{tikzcd} \tag{3.8}
$$

where the left leg j is a universally dense open immersion and f is allowed to be any morphism from U to Y. Upon Kan extension, prestack $q_!\mathrm{GenMaps}(X,Y)$ classifies spans such as (3.8) where we identify maps that agree on smaller and smaller open subsets.

More precisely, we will display the formula for this left Kan extension. Let $T \in \mathrm{Aff}$, then we have the pullback $(\mathrm{Dom}_X)_T := \mathrm{Dom}_X \times_{\mathrm{Aff}} \{T\}$. Suppose that we have a presheaf

$$\mathscr{F} : \mathrm{Dom}_X^{\mathrm{op}} \to \mathrm{Spc}, \tag{3.9}$$

then we have that

$$q_!\mathscr{F}(T) \simeq \mathrm{colim}((\mathrm{Dom}_X)_T^{\mathrm{op}} \hookrightarrow \mathrm{Dom}_X^{\mathrm{op}} \to \mathrm{Spc})); \tag{3.10}$$

see [**8**, Page 7].

3.2.0.2. Here is an example of GenMaps that has a very concrete interpretation, especially on the level of k-points.

Example 3.2.2. [The space of rational functions] Recall the following terminology: if S is a base scheme and $p : X \to S$ is an S-scheme and Y is another scheme, then an *explicit S-rational function* is a span

$$
\begin{tikzcd}
 & U \arrow[dl, "j"'] \arrow[dr, "f"] & \\
X & & \mathbf{A}^1,
\end{tikzcd}
\tag{3.11}
$$

where U is a S-universally dense subscheme of X. An *S-rational function* is then an equivalence class of explicit S-rational functions where we take the equivalence relation generated by two explicit S-rational functions agreeing on a smaller universally dense subset. In other words, the set $\mathrm{GenMaps}(X, \mathbf{A}^1)(S)$ classifies S-rational functions on X_S. Suppose now that X is an S-scheme where S is the spectrum of the base field, then the k-points

$$\mathrm{GenMaps}(X, \mathbf{A}^1)(k) = \operatorname*{colim}_{U \subset U'} \{X \hookleftarrow U \to \mathbf{A}^1\}. \tag{3.12}$$

naturally identifies with the vector space $k(X)$ of rational functions on X.

Before proceeding further, we will need to understand some "invariant" behavior of GenMaps better.

3.2.0.3. Suppose that $U \subset X$ is a dense open subset of X. We would expect that $\mathrm{GenMaps}(U, Y)$ and $\mathrm{GenMaps}(X, Y)$ to be equivalent as prestacks. Indeed, there is a functor $\mathrm{Dom}_U \to \mathrm{Dom}_X$ because a universally dense subset of U is automatically a universally dense subset of X.

Proposition 3.2.3. *Suppose that $U \subset X$ is dense open immersion of schemes, then the functor* $\mathrm{Dom}_U \to \mathrm{Dom}_X$ *induces an equivalence of prestacks* $\mathrm{GenMaps}(U, Y) \to \mathrm{GenMaps}(X, Y)$.

Proof. It suffices to check that for all $T \in \mathrm{Aff}$, the map of sets

(3.13)
$$\operatorname*{colim}_{(T',V)\in(\mathrm{Dom}_U)_T} \mathrm{GenMaps}(U,Y)_{\mathrm{Dom}_U}(T',V) \to \operatorname*{colim}_{(T',V)\in(\mathrm{Dom}_X)_T} \mathrm{GenMaps}(U,Y)_{\mathrm{Dom}_U}(T',V)$$

is an isomorphism. But this follows from the fact that $(\mathrm{Dom}_U)_T \hookrightarrow (\mathrm{Dom}_X)_T$ is a cofinal inclusion. □

3.2.0.4. Next, we ask about descent properties of the prestack $\mathrm{GenMaps}(X,Y)$. The following proposition captures the descent properties of the space of generic maps

Proposition 3.2.4. *Suppose that* Y *is an fppf (resp. fpqc) sheaf, then* $\mathrm{GenMaps}(X,Y) : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Spc}$ *is also an fppf (resp. fpqc) sheaf.*

Proof. Since Y satisfies fppf (resp. fpqc) descent, we need only check that universally dense opens satisfy fppf (resp. fpcq) descent. To do so we use Proposition 3.1.5 characterizing universally dense opens. Indeed, let $T \to S$ be an fppf (resp. fpqc) cover (in fact the argument works so long as the map is surjective on closed points) and suppose that U_T is universally dense in $X \times T$. Let $\{s\} \hookrightarrow S$ be a closed point, then $\{s\} \times_S T \hookrightarrow T$ is a closed subset. Since U_T is universally dense, for any closed point $\{t\} \hookrightarrow \{s\} \times_S T$, $U_T \times_T \{t\}$ is nonempty and thus $U \times_S \{s\}$ is nonempty. □

As a special case, if Y is representable by a scheme, then $\mathrm{GenMaps}(X,Y)$ has fppf and fpqc descent.

3.2.0.5. We will also need to understand closed gluing in GenMaps. We will quickly review this notion since it might be less familiar to the reader but see the discussion in [**26**, Appendix A.2].

Definition 3.2.5. Suppose that C is an ∞-category with finite limits. A functor $F : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{C}$ satisfies *closed gluing* if it takes a pushout diagram in Sch

(3.14)
$$\begin{tikzcd} D \arrow[r, "i"] \arrow[d, "i'"'] & Z' \arrow[d] \\ Z \arrow[r] & Z \coprod_D Z' = X, \end{tikzcd}$$

where $i,' i'$ are closed immersions to a Cartesian square

(3.15)
$$\begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(Z') \arrow[d] \\ F(Z) \arrow[r] & F(C), \end{tikzcd}$$

Remark 3.2.6. According to [**29**] the maps $Z \to X, Z' \to X$ in (3.16) are also closed immersions.

Example 3.2.7. [Closed gluing and Milnor patching] In the affine case the square (3.16) is usually called a *Milnor square* which is a commutative square of rings

(3.16)
$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & A/I \arrow[d] \\ B \arrow[r] & B/I, \end{tikzcd}$$

where the map f maps the ideal I of A isomorphically onto an ideal of B which we also call I; the closed gluing condition is often also called *Milnor patching.* A functor that satisfies Milnor patching (and hence closed gluing since it also has Zariski descent) is given by $\mathrm{Vect} : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{Spc}$, the stack of finite dimensional vector bundles. A very general version of this statement can be found in [**50**, Theorem 16.2.0.1].

We further remark that, in general, algebraic K-theory does not satisfy Milnor patching unless the multiplication map $B \otimes_A B \to B$ is an isomorphism. This is equivalent to the notion of Tor-unitality of Suslin and Wodzicki [**67**]; while the modern formulation is given by Tamme in [**69**].

3.2.0.6. A consequence of closed gluing is the following simple but useful lemma about detecting $\mathrm{L}_{\mathbf{A}^1}$-equivalences. We recall that there is a closed immersion of affine schemes

$$i_n : \partial\Delta^n_{\mathbf{Z}} \subset \Delta^n_{\mathbf{Z}}. \tag{3.17}$$

Lemma 3.2.8. *Let $f : F \to G$ be a map in* $\mathrm{P}(\mathrm{Aff})$ *such that F and G have closed gluing. Suppose that the following condition holds of f:*

- *for every $S \in \mathrm{Aff}$ and every $n \geq 0$, the map induced map*

$$F^{\partial\Delta^n} \times_{G^{\partial\Delta^n}} G^{\Delta^n} \leftarrow F^{\Delta^n}, \tag{3.18}$$

  is an epimorphism in Spc*.*

Then f is an $\mathrm{L}_{\mathbf{A}^1}$-equivalence.

Proof. By the discussions in Proposition B.0.3, $\mathrm{L}_{\mathbf{A}^1}$ is calculated using the Suslin construction. Hence f is a $\mathrm{L}_{\mathbf{A}^1}$-equivalence if and only if the geometric realization of the map of simplicial objects in the ∞-topos $\mathrm{P}(\mathrm{Aff})$

$$|\mathrm{Sing}^{\mathbf{A}^1} f| : |\mathrm{Sing}^{\mathbf{A}^1} F| \to |\mathrm{Sing}^{\mathbf{A}^1} G|, \tag{3.19}$$

is an equivalence. Since $\mathrm{P}(\mathrm{Aff})$ is a hypercomplete ∞-topos, $|\mathrm{Sing}^{\mathbf{A}^1} f|$ is an equivalence if and only it is ∞-connective.

Now, suppose that the condition holds. According to [**50**, Theorem A.5.3.1], if the map of simplicial objects $\mathrm{Sing}^{\mathbf{A}^1} f$ is a trivial Kan fibration [**50**, Definition A.5.2.1] then, $|\mathrm{Sing}^{\mathbf{A}^1} f|$ is ∞-connective so we need to verify that the map (3.18) being an epimorphism implies that $\mathrm{Sing}^{\mathbf{A}^1} f$ is a trivial Kan fibration. The affine scheme $\partial\Delta^n$ is the pushout of affine schemes $\Delta^{n-1} \coprod_{\Delta^{n-2}} \Delta^{n-1} \coprod_{\Delta^{n-2}} \cdots \Delta^{n-1}$ and since F has closed gluing we have an equivalence

$$\mathrm{Sing}^{\mathbf{A}^1} F^{\partial\Delta^n}(-) \simeq (\mathrm{Sing}^{\mathbf{A}^1} F(-))^{\partial\Delta^n}; \tag{3.20}$$

and similarly for $\mathrm{Sing}^{\mathbf{A}^1} G$ and we are done.

□

3.2.0.7. The next proposition helps us access the motivic homotopy type of generic maps.

Proposition 3.2.9. *Suppose that Y satisfies closed gluing, then so does* $\mathrm{GenMaps}(X, Y)$.

Proof. Since Y has closed gluing, it suffices to check that dense open immersions have closed gluing. To do so we begin with an affine scheme $T = T_0 \coprod_{T_{01}} T_1$ where $T_{01} \hookrightarrow T_i, i = 0, 1$ are closed immersions. Then $X \times T_0 \coprod_{T_{01}} T_1 \simeq X_{T_0} \coprod_{X_{T_{01}}} X_{T_1}$ by universality of colimits. By [**29**, Lemma 4.4] any flat morphism $U \to X \times T_0 \coprod_{T_{01}} T_1$ decomposes as $U_0 \coprod_{U_{01}} U_1$ where $U_* = U \times_X X_{T_*}$ where $* = 0, 1, 01$. Moreoever, each U_* is universally dense in X_{T_*} by the universally dense assumption on U.

□

3.2.0.8. One of the technical advantages of the model of generic maps as rational maps is the following "descent on the target statement" proved by Barlev in [**8**].

Proposition 3.2.10. *Suppose X is an scheme, then the functor*

$$\mathrm{GenMaps}(X, -) : \mathrm{Sch} \to \mathrm{Set} \subset \mathrm{Spc} \tag{3.21}$$

is a Zariski cosheaf.

Proof. This is essentially [**8**, Lemma 6.2.5]. We give a slightly different formulation of the proof. Recall that the Zariski topology is generated by a cd-structure (in the sense of

[**75**]; see also the formulation in [**2**, Section 2]) where the squares are Cartesian squares

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & V \arrow[d] \\ U \arrow[r] & Y \end{tikzcd} \tag{3.22}$$

such that the maps $V, U \to Y$ are open immersions and $U \cup V = X$. Hence, being a Zariski cosheaf just means that the square (3.22) gets taken to a pushout under application of $\mathrm{GenMaps}(X, -)$

$$\begin{tikzcd} \mathrm{GenMaps}(X, U \cap V) \arrow[r] \arrow[d] & \mathrm{GenMaps}(X, V) \arrow[d] \\ \mathrm{GenMaps}(X, U) \arrow[r] & \mathrm{GenMaps}(X, Y) \end{tikzcd} \tag{3.23}$$

To see that (3.23) is a pushout, we first note that the vertical maps in (3.23) are monomorphisms of simplicial sets and are thus cofibrations. Hence the pushout can be checked in the category of sets. In this case, suppose that we have a T-point in $\mathrm{GenMaps}(X, Y)$ presented by a $(U' \subset X_T, f : U' \to Y)$. Then for $W = U, V$ or $U \cap V$ consider the open subschemes $U' \cap (X_T \times_Y W) \subset X_T$. These open subschemes are universally dense open subsets (which can be checked on closed points by Lemma 3.1.5) of X_T. On these domains we have map $f|_{U' \cap (X_T \times_Y W)} : U' \cap (X_T \times_Y W) \to Y$ which glues to the original map f. This verifies that the diagram (3.23) is indeed a pushout in the category of sets.

□

Remark 3.2.11. Since the ∞-topos of Zariski sheaves is hypercomplete, we conclude that $\mathrm{GenMaps}(X, -)$ is a *hypercosheaf*: if $Y_\bullet \to Y$ is a Zariski hypercover, then the canonical map

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{GenMaps}(X, Y_\bullet) \to \mathrm{GenMaps}(X, Y) \tag{3.24}$$

in $\mathrm{P}(\mathrm{Aff})$ is an $\mathrm{L}_{\mathrm{Zar}}$-equivalence.

3.3. The motivic homotopy type of generic maps

We will now work over a base field k. We will now consider Example 3.2.2 from the point of view of motivic homotopy theory. In particular, we want to prove that $\mathrm{GenMaps}(X, \mathbf{A}^1)$ is indeed $\mathrm{L}_{\mathbf{A}^1}$-equivalent to the point whenever X is irreducible. We begin with some elementary extension lemmas.

3.3.0.1. We are now ready to prove our first contractibility statement

Proposition 3.3.1. *Suppose that X is a k-scheme, then the map*

$$\mathrm{Sing}^{\mathbf{A}^1}\mathrm{GenMaps}(X, \mathbf{A}^d) \to \mathrm{Sing}^{\mathbf{A}^1}\,\mathrm{Spec}\,k \tag{3.25}$$

is a trivial Kan fibration in $\mathrm{P}(\mathrm{Aff}_k)$.

Proof. The claim is that for any $T \in \mathrm{Aff}_k$, we have a solution to the following lifting problem

$$\begin{tikzcd} \partial\Delta^n_T \arrow[r, "g"] \arrow[d] & \mathrm{GenMaps}(X, \mathbf{A}^d) \arrow[d] \\ \Delta^n_T \arrow[r] \arrow[ur, dotted, "\tilde{g}"] & \mathrm{Spec}\,k. \end{tikzcd} \tag{3.26}$$

Since $\mathrm{GenMaps}(X, \mathbf{A}^d)$ (by Proposition 3.2.9) and $\mathrm{Spec}\,k$ both have closed descent, it suffices to prove that the map

$$\mathrm{GenMaps}(X, \mathbf{A}^d)(\partial\Delta^n_S) \leftarrow \mathrm{GenMaps}(X, \mathbf{A}^d)(\Delta^n_S), \tag{3.27}$$

is an epimorphism.

Concretely, we have the following situation: the map g classifies rational maps $(g^0_1, \cdots, g^0_d)$ on $X \times \partial\Delta^n \times T$ such that each g_i is defined on a universally dense open

$U_i^0 \subset X \times \partial\Delta^n \times T$:

$$g_i : U_i^0 \to \mathbf{A}^1. \tag{3.28}$$

Using Lemma 3.3.2 below, we may extend the functions g_i^0 to g_i defined on universally dense opens $U_i \subset X \times \Delta^n \times T$ which defines the desired lift in (3.26).

□

Lemma 3.3.2. *Let $S_0 \subset S$ be a closed immersion of finite presentation and Suppose that X is an S-scheme. Given an S_0-rational function f_0 (Example 3.2.2) on X_{S_0}, we can extend it to an S-rational function f on X_S.*

Proof. Let $U^0 \subset X_{S_0}$ be the domain of definition of f_0 so that $U^0 \subset X_{S_0}$ is universally dense. First we choose $U \subset X_S$ a universally dense open subset of X which restricts to U_0, i.e., $U \times_S S_0 = U_{S_0}$. In this case, $U^0 \to U$ is a closed immersion. Furthermore by shrinking U^0 and thus U we may assume that $U_0 = \text{Spec } A/I \subset U = \text{Spec } A$ is affine. In this case, we can just extend f^0 defined on U_0 to the U by choosing any lift in the following diagram (which exists since $\mathbf{Z}[x]$ is free and $A \to A/I$ is epi).

$$\begin{tikzcd} & A \arrow[d] \\ \mathbf{Z}[x] \arrow[r] \arrow[ur, dotted] & A/I \end{tikzcd} \tag{3.29}$$

□

3.3.0.2. Lemma 3.2.8 then tells us

Proposition 3.3.3. *Suppose that X is a k-scheme and $d \geq 0$, then the canonical map*

$$\boldsymbol{\Psi}(\text{GenMaps}(X, \mathbf{A}^d)) \to \text{Spec } k \tag{3.30}$$

is an $\mathrm{L}_{\mathrm{mot}}$*-equivalence.*

3.3.0.3. Now we want to improve the contractibility result in Proposition 3.3.3 to open subsets of $\mathbf{A}^n$ *at least when the domain is a curve.* Using the codescent result in Proposition 3.2.10, we can then boostrap the contractibility result to include schemes which can be covered using open subsets of $\mathbf{A}^n$. To do this, we will use the following moving lemma due to Suslin [**66**]. This is where we have to restrict to the base field and thus we work with k-prestacks, $\mathrm{P}(\mathrm{Aff}_k)$ and the functor $\mathbf{\Psi} : \mathrm{P}(\mathrm{Aff}_k) \to \mathrm{H}(k)$.

3.3.0.4. The heart of our argument uses the following "moving lemma" due to Suslin.

Theorem 3.3.4. [Suslin] *Let S be an affine scheme of finite type over a field k, and suppose that we are given the following data*

(1) $V \subset S \times \mathbf{A}^n$ *a closed subscheme.*
(2) Z *an effective divisor of* $\mathbf{A}^n$.
(3) $\phi : S \times Z \to S \times \mathbf{A}^n$ *a map over* S.
(4) $t \geq 0$ *such that* $\dim V \leq n + t$.

Then there exists an filler

$$\begin{tikzcd} S \times \mathbf{A}^n \arrow[r, dotted, "\Phi"] & S \times \mathbf{A}^n \\ S \times Z \arrow[u] \arrow[ur, "\phi"'] & \end{tikzcd} \tag{3.31}$$

such that the map

$$\Phi \mid_{\mathbf{A}^n \setminus Z} : \Phi^{-1}(V) \mid_{\mathbf{A}^n \setminus Z} \to \mathbf{A}^n \setminus Z \tag{3.32}$$

is dimension $\leq t$.

Remark 3.3.5. Theorem 3.3.4 was proved in [**66**] as the main technical ingredient to proving that, for high enough degree, Bloch's higher Chow groups agree with étale cohomology; more precisely if $i \geq \dim X$ and X is a smooth scheme over an algebraically closed field of characteristic zero then

$$\mathrm{CH}^i(X, n, \mathbf{Z}/m) \simeq \mathrm{H}^{2(d-i)+n}_{\text{ét}}(X, \mathbf{Z}/m(d-i)). \tag{3.33}$$

This theorem is now a special case of the Beilinson-Lichtenbaum conjecture as proved by Voevodsky. Suslin's generic equidimensionality theorem is also used in [**51**] to prove a comparison theorem betweem Bloch's higher Chow groups and motivic cohomology (as defined via the mapping spaces in DM). In particular, it is used as a "moving lemma" to prove that Bloch's complex as defined in [**11**] and Friedlander-Suslin's equidimensional cycles [**30**] are quasiisomorphic, i.e., one can perturb any cycle that intersects $X \times \Delta^n$ properly (see [**51**, Definition 17A.1]) to one that is also equidimensional over Δ^n. This geometric picture was what prompted the author of the present to investigate the connections with the contractibility of rational maps.

3.3.0.5. We now come to the main theorem about the space of generic maps. We fix a connected curve C (possibly not smooth and possibly open) over a field k.

Proposition 3.3.6. *Suppose that Y is representable by an open subset of affine space and suppose that C is a k-curve. The map* $\mathrm{Sing}^{\mathbf{A}^1}\mathrm{GenMaps}(C, Y) \to \mathrm{Sing}^{\mathbf{A}^1}\operatorname{Spec} k$ *is a trivial Kan fibration of simplicial objects in* $\mathrm{P}(\mathrm{Aff}_k)$.

Proof. Using Proposition 3.2.3, we may assume that C is an affine curve. Since Y is representable by an open subset of $\mathbf{A}^n$, we fix the open immersion of schemes $Y \subset \mathbf{A}^n$. The claim is that for any $T \in \mathrm{Aff}_k$, we have a solution to the following lifting problem

$$\begin{tikzcd} \partial\Delta^n_T \arrow[r, "g"] \arrow[d] & \mathrm{GenMaps}(C, Y) \arrow[d] \\ \Delta^n_T \arrow[r] \arrow[ur, dotted, "\tilde{g}"] & \operatorname{Spec} k. \end{tikzcd} \tag{3.34}$$

Now, since GenMaps(C, Y) has closed gluing (by Proposition 3.2.9), the map g classifies a rational map displayed as a span

$$
\begin{tikzcd}
 & U \arrow[dl, "j"'] \arrow[dr, "g"] & \\
C \times \partial\Delta^n \times T & & Y \subset \mathbf{A}^n,
\end{tikzcd}
\tag{3.35}
$$

where j is a universally dense open immersion; we let D be a complement of U. The goal is to extend the span in (3.35) in the following way:

$$
\begin{tikzcd}
 & U \arrow[dl, "j"'] \arrow[dr, "g"] \arrow[dd, dotted] & \\
C \times \partial\Delta^n \times T \arrow[dd] & & Y \subset \mathbf{A}^n \arrow[dd, "="] \\
 & W \arrow[dl, dotted, "j'"'] \arrow[dr, dotted, "\tilde{g}"] & \\
C \times \Delta^n \times T & & Y \subset \mathbf{A}^n,
\end{tikzcd}
\tag{3.36}
$$

where j' is again a universally dense open immersion. To do so, we first extend the map g to a diagram

$$
\begin{tikzcd}
 & U \arrow[r, "g"] \arrow[dl] \arrow[d] & Y \arrow[r] & \mathbf{A}^n \\
C \times \partial\Delta^n \times T \arrow[d] & U' \arrow[urr, "\tilde{g}'"'] \arrow[dl] & & \\
C \times \Delta^n \times T & & &
\end{tikzcd}
\tag{3.37}
$$

using Lemma 3.3.2. Choose a closed complement $Z \subset \mathbf{A}^n$ to Y (any scheme structure) so that we have a closed immersion $\tilde{g}'^{-1}(Z) \subset X \times \Delta^n \times T$. We would like to define U'

as the open complement of the closed subschemes $\tilde{g}'^{-1}(Z)$ and D. However, the problem is that U' might not be generically open. What could happen is that at some point $(v,t) \in \Delta^n \setminus \partial\Delta^n \times T$, the inclusion $\tilde{g}'^{-1}(Z)_{(v,t)} \subset C_{(v,t)}$ is *not proper* and hence the complement of $\tilde{g}'^{-1}(Z) \cup D$ at this point is empty.

To get around this problem, we apply Suslin's Theorem 3.3.4. First notice that since g misses Z, the extension $\tilde{g}$ misses points of Z as well hence the inclusion $\tilde{g}^{-1}(Z) \subset C \times \mathbf{A}^n \times T$ is a proper closed subset. Therefore, the dimension of $\tilde{g}^{-1}(Z)$ is $< \dim C + n + \dim T = 1 + n + \dim T$ and so

$$\dim \tilde{g}^{-1}(Z) \leq n + \dim T. \tag{3.38}$$

Now, we apply Suslin's Theorem 3.3.4 using the following hypotheses:

(1) S is the affine scheme $C \times T$ (remember that we assumed C is an affine curve),
(2) $\tilde{g}'^{-1}(Z) \subset C \times \mathbf{A}^n \times T$ the closed subscheme,
(3) $\partial\Delta^n$ the effective divisor of $\mathbf{A}^n$ and ϕ is the canonical closed immersion $i : \partial\Delta^n \hookrightarrow \Delta^n$ and,
(4) t is $\dim T$ so that $\dim \tilde{g}^{-1}(V) \leq n + t$.

The thesis of Theorem 3.3.4 is that there is a map $\mathbf{\Phi} : C \times \Delta^n \times T \to C \times \Delta^n \times T$ such that $\mathbf{\Phi}^{-1}(\tilde{g}'^{-1}(Z))$ is of dimension $\leq t$ at every point $\Delta^n \setminus \partial\Delta^n$. Furthermore, since $\mathbf{\Phi}$ agrees with closed immersion i and $\tilde{g}^{-1}(Z) \cap \partial\Delta^n = \emptyset$ we also note that $\mathbf{\Phi}^{-1}(\tilde{g}^{-1}(V)) \cap \partial\Delta^n = \emptyset$.

From this we first conclude that at every point $s \in \Delta^n \setminus \partial\Delta^n$ we have a closed immersion over T

$$\mathbf{\Phi}^{-1}(\tilde{g}^{-1}(V))_s \hookrightarrow C \times T \times \{s\}. \tag{3.39}$$

where the $\mathbf{\Phi}^{-1}(\tilde{g}^{-1}(V))_s$ is of dimension $\leq t$ and $C \times T \times \{s\}$ is of dimension $= t + 1$, hence we have a closed immersion of codimension at least[2] 1. In other words, the map $\mathbf{\Phi}^{-1}(\tilde{g}'^{-1}(Z))_s \to T$ is equidimensional of relative dimension zero and is thus quasi-finite.

[2]Without Suslin's Theorem, the codimension could be zero!

We claim that at every closed point $t \in T$, the inclusion of fibers $\Phi^{-1}(\tilde{g}'^{-1}(Z))_s \times_T \{t\} \hookrightarrow$ $\mathrm{C} \times \{t\} \times \{s\}$ is proper [3]. Since C is a curve it thus suffices to prove that the map $\Phi^{-1}(\tilde{g}'^{-1}(Z))_s \to T$ is, in fact, finite [4].

To do so, since $\mathrm{GenMaps}(X, Y)$ is a Nisnevich sheaf by Lemma 3.2.4, we may assume that T is a Henselian local ring of dimension t (we could have assumed so at the beginning of the proof). Since $\Phi^{-1}(\tilde{g}'^{-1}(Z))_s \subset C \times T \times \{s\}$ is nonempty and closed, the intersection of $\Phi^{-1}(\tilde{g}'^{-1}(Z))_s$ with a $\{c\} \times T$ for some closed point c is a closed subset of T and must thus intersect with the closed point of T, hence the map $\Phi^{-1}(\tilde{g}'^{-1}(Z))_s \to T$ hits the closed point of T. According to [**56**, Chapter 1, Theorem 4.2], we may conclude that $\Phi^{-1}(\tilde{g}'^{-1}(Z))_s \to T$ is indeed finite as desired.

Hence at every closed point $(s, t) \in \Delta^n \times T$, we deduce that the fiber $(\Phi^{-1}(\tilde{g}'^{-1}(Z)) \cup D)_{(s,t)}$ is a proper subset of $C_{(s,t)}$ ($D_{(s,t)}$ is already a proper subset by the starting universally dense assumption on $U \subset C \times \partial\Delta^n \times T$) and hence the complement of $\Phi^{-1}(\tilde{g}'^{-1}(Z)) \cup D$ is generically dense since the fiber at every closed point (s, t) of $\Delta^n \times T$ is the complement of a closed, proper subset of C. We set this to be W. By design, map $\tilde{g} \circ \Phi$ restricted to V misses Z and thus we are done. $\square$

Remark 3.3.7. The hypothesis that C is a curve is used in the estimate (3.38). If C was a $k > 1$-dimensional scheme instead, then $\dim\, \tilde{g}^{-1}(V) \leq n + \dim\, T + k - 1$, in which case we will not have the control over the dimension of $\Phi^{-1}(\tilde{g}^{-1}(V))$ over $\Delta^n \setminus \partial\Delta^n$ as produced by Suslin's theorem as applied in the subsequent analysis.

3.3.0.6. We now obtain

Theorem 3.3.8. *Suppose that Y is a a connected, separated scheme which has a Zariski cover $\{U_\alpha\}$ where U_α is a dense open subset of $\mathbf{A}^{n_\alpha}$ and suppose that C is a k-curve. The map* $\mathrm{GenMaps}(C, Y) \to \mathrm{Spec}\ k$ *is an* $\mathrm{L}_{\mathrm{mot}}$*-equivalence.*

[3] In the sense that the inclusion is not an equality; not the sense that the morphism is proper (which of course it is because we have a closed immersion).

[4] So, proper in the sense of a morphism being proper!

Proof. Let $U := \coprod U_\alpha$. Since the Nisnevich topology is finer than the Zariski topology, Remark 3.2.11 tells us that the map

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}} \mathrm{GenMaps}(X, \check{C}_Y(U)) \to \mathrm{GenMaps}(X, Y) \tag{3.40}$$

is an $\mathrm{L}_{\mathrm{mot}}$-equivalences whence it suffices to check that the $\mathrm{colim}_{\Delta^{\mathrm{op}}} \mathrm{GenMaps}(X, \check{C}_Y(U))$ is contractible. By the assumption on U'_αs and using Proposition 3.2.10 to see that $\mathrm{GenMaps}(X, -)$ preserves coproduct decomposition of schemes, we conclude the Theorem using Proposition 3.3.6, the fact that trivial Kan fibrations are ∞-connective [**50**, Theorem A.5.3.1] and Proposition 2.1.3.

$\square$

CHAPTER 4

The space of rational maps

We now deduce the contractibility of another version of the space of rational maps using techniques which are closer in spirit to [**31**], [**32**]; we denote this space of rational maps as RatMaps.

Let us explain the main idea, following [**32**, Example 3.3.3]. The set-up is as follows: we have a smooth complete curve C over a field k and $U \subset \mathbf{A}^n$ an open dense subscheme of affine space. We have an equivalence $\mathrm{RatMaps}(C, \mathbf{A}^n) \simeq \mathrm{RatMaps}(C, \mathbf{A}^1)^{\times n}$ so contractibility of this space is implied by the contractibility of $\mathrm{RatMaps}(C, \mathbf{A}^1)$. This latter gadget is an "algebro-geometric incarnation" of the function field of the curve $k(C)$ which can be thought of as an infinite-dimensional k-vector space and hence this object should be $\mathbf{A}^1$-contractible. The sub-prestack $\mathrm{RatMaps}(C, U) \subset \mathrm{RatMaps}(C, \mathbf{A}^n)$ then has a complement which is, roughly, $\mathrm{RatMaps}(C, Z)$ where $Z \subset \mathbf{A}^n$ is some complement of U. This prestack is then of infinite codimension inside $\mathrm{RatMaps}(C, \mathbf{A}^n)$ and hence have equivalent homology.

This last point is not a condition that can be meaningfully expressed in the unstable motivic homotopy category $\mathrm{H}(k)$. However, in Voevodsky's category of motives $\mathrm{DM}(k)$, and more generally in the category of modules over oriented motivic ring spectra [**20**], we have Gysin triangles

$$\mathrm{M}(X \setminus Z) \to \mathrm{M}(X) \to \mathrm{M}(Z)(c)[2c] \tag{4.1}$$

where c is the codimension of Z in X and $Z \hookrightarrow X$ is a closed immersion of smooth k-schemes. We see that $\mathrm{M}(X \setminus Z) \to \mathrm{M}(X)$ becomes "increasingly more and more equivalent" as we let $c \to \infty$ and the object $\mathrm{M}(Z)(c)[2c]$ becomes more "connective." This was formalized in our discussions in §2.2. More precisely in Lemma 2.2.9.

We also remark that this version of the rational maps is the one used to approximate $\mathrm{Bun}_G(C)$ using the Beilinson-Drinfeld Grassmanian. In particular it appears (fppf-locally) as the fiber of the "uniformization map" $\mathrm{approx} : \mathrm{Gr}_G(C) \to \mathrm{Bun}_G(C)$ [**32**], [**31**]. The motivic consequences of our results to $\mathrm{Bun}_G(C)$ will be explored in a sequel to this thesis.

4.1. The Ran space and its variants

Let us recall the following prestack.

Definition 4.1.1. Let $X \in \mathrm{Sch}$ be separated. The *Ran space of* X is the prestack

(4.2)

$$\mathrm{Ran}(X) : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Set} \subset \mathrm{Spc}, T \mapsto \{I \subset X(T) : I \text{ is a finite non-empty set of } X(T)\}.$$

4.1.0.1. The next proposition summarizes the basic nature of of $\mathrm{Ran}(X)$. We recall that an *ind-scheme* (sometimes called a *strict* ind-scheme) is a formal filtered colimit of schemes

$$\operatorname*{colim}_I X_i \in \mathrm{P}(\mathrm{Aff}) \tag{4.3}$$

where the transition maps $X_i \to X_j$ are closed immersions; these form a category $\mathrm{Ind}\,\mathrm{Sch}$ which is a full subcategory of prestacks. According to [**31**] a *pseudo ind-scheme* is a prestack $\mathscr{X} \in \mathrm{P}(\mathrm{Aff})$ which can be written as

$$\operatorname*{colim}_I X_i \in \mathrm{P}(\mathrm{Aff}) \tag{4.4}$$

where I is a small diagram and X is a functor $X : I \to \mathrm{Ind}\,\mathrm{Sch}$ subject to the condition

- Each transition map $X_i \to X_j$ is ind-proper (see [**31**, 2.1.5] for this terminology and its cousins).

The prestack $\mathrm{Ran}(X)$ is a pseudo ind-scheme. The diagram that presents $\mathrm{Ran}(X)$ is the (opposite category) of finite sets and surjections $\mathrm{Fin}^{\mathrm{surj}}$ where the objects are finite sets and the morphisms are surjective maps.

Proposition 4.1.2. *Suppose that* $X \in \mathrm{Sch}$ *is separated, then* $\mathrm{Ran}(X)$ *is a pseudo ind-scheme. More precisely,* $\mathrm{Ran}(X)$ *is the colimit in* $\mathrm{P}(\mathrm{Aff})$

$$\operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{surj,op}}} X^I. \tag{4.5}$$

Proof. This is well-known, but a proof can be found in [**8**, Appendix B]. The basic idea is as follows: since colimits are taken pointwise in $\mathrm{P}(\mathrm{Aff})$, we get that

$$(\operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{surj,op}}} X^I)(S) \simeq \operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{surj,op}}} \mathrm{Hom}(S, X)^I. \tag{4.6}$$

where Hom above is the set of scheme maps from S to X. Now, in general

$$\operatorname*{colim}_{I \in \mathrm{Fin}^{\mathrm{surj,op}}} A^I, \tag{4.7}$$

where A is a set, computes the set of non-empty finite subsets of A. The point of the argument in [**8**, Appendix B] is to show that the resulting colimit is still discrete.

□

Remark 4.1.3. As remarked in [**32**, Warning 2.4.4], $\mathrm{Ran}(X)$ does not satisfy étale descent. The prestack is not even in $\mathrm{P}_\Sigma(\mathrm{Aff})$: if A and B are sets, then taking the product of a finite subset of A with a finite subset of B in $A \times B$ forms a finite subset of $A \times B$, but not all finite subsets of $A \times B$ are obtained this way. We also remark that $\mathrm{Fin}^{\mathrm{surj,op}}$ has contractible classifying space (it has an initial object) but is *not* sifted.

4.1.0.2. The model of the space of rational maps that we are interested is a prestack over the Ran space. This will be (in a sense we will make precise later) different from the formulation using generic maps, which we studied in the previous sections. In the present situation, we are restricting the domains of definitions of the rational functions to graph complements. The upshot is that this space will be presented by a more reasonable colimit made up of ind-schemes. As usual, we denote the unstraightening of $\mathrm{Ran}(X)$ as the Cartesian fibration $\mathrm{Aff}_{/\!/\mathrm{Ran}(X)} \to \mathrm{Aff}$.

The category $\mathrm{Aff}_{/\!/\mathrm{Ran}(X)}$ is a discrete category which is easy to describe:

- the objects are $(S, I \subset X(S))$ where S is an affine scheme and $I \subset X(S)$ is a finite nonempty subset,
- the morphisms, displayed as $(T, J \subset X(T)) \to (S, I \subset X(S))$, is a map of affine schemes $p : T \to S$ such that under the map $p^* : X(S) \to X(T)$, we get that $p^*I = J$.

Remark 4.1.4. There are variants of Ran where, in the description of the morphisms, instead of requiring that $p^*I = J$ we require $p^*I \subset J$. In [**8**], this is written as Dom_X^Γ and the corresponding Cartesian fibration over Aff sits in between (4.8) below; see [**8**, Construction 5.2.1].

4.1.0.3. We have a functor

$$c : \mathrm{Aff}_{/\!/\mathrm{Ran}(X)} \to \mathrm{Dom}_X \tag{4.8}$$

which takes $(S, I \subset X(S))$ to $(S, X_S \setminus \Gamma_I \subset X \times S)$ where Γ_I is the union of the graph of the morphisms in I.

Construction 4.1.5. Let X be a separated scheme and let $Y \in \mathrm{P}(\mathrm{Sch})$ be a presheaf. We define the presheaf

$$\mathrm{RatMaps}(X, Y) : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Spc} \tag{4.9}$$

as the straightening of the Cartesian fibration $\mathrm{RatMaps}(X, Y)_{\mathrm{Ran}(X)} \to \mathrm{Aff}_{/\!/\mathrm{Ran}(X)} \to \mathrm{Aff}$ defined via the pullback

$$
\begin{tikzcd}
\mathrm{RatMaps}(X,Y)_{\mathrm{Ran}(X)} \arrow[r] \arrow[d] & \mathrm{GenMaps}(X,Y)_{\mathrm{Dom}_X} \arrow[d] \\
\mathrm{Aff}_{/\!/\mathrm{Ran}(X)} \arrow[r, "c"] \arrow[d] & \mathrm{Dom}_X \\
\mathrm{Aff} &
\end{tikzcd}
\tag{4.10}
$$

Informally, a point in the space $\mathrm{RatMaps}(X,Y)(T)$ classifies $(I \subset X(T), f)$ where I is a finite non-empty subset of $X(T)$ and f is a map from X_T to Y defined away from the graph of X_T, i.e. a morphism $f : X_T \setminus \Gamma_I \to Y$.

4.1.0.4. As the map $\mathrm{RatMaps}(X,Y)_{\mathrm{Ran}(X)} \to \mathrm{Aff}_{/\!/\mathrm{Ran}(X)}$ in (4.10) is a map of Cartesian fibrations (in the sense that it preserves Cartesian arrows) over Aff we have a canonical morphism of prestacks $\mathrm{RatMaps}(X,Y) \to \mathrm{Ran}(X)$.

We also have the following colimit formulation of the space of rational maps. We have the canonical map $X^I \to \mathrm{Ran}(X)$ from its formulation as a colimit, We write $\mathrm{RatMaps}(X,Y)_I := X^I \times_{\mathrm{Ran}(X)} \mathrm{RatMaps}(X,Y)$. For a finite surjective morphism $I \to J$ we then obtain a canonical map of prestacks $\mathrm{RatMaps}(X,Y)_J \to \mathrm{RatMaps}(X,Y)_I$.

Lemma 4.1.6. *We have a canonical equivalence*

$$
\mathrm{RatMaps}(X,Y) \simeq \operatorname*{colim}_{\mathrm{Fin}^{\mathrm{surj,op}}} \mathrm{RatMaps}(X,Y)_I. \tag{4.11}
$$

Proof. This follows by universality of colimits:

$$
\begin{aligned}
\operatorname*{colim}_{\mathrm{Fin}^{\mathrm{surj,op}}} (X^I \times_{\mathrm{Ran}(X)} \mathrm{RatMaps}(X,Y)) &\simeq \operatorname*{colim}_{\mathrm{Fin}^{\mathrm{surj,op}}} X^I \times_{\mathrm{Ran}(X)} \mathrm{RatMaps}(X,Y) \\
&\simeq \mathrm{Ran}(X) \times_{\mathrm{Ran}(X)} \mathrm{RatMaps}(X,Y) \\
&\simeq \mathrm{RatMaps}(X,Y).
\end{aligned}
$$

$\square$

4.1.0.5. The E-motive of $\mathrm{RatMaps}(X, Y)$. We now set the stage for the computation of $\mathrm{RatMaps}(X, Y)$ — as far as we know this is the maximal generality where the computations can be performed. For the remainder of this section, we work over a perfect field k — so our prestacks are defined over Aff_k. Suppose that $\mathrm{E} \in \mathrm{CAlg}(\mathrm{SH}(k))$, and consider the functor constructed in §2.2

$$\mathrm{M_E}(-) : \mathrm{P}(\mathrm{Aff}) \longrightarrow \mathrm{Mod_E}(k), \tag{4.12}$$

which associates to a prestack $\mathscr{X}$ its E-motive $\mathrm{M_E}(\mathscr{X})$.

4.1.0.6. Suppose now that C is a smooth, complete curve over k. Suppose that Y is an arbitrary k-scheme, then the map

$$\pi : \mathrm{RatMaps}(C, Y) \to \mathrm{Ran}(C) \tag{4.13}$$

induces a map on the level of E-motives

$$\mathrm{M_E}\pi : \mathrm{M_E}(\mathrm{RatMaps}(C, Y)) \to \mathrm{M_E}(\mathrm{Ran}(C)). \tag{4.14}$$

According to [**31**] and [**32**] the map (4.13) induces an equivalence on ℓ-adic homology. This is the kind of theorems that we are working towards for (4.14), but as we will see there are actual obstructions to proving statements like this on the nose for $\mathrm{M_E}$.

4.1.0.7. Nonetheless, we proceed. To begin, we consider the map $\mathrm{RatMaps}(C, Y)_I \to C^I$ for $I \in \mathrm{Fin}^{\mathrm{surj}}$. In the case that $Y = \mathbf{A}^n$ we have a the following

Proposition 4.1.7. *The map* $\mathrm{M_E}\mathrm{RatMaps}(C, \mathbf{A}^n)_I \to \mathrm{M_E}C^I$ *is an equivalence*

Proof. In fact the equivalence occurs on the level of $\mathrm{H}(k)$. For simplicity, we let $n = 1$ which already has all the ideas involved. There is a filtration on $\mathrm{RatMaps}(C, \mathbf{A}^1)$

$$\mathrm{RatMaps}(C, \mathbf{A}^1)^{\leq 0}_I \subset \mathrm{RatMaps}(C, \mathbf{A}^1)^{\leq 1}_I \subset \cdots \subset \mathrm{RatMaps}(C, \mathbf{A}^1)^{\leq d}_I \subset \cdots. \tag{4.15}$$

where a lift

$$
\begin{tikzcd}
 & \mathrm{RatMaps}(C,\mathbf{A}^1)_I^{\le d} \arrow[d] \\
T \arrow[r, "t"'] \arrow[ur, dotted] & C^I
\end{tikzcd}
\tag{4.16}
$$

classifies $(I \subset C(T), f)$ where I is a finite nonempty subset (classified by t) and f is a rational function on C_T with poles of degree d along Γ_I. Such a data is classified by the set $H^0(C_T; \mathscr{O}_{C_T}(d\Gamma_I))$. Now, by the Riemann-Roch theorem, as soon as $d > 2g-2$ the cohomology group $H^1(C_T; \mathscr{O}_{C_T}(d\Gamma_I))$ disappears so that $\mathrm{RatMaps}(C,\mathbf{A}^1)_I^{\le d} \to C^I$ is representable by a vector bundle of rank $d-1+g$. Hence, for each d, the map $\boldsymbol{\Psi}(\mathrm{RatMaps}(C,\mathbf{A}^1)_I^{\le d}) \to C^I$ is an $\mathrm{L}_{\mathbf{A}^1}$-equivalence. Since $\boldsymbol{\Psi}(\mathrm{RatMaps}(C,\mathbf{A}^1)) \simeq \operatorname{colim} \boldsymbol{\Psi}(\mathrm{RatMaps}(C,\mathbf{A}^1)^{\le d})$, as $\boldsymbol{\Psi}$ preserves colimits, we are done in this case.

For the general case we note that $\mathrm{RatMaps}(C,\mathbf{A}^n)_I \simeq \mathrm{RatMaps}(C,\mathbf{A}^1)_I \times_{X^I} \cdots \times_{X^I} \mathrm{RatMaps}(C,\mathbf{A}^1)_I$ in $\mathrm{P}(\mathrm{Aff}_k)$ and we can carry out the same argument with the filtration

$$
\mathrm{RatMaps}(C,\mathbf{A}^n)_I^{\le d} :=\subset \mathrm{RatMaps}(C,\mathbf{A}^1)_I^{\le d} \times_{X^I} \cdots \times_{X^I} \mathrm{RatMaps}(C,\mathbf{A}^1)_I^{\le d}
\tag{4.17}
$$

□

As a result, if we take the colimit along $\mathrm{Fin}^{\mathrm{surj,op}}$ we obtain

Corollary 4.1.8. *The map* (4.14) *is an equivalence when* $Y = \mathbf{A}^n$.

4.1.0.8. Now suppose that $Y \subset \mathbf{A}^n$ is a dense open subset. Let $I \in \mathrm{Fin}^{\mathrm{surj}}$ be fixed and let $Z \subset \mathbf{A}^n$ be a fixed closed complement to Y with any scheme structure. Then we have a monomorphism of prestacks

$$
i_Z : \mathrm{RatMaps}(C,Z) \hookrightarrow \mathrm{RatMaps}(C,\mathbf{A}^n),
\tag{4.18}
$$

i.e., for any test scheme T, $\mathrm{RatMaps}(C,Z)(T)$ is spanned by those rational maps $(I \subset X(T), f)$ where $f : X_T \setminus \Gamma_I \to \mathbf{A}^n$ factors through Z. The key geometric fact about the inclusion i_Z is the following Lemma

Lemma 4.1.9 (Gaitsgory [**31**], Gaitsgory-Lurie [**32**])**.** *Suppose that* $d > 2g-2$ *so that* $\mathrm{RatMaps}(C,\mathbf{A}^n)_I^{\leq d} \to X^I$ *is a vector bundle. Then the map* $\mathrm{RatMaps}(C,Z)_I^{\leq d} \hookrightarrow \mathrm{RatMaps}(C,\mathbf{A}^n)_I^{\leq d}$ *is a closed immersion of schemes whose codimension of is bounded below by*

$$n(d'-(1+g))-(n-1)d'+C = d'-n(1+g)+N. \tag{4.19}$$

where N *is a constant that is independent of* d'*.*

Proof. This is [**31**, Lemma 4.4.5]. We sketch the proof for completeness (and also because the author is a fan of the argument). By Noether normalization, we may choose a projection $\pi : \mathbf{A}^n \to \mathbf{A}^{n-1}$ which is finite on $Z \subset \mathbf{A}^n$. This then induces a map $\pi : \mathrm{RatMaps}(C,Z) \to \mathrm{RatMaps}(C,\mathbf{A}^{n-1})$. The claim is then for fixed I and for d large enough, the map $\mathrm{RatMaps}(C,Z)_I^{\leq d} \to \mathrm{RatMaps}(C,\mathbf{A}^{n-1})_I^{\leq d}$ is finite. If this was the case then we are looking at a finite morphism between schemes over X^I where the latter is a vector bundle of rank the dimension of $H^0(C,\mathscr{O}(D))^{n-1}$ where D is divisor of degree d, i.e., it is of dimension $n-1(d+1-g) = (n-1)d+(n-1)(1-g)$ and so the constant we take is $(n-1)(1-g)$.

Now, the map of schemes $\mathrm{RatMaps}(C,Z)_I^{\leq d} \to \mathrm{RatMaps}(C,\mathbf{A}^{n-1})_I^{\leq d}$ is affine and so we will obtain a finite morphism if it is also proper [**65**, Tag 01WG]. To check properness we need only check the valuative criterion for properness which unpacks to the the following:

- Suppose that X is an affine regular curve and $V \subset X$ is the complement of a point of X, the following lifting problem has a solution

(4.20)
$$\begin{tikzcd} V \times (C \setminus \Gamma_I) \arrow[r, "g"] \arrow[d] & Z \arrow[d] \\ X \times (C \setminus \Gamma_I) \arrow[r, "f"] \arrow[ur, dotted] & \mathbf{A}^{n-1} \end{tikzcd}$$

Now the map $Z \to \mathbf{A}^{n-1}$, being finite, is proper, so we can always extend g to an open subscheme of $X \times C \setminus \Gamma_I$ whose complement is is of codimension 2. But now the scheme $X \times C \setminus \Gamma_I$ is normal and we have a map, defined on an open subscheme whose complement is dimension 2, going to an affine scheme. Using the algebraic version of Hartog's theorem (see, for example, [**36**, Theorem 6.45]) we may uniquely extend to all of $X \times C \setminus \Gamma_I$.

$\square$

The complement of the map (4.18) is not quite $\mathrm{RatMaps}(C, Y)$. We record it a description of it as a lemma.

Lemma 4.1.10. *The complement of the monomorphism* (4.18) *classifies, for any* $T \in \mathrm{Aff}_k$*,*

(4.21)
$$\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)(T) := \{(I \subset C(T), f) : f \textit{ takes a generic point of } C_T \setminus \Gamma_I \textit{ to } U\}.$$

In particular, if if $d > 2g-2$, then $\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I^{\leq d} \subset \mathrm{RatMaps}(C, \mathbf{A}^n)_I^{\leq d}$ *is an open subscheme of a vector bundle over C^I.*

4.1.0.9. We now come to the first contractibility theorem. We have a map f_Y : $\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n) \subset \mathrm{RatMaps}(C, \mathbf{A}^n) \to \mathrm{Ran}(C)$.

Theorem 4.1.11. *Let C be a smooth, complete curve over a field k and let $Y \subset \mathbf{A}^n$. Then the map*

(4.22)
$$\mathrm{M_E} f_Y : \mathrm{M_E}\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n) \to \mathrm{M_E}\mathrm{Ran}(C)$$

is an equivalence.

Proof. We begin by fixing an $I \in \mathrm{Fin}^{\mathrm{surj}}$. We first claim that the map

$$\mathrm{M_E} f_{Y,I} : \mathrm{M_E}\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I \to \mathrm{M_E} C^I \tag{4.23}$$

is an equivalence. We define

$$\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I^{\leq d} := \mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I \cap \mathrm{RatMaps}(C, \mathbf{A}^n)_I^{\leq d} \tag{4.24}$$

so that $\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I^{\leq d}$ is indeed the open complement of the closed subscheme (when d is large enough) of $\mathrm{RatMaps}(C, Z)_I^{\leq d}$ where Z is a chosen closed complement of Y. We also have a filtration

$$\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I^{\leq 0} \subset \cdots \subset \mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I^{\leq d} \subset \cdots . \tag{4.25}$$

which is compatible with (4.15) and in the colimit we obtain the open immersion

$$\operatorname{colim} \mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I^{\leq d} \simeq \mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I \hookrightarrow \operatorname{colim} \mathrm{RatMaps}(C, \mathbf{A}^n)_I^{\leq d}. \tag{4.26}$$

Using Lemma 2.2.9 we conclude by Lemma 4.1.9 that we have an equivalence

$$\mathrm{M_E}\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I \simeq \mathrm{M_E}\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)_I, \tag{4.27}$$

and we conclude using Corollary 4.1.8. Taking the colimit along $\mathrm{Fin}^{\mathrm{surj}}$ we obtain the theorem since all functors in sight commute with colimits. $\square$

4.2. Contractibility of the Ran space

We have proved the equivalence

$$\mathrm{M_E}(\mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)) \xrightarrow{\simeq} \mathrm{M_E}(\mathrm{Ran}(C)) \tag{4.28}$$

in Theorem 4.1.11. For this result to be useful, or at least to generalize the results of [**31**], [**32**] we would require two statements:

(1) that the map $\mathrm{RatMaps}(C, Y) \subset \mathrm{RatMaps}(C, Y \subset \mathbf{A}^n)$ induces an equivalence upon applying $\mathrm{M_E}$ and,
(2) that the canonical map $\mathrm{M_E}\mathrm{Ran}(C) \to \mathrm{M_E}(\mathrm{Spec}\ k) = \mathrm{E}$ is an equivalence, i.e., the E-motive of the Ran space is that of a point.

As we will see the second statement, or a variant of it, will imply the first. In general one should not expect $\mathrm{M_E}(\mathrm{Ran}(C))$ to be equivalent to E as the next example shows.

Example 4.2.1. Let X be a k-scheme, then by definition and the compactness of the unit object in $\mathrm{M_E}$ we have an equivalence $\mathrm{H}^{\mathrm{E}}_{0,0}(\mathrm{Ran}(X)) \xrightarrow{\simeq} \mathrm{colim}_{I \in \mathrm{Fin}^{\mathrm{surj,op}}} \mathrm{H}^{\mathrm{E}}_{0,0}(X^I)$. Suppose that $\mathrm{E} = \mathrm{M}\mathbf{Z}$ be the motivic cohomology spectrum, then $\mathrm{H}^{\mathrm{M}\mathbf{Z}}_{0,0}(X) \simeq \mathrm{CH}_0(X)$ when X is projective by [**51**, Page 47] and [**51**, Proposition 14.18]. There are many examples where $\mathrm{colim}\, \mathrm{CH}_0(X^I)$ is not $\mathbf{Z}$ (which it would be if $\mathrm{M}_{\mathrm{M}\mathbf{Z}}\mathrm{Ran}(X)$ is equivalent to $\mathrm{M}_{\mathrm{M}\mathbf{Z}}(\mathrm{Spec}\ k)$). For the first example let X be an elliptic curve over k. Then $\mathrm{colim}_I\, \mathrm{CH}_0(X^I) \cong \mathrm{colim}_I\, X^I(k)$ as abelian groups which are way larger than $\mathbf{Z}$.

4.2.0.1. The lack of contractibility of the Ran space, or rather, the lack of connectedness of the Ran space is actually related to the issue of descent discussed in Remark 4.1.3. The usual argument [**32**, Proposition 2.4.8] proves, at least morally, that the étale motive of the Ran space is connected. More precisely it was proved in *loc. cit.* that $\mathrm{H}_{\text{ét},0}(-; \Lambda)$ is equivalent to $\mathrm{H}_{\text{ét},0}(\mathrm{Spec}\ k; \Lambda)$ for any coefficient ring Λ. This is actually the only obstruction to the contractibility of the Ran space.

4.2.0.2. To clarify this issue and proceed further, we shall explain how to contract the Ran space in $\mathrm{H}(S)$.

Definition 4.2.2. Let $\mathscr{X} \in \mathrm{P}(\mathrm{Aff}_S)$. We say that it is an *idempotent prestack* if there exists a map $m : \mathscr{X} \times \mathscr{X} \to \mathscr{X}$ which endows it with the structure of a commutative monoid object in $\mathrm{Ho}(\mathrm{P}(\mathrm{Aff}_S))$ subject to the following condition:

- The composite of the diagonal map $\Delta : \mathscr{X} \to \mathscr{X} \times \mathscr{X}$ with m is equivalent to the identity, i.e. we have a homotopy

$$m \circ \Delta \xrightarrow{\alpha, \simeq} \mathrm{id}_{\mathscr{X}} . \tag{4.29}$$

Proposition 4.2.3. *Let S be a quasicompact and locally Noetherian and suppose that $\mathscr{X}$ is idempotent prestack which is also $\mathbf{A}^1$-connected, then the canonical map to the terminal object in* $\mathrm{P}(\mathrm{Sm}_S)$, $\Psi(\mathscr{X}) \to S$ *is an* $\mathrm{L}_{\mathrm{mot}}$*-equivalence.*

Proof. We shall drop the Ψ throughout the proof so that $\mathscr{X}$ just means $\Psi(\mathscr{X})$. We claim that for $n \geq 0$, and any base point $x \in \mathscr{X}(S)$ (of which there is a single choice up to $\mathrm{L}_{\mathrm{mot}}$-equivalence due to the assumptions!), the Nisnevch sheaf $\pi_n^{\mathrm{Nis},\mathbf{A}^1}(\mathscr{X}, x)$ is isomorphic to the terminal object object in the topos $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)^{\heartsuit}$. for all $n \geq 0$. The hypotheses assures the hypercompleteness of the ∞-topos $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{Sm}_S)$ (see [**50**, Corollary 3.7.7.3] for an overkill statement) and so this is enough for the desired statement.

The claim is indeed true for $\pi_0^{\mathrm{Nis},\mathbf{A}^1}(\mathscr{X})$ by assumption. Furthermore, we see that the set $\pi_0^{\mathbf{A}^1}(\mathscr{X})(S)$ is not empty and is a singleton, so we may choose an S-point of $\mathscr{X}$, $x : S \to \mathscr{X}$ with respect to which we compute the $\mathbf{A}^1$-homotopy sheaves of $\mathscr{X}$.

We check that $\pi_n^{\mathrm{Nis},\mathbf{A}^1}(\mathscr{X}, x)(T)$ is a singleton for every S-scheme T, i.e., we check the claim sectionwise. For convenience, we define $V_n := \pi_n^{\mathbf{A}^1}(\mathscr{X}, x)(T)$. For all $n \geq 1$, the set V_n is an abelian groups in two ways — using the fact that $\mathscr{X}$ is, in particular, an H-space (i.e. we use the induced map m_*) and the natural abelian group structure on homotopy sheaves (which we denote by $+$). By the usual Eckman-Hilton argument, the structures agree.

We will see that for all $\eta \in V_n$, we get that $\eta = 0$. First, observe that in the abelian group V_n, we have that $m_*(\eta, 0) = \eta = m_*(0, \eta)$. Furthermore, by the idempotent condition we have that $\eta = m_*(\eta, \eta)$. Therefore we conclude that $\eta = m_*(\eta, \eta) = m_*(0, \eta) + m_*(\eta, 0) = 2\eta$, from which we conclude that $\eta = 0$.

$\square$

4.2.0.3. Proposition 4.2.3 encourages us to construct a version of the Ran space which is idempotent but also $\mathbf{A}^1$-connected. For many reasons, we would not want to just take the connected cover $\tau_{\geq 1}\mathrm{Ran}(X)$ — for one, prestacks living over this prestack, such as $\tau_{\geq 1}\mathrm{RatMaps}(X, Y)$ lose all their moduli interpretations.

Instead, we will first construct a version of the Ran space which is idempotent, $\mathbf{A}^1$-connected and is a certain sheafification of the Ran space. The corresponding sheafification of $\mathrm{RatMaps}(C, Y)$ will then be equivalent to $\mathrm{Ran}(C)$, after applying M_{E}. We will then explain how this is a refinement of the results of [**31**] and [**32**]

4.2.0.4. Here is the version of the Ran space we will be studying.

Definition 4.2.4. Let $X \in \mathrm{Sch}$. The *cycle*-Ran *space of* X is the prestack

$$\mathrm{Ran}^{\mathrm{cyc}}(X) : \mathrm{Aff}^{\mathrm{op}} \to \mathrm{Set} \subset \mathrm{Spc} \tag{4.30}$$

$$T \mapsto \{Z \subset X \times_S T \text{ a closed subset} : \text{the map } Z \to T \text{ is finite and surjective}\}.$$

We remark that $Z \subset X \times_S T$ being a closed subset means that Z is equipped with the reduced scheme structure. The scheme Z need not be irreducible — for examples just take the graph of T-points of X, i.e., points coming from $\mathrm{Ran}(X)$. The functoriality of $\mathrm{Ran}^{\mathrm{cyc}}(X)$ comes from the fact that being a finite morphism and being surjective is stable under arbitrary pullbacks of affine schemes (or any scheme). We can also describe the associated Cartesian fibration $\mathrm{Aff}_{//\mathrm{Ran}^{\mathrm{cyc}}(X)} \to \mathrm{Aff}$:

- the objects are $(S, Z \subset S \times X)$ where S is an affine scheme and $Z \subset S \times X$ is a closed subset such that the map $Z \to S$ is finite and surjective.
- the morphisms, displayed as $(T, Z') \to (S, Z)$, is a map of affine schemes $p : T \to S$ such that under the map $p^*Z = Z'$.

There is an obvious morphism of prestacks

$$t : \mathrm{Ran}(X) \to \mathrm{Ran}^{\mathrm{cyc}}(X), (S, I) \mapsto (S, \Gamma_I). \tag{4.31}$$

4.2.0.5. We will prove that the prestack $\mathrm{Ran}^{\mathrm{cyc}}(X)$ is actually a stack in the h-topology; in fact it is the h-sheafification of the Ran space. For safety we now restrict ourselves to a Noetherian base S [1]. In this context, the h-topology on Sch_S was first introduced by Voevodsky in [**70**]. For our purposes and the reader's peace of mind we give some impressions of how this topology looks like:

(1) The h-topology is the Grothendieck topology associated to the pretopology where the covering families are finite collections $\{p_i : U_i \to X\}$ of finite type morphisms such that $\coprod_i U_i \to X$ is a universal topological epimorphism [**70**, Definition 3.1.2].
(2) It is finer than the étale or the proper topology on Sch_S and is not subcanonical (representable presheaves are not h-sheaves).
(3) It is generated by open coverings as well as coverings of shape $\{p : Y \to X\}$ where p is proper and surjective (see [**70**, Proposition 3.1.3] and [**62**, Theorem 8.4] for the generality suitable for our situation).
(4) The typical example of an h-cover includes faithfully flat morphisms and proper surjections.

Proposition 4.2.5. *The prestack* $\mathrm{Ran}^{\mathrm{cyc}}(X)$ *is an h-sheaf. Furthermore, the map*

$$\mathrm{Ran}(X) \to \mathrm{Ran}^{\mathrm{cyc}}(X) \tag{4.32}$$

is an L_{h}*-equivalence.*

Proof. By [**62**, Theorem 8.4] we need to check Zariski descent for $\mathrm{Ran}^{\mathrm{cyc}}(X)$ and descent with respect to a single proper surjective morphism $\{T \to S\}$.

First we check that $\mathrm{Ran}^{\mathrm{cyc}}(X)$ is in $\mathrm{P}_\Sigma(\mathrm{Aff})$. To do so take $T_1, T_2 \in \mathrm{Aff}$ and let $T := T_1 \coprod T_2$. We have a map

$$\mathrm{Ran}^{\mathrm{cyc}}(X)(T) \to \mathrm{Ran}^{\mathrm{cyc}}(X)(T_1) \times \mathrm{Ran}^{\mathrm{cyc}}(X)(T_2) \tag{4.33}$$

[1]Often the statements below are modified by adding “(locally) of finite presentations,” see [**62**].

defined by taking a $Z \subset X \times_S T \simeq X \times_S (T_1 \coprod T_2) \mapsto (Z \times_S T_1, Z \times_S T_2)$. But now we have that $Z = Z \times_S T_1 \coprod Z \times_S T_2$ since coproducts are disjoint in Sch and thus we have the inverse map

$$\mathrm{Ran}^{\mathrm{cyc}}(X)(T) \leftarrow \mathrm{Ran}^{\mathrm{cyc}}(X)(T_1) \times \mathrm{Ran}^{\mathrm{cyc}}(X)(T_2) \tag{4.34}$$

by taking coproducts.

Now suppose that U is connected and suppose that $U_1, U_2 \subset U$ are open subschemes, we need to check that the following is equalizer diagram

$$\mathrm{Ran}^{\mathrm{cyc}}(X)(U) \to \mathrm{Ran}^{\mathrm{cyc}}(X)(U_1 \coprod U_2) \rightrightarrows \mathrm{Ran}^{\mathrm{cyc}}(X)(U_1 \cap U_2). \tag{4.35}$$

Indeed, if $Z_1 \subset X \times U_1$ and $Z_2 \subset X \times U_2$ are closed subsets which agrees on the intersection, we obtain a unique closed subset Z restricting to Z_i and $X \times U_i$. The only thing we need to note is that $Z \to U$ is finite and surjective if $Z_i \to U_i$ are. This is clear for surjective. Now, being finite is equivalent to quasi-finite and proper [**38**, Chapter III, Exercise 11.2]. The requisite property is clear for quasi-finite and we conclude using the fact that being proper is local on the base [**38**, Chapter II, Corollary 4.8.f]. That ends the proof of Zariski descent.

Now, suppose that $p : T \to S$ is a proper morphism, we need to check that the following is an equalizer diagram

$$\mathrm{Ran}^{\mathrm{cyc}}(X)(S) \to \mathrm{Ran}^{\mathrm{cyc}}(X)(T) \rightrightarrows \mathrm{Ran}^{\mathrm{cyc}}(X)(T \times_S T). \tag{4.36}$$

But this follows because finite morphisms are, in particular, proper and hence has descent along proper maps.

Finally, we want to prove that the map $\mathrm{Ran}(X) \to \mathrm{Ran}^{\mathrm{cyc}}(X)$ is an $\mathrm{L_h}$-equivalence. To do so, we fix a point $t : T \to \mathrm{Ran}^{\mathrm{cyc}}(X)$. Then we would like to show that there exists an h-cover $\tilde{T} \to T$ and a solution to the following lifting problem

$$
\begin{tikzcd}
 & & \mathrm{Ran}(X) \arrow[d] \\
\tilde{T} \arrow[r] \arrow[urr, dotted, "\tilde{t}"] & T \arrow[r, "t"] & \mathrm{Ran}^{\mathrm{cyc}}(X)
\end{tikzcd} \quad . \tag{4.37}
$$

Concretely t classifies a subset $Z \subset T \times X$ where the map $Z \to T$ is finite and surjective. We want to find an h-cover $\tilde{T}$ such that $Z_{\tilde{T}}$ is the union of a graphs of morphisms from $\tilde{T} \to X$. Over connected components of T, the map $Z \to T$ has different degrees. Since Zariski covers are h-covers we may assume that $Z \to T$ has a constant degree d. In this case, the morphism $Z \to T$ itself an h-cover of T since proper morphisms are h-covers. Pulling Z back across itself, we obtain a section $\Delta : Z \to Z \times_T Z$. The complement is of degree $d-1$ over Z. We repeat this procedure d-times to obtain a proper morphism $\tilde{T} \to T$ equipped with d-sections $\tilde{T} \to Z_{\tilde{T}}$, and thus obtaining a point $\tilde{T} \to \mathrm{Ran}(X)$ as desired.

□

4.2.0.6. Now we restrict to working over a perfect field k. Our goal is to prove the following fundamental fact about $\mathrm{Ran}^{\mathrm{cyc}}(X)$:

Theorem 4.2.6. *Let X be a connected quasiprojective k-scheme, then* $\mathrm{Ran}^{\mathrm{cyc}}(X)$ *is* $\mathbf{A}^1$*-connected and is thus* $\mathrm{L}_{\mathrm{mot}}$*-equivalent to* Spec k.

As a corollary

Corollary 4.2.7. *Let X be a connected k-scheme then the canonical map* $\mathrm{Ran}^{\mathrm{cyc}}(X) \to$ Spec k *is an* $\mathrm{L}_{\mathrm{mot}}$*-equivalence. In particular* $\mathrm{L}_{\mathrm{h}}\mathrm{Ran}(X) \to$ Spec k *is an* $\mathrm{L}_{\mathrm{mot}}$*-equivalence.*

Proof. Clearly, $\mathrm{Ran}^{\mathrm{cyc}}(X)$ is an idempotent prestack: we have a multiplication map

$$
\mathrm{Ran}^{\mathrm{cyc}}(X)(T) \times \mathrm{Ran}^{\mathrm{cyc}}(X)(T) \to \mathrm{Ran}^{\mathrm{cyc}}(X)(T); (Z, Z') \mapsto Z \cup Z. \tag{4.38}
$$

such that $m\Delta = \mathrm{id}$ on the nose. By Theorem 4.2.6, the hypotheses of Proposition 4.2.3 applies and so is its conclusion. The last statement is an immediate consequence of Proposition 4.2.5.

$\square$

4.2.0.7. $\mathbf{A}^1$-connectedness. We require some preliminaries on $\mathbf{A}^1$-connected spaces in order to prove Theorem 4.2.6. First an interlude on the size of the fields we are working over

Remark 4.2.8. One thorny aspect of the subject of motivic homotopy theory is the size of the field. This issue stems from the usage of a fundamental theorem of Gabber, which we review the statements of in Appendix A. To the best knowledge of the author, the paper of Hogadi and Kulkarni [**39**] supplies a proof of Gabber's theorem using techniques from Poonen's finite fields Bertini theorem [**58**] while [**57**] resolves its ramifications in motivic homotopy theory. For the purposes of this paper, we **assume that Gabber's theorem holds over finite fields**. If the reader wishes to be safe, she can assume that we are working over an infinite field.

4.2.0.8. Recall a more "naive" notion of $\mathbf{A}^1$-connectedness.

Definition 4.2.9. We say that $\mathscr{X} \in \mathrm{P}(\mathrm{Sm}_k)$ is $\mathbf{A}^1$-chain connected if for every finitely generated separable field extension L of k, the set

$$\pi_0(\mathrm{Sing}^{\mathbf{A}^1}(\mathscr{X})(L)) = *. \tag{4.39}$$

Note that $\pi_0(\mathrm{Sing}^{\mathbf{A}^1}(\mathscr{X})(L))$ is the equalizer

$$\pi_0(\mathscr{X}(\Delta^1_L)) \rightrightarrows \pi_0(\mathscr{X}(L)) \to \pi_0(\mathrm{Sing}^{\mathbf{A}^1}(\mathscr{X})(L)). \tag{4.40}$$

Unpacking this, $\mathscr{X} \in \mathrm{P}(\mathrm{Sm}_k)$ is $\mathbf{A}^1$-chain connected if and only if:

(1) The space $\mathscr{X}(L)$ is nonempty and,

(2) Given points $x, y \in \mathscr{X}(L)$, we may find $x_0, ..., x_n$ and maps $f_i : \mathbf{A}^1 \to \mathscr{X}(L)$ such that $f_i(0) = x_i$ and $f_i(1) = x_{i+1}$; In other words, we may be able to connect points insider $\mathcal{X}$ using $\mathbf{A}^1$-paths.

4.2.0.9. The next theorem is essentially due to Morel and is stated in, say [**1**].

Theorem 4.2.10. *Suppose $\mathscr{X} \in \mathrm{P}(\mathrm{Sm}_k)$ and $\mathscr{X}$ is $\mathbf{A}^1$-chain connected, then $\mathscr{X}$ is $\mathbf{A}^1$-connected.*

Here we piece together a complete proof of this theorem. To begin, we need an intermediate definition between $\mathbf{A}^1$-chain connected and $\mathbf{A}^1$-connected; this definition can be found in [**52**, Definition 3.3.5].

Definition 4.2.11. Suppose that $\mathscr{X} \in \mathrm{P}(\mathrm{Sm}_k)$. We say that $\mathscr{X}$ is *weakly* $\mathbf{A}^1$-*connected* if for every $T \in \mathrm{Sm}_k$ which is irreducible then on the function field of T we have $\pi_0^{\mathbf{A}^1}(\mathscr{X})(\mathrm{Spec}\ k(T)) = *$.

The condition above is implied by being $\mathbf{A}^1$-connected, however it is *a priori* weaker as we are only checking sections of the homotopy sheaves at generic points of varieties. Our goal is to reverse this implication.

4.2.0.10. The next Lemma is [**52**, Lemma 3.3.6]. However, the proof there contains a mistake which can be corrected using arguments in a later paper of Morel [**53**]. For convenience and to avoid confusion, we reproduce the proof. This is the part where we need to appeal to Gabber's lemma.

Lemma 4.2.12. [Morel, [**52**]] *Assume that $\mathscr{X}$ is $\mathrm{L}_{\mathrm{mot}}$-local so that $\mathscr{X} \simeq \mathrm{L}_{\mathrm{mot}}\mathscr{X}$. Suppose that $\mathscr{X}$ is weakly $\mathbf{A}^1$-connected, then $\mathscr{X}$ is $\mathbf{A}^1$-connected.*

Proof. Let $Y \in \mathrm{Sm}/k$ and $s : Y \to \pi_0^{\mathrm{Nis}}(\mathscr{X})$ be an element. Since $\pi_0^{\mathrm{Nis}}(\mathscr{X})$ has, in particular, Zariski descent we may assume that Y is irreducible. We wish to show that s is indeed trivial. In order to do this, it suffices to prove that there exists a Nisnevich cover $V \to Y$ such that the composite $V \to Y \to \pi_0^{\mathrm{Nis}}(\mathscr{X})$ is trivial.

We first note that the map of simplicial sheaves $\mathscr{X} \to \pi_0(\mathscr{X})$ is an epimorphism (where the latter is treated as a discrete one), hence there exists a cover V of Y for which we have the following commutative diagram (not necessarily Cartesian):

$$\begin{tikzcd} V \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ Y \arrow[r] & \pi_0^{\mathrm{Nis}}\mathscr{X} \end{tikzcd} \tag{4.41}$$

Hence, we may assume that the section s lifts to $\mathscr{X}$.

Now the goal is to show that the composite

$$Y \to \mathscr{X} \to \pi_0(\mathscr{X}) \tag{4.42}$$

is indeed trivial. The section $\mathscr{X}(k(Y))$ is calculated as the colimit $\mathrm{colim}_{V \subset Y} \mathscr{X}(V)$, where the colimit runs over the filtered category of Zariski open subsets of Y. Furthermore since sheafification does not change stalks and fields are Henselian local rings, we have that $\pi_0^{\mathrm{Nis}}(\mathscr{X})(L) \cong \pi_0(\mathrm{colim}_{V \subset Y} \mathscr{X}(V)) \cong \mathrm{colim}_{V \subset Y} \pi_0(\mathscr{X}(V))$. By assumption there exists a dense open subset $W \subset Y$ for which the composite $W \to Y \to \mathscr{X}$ is homotopic to a constant map; say at a point $x \in \mathscr{X}$.

Therefore the map $W \to Y \to \mathscr{X}$ induces a map $Y/W \to \mathscr{X}$ that factors $Y \to \mathscr{X}$. We claim that $\mathrm{L}_{\mathrm{mot}}(Y \to Y/W)$ is nullhomotopic. This will suffice because the diagram:

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & \mathscr{X} \\ Y/W \arrow[ur] & \end{tikzcd} \tag{4.43}$$

factors through:

$$
\begin{tikzcd}
\mathrm{L}_{\mathrm{mot}}Y \arrow[r] \arrow[d] & \mathscr{X} \\
\mathrm{L}_{\mathrm{mot}}Y/W \arrow[ur] &
\end{tikzcd}
\tag{4.44}
$$

because $\mathscr{X}$ is $\mathrm{L}_{\mathrm{mot}}$-local and thus we have proved that the map $Y \to \mathscr{X}$ is homotopy to a constant map.

To prove the claim, we take a Zariski cover of Y, $\{U_\alpha\}$ and prove it for each $U_\alpha \to U_\alpha/W \cap U_\alpha$. This is where Gabber's lemma comes in: for each $y \in Y$ we may choose U satisfying Gabber's lemma (Theorem A.0.1). In other words:

- We find a map $\phi : U \to \mathbf{A}^1_V$ where $V \subset \mathbf{A}^{d-1}$ where $d = \dim Y$ such that for $Z_U := U \setminus (U \cap W)$ we have that $\phi^{-1}(\phi(Z_U)) = Z_U$

Hence, we get an equivalence:

$$
\frac{U}{U \setminus Z_U} \simeq \frac{\mathbf{A}^1_V}{\mathbf{A}^1_V \setminus Z_U}. \tag{4.45}
$$

Therefore, in order to show that $\mathrm{L}_{\mathrm{mot}}(U \to U/(W \cap U))$ is nullhomotopic, we need only show that $\mathrm{L}_{\mathrm{mot}}(\mathbf{A}^1_V \to \mathbf{A}^1_V/\mathbf{A}^1_V \setminus Z_U)$ is nullhomotopic. This is done in the Lemma 4.2.13 below.

□

Lemma 4.2.13. *Let $(U, V, \phi = (\psi, v))$ be as in Gabber's Lemma (Theorem A.0.1), then the map $\mathbf{A}^1_V \to \frac{\mathbf{A}^1_V}{\mathbf{A}^1_V \setminus Z_U}$ is $\mathbf{A}^1$-nullhomotopic (i.e. nullhomotopic after $\mathrm{L}_{\mathbf{A}^1}$-localization).*

Proof. First, note that $F := \psi(Z \cap U) \subset V$ is closed so that the map $Z_U \to \mathbf{A}^1_V \to \mathbf{P}^1_V$ is still closed and misses the section at ∞. In order to prove that the desired map is nullhomotopic, one observes that the map factors through $\mathbf{A}^1_V \to \frac{\mathbf{A}^1_V}{\mathbf{A}^1_V \setminus \mathbf{A}^1_F}$ so that we need only prove that $\frac{\mathbf{A}^1_V}{\mathbf{A}^1_V \setminus \mathbf{A}^1_F} \to \frac{\mathbf{A}^1_V}{\mathbf{A}^1_V \setminus Z_U}$ is $\mathbf{A}^1$-nullhomotopic.

Now there exists a weak equivalence: $\frac{\mathbf{A}^1_V}{\mathbf{A}^1_V \backslash Z_U} \simeq \frac{\mathbf{P}^1_V}{\mathbf{P}^1_V \backslash Z_U}$, the idea is that upon $\mathbf{A}^1$-localization, we may "move" the image of map $\frac{\mathbf{A}^1_V}{\mathbf{A}^1_V \backslash \mathbf{A}^1_F} \to \frac{\mathbf{P}^1_V}{\mathbf{P}^1_V \backslash Z_U}$ inside the "denominator" of the target. Let us carefully execute this.

We have an $\mathbf{A}^1$-weak equivalence $\frac{V}{(V\backslash F)} \to \frac{\mathbf{A}^1_V}{\mathbf{A}^1_V \backslash \mathbf{A}^1_F}$ so it suffices to prove the claim for the composite $\frac{V}{V\backslash F} \to \frac{\mathbf{P}^1_V}{\mathbf{P}^1_V \backslash Z_U}$. This map is induced by the map $V \to \mathbf{P}^1_V$ which includes the zero section (being more detailed: one observes that $F \subset \mathbf{P}^1_V$ is a closed immersion and $F \subset Z_U$ by the definition of the map in Gabber's lemma so that $V \setminus F$ is an open that lies inside $\mathbf{P}^1_V \setminus Z_U$). But now we note that the zero section is $\mathbf{A}^1$-homotopic to the ∞-section, whence $s_\infty(V) \subset \mathbf{P}^1_V \setminus Z_U$.

□

Proof of 4.2.10. After Lemma 4.2.12 we need only check that $\mathscr{X}$ is weakly $\mathbf{A}^1$-connected. Let L be a separable finitely generated extension of k, then we have a surjection

$$\pi_0(\mathrm{Sing}^{\mathbf{A}^1}(\mathscr{X})(L)) \to \pi_0(\mathrm{L}_{\mathrm{mot}}(\mathscr{X})(L)) = \pi_0^{\mathbf{A}^1}(\mathscr{X})(L) \tag{4.46}$$

using the unstable $\mathbf{A}^1$-connectivity theorem [**55**, Corollary I.3.22] and the fact that fields are Henselian local rings and hence we are done. □

4.2.0.11. Divisor spaces and $\mathrm{Ran}^{\mathrm{cyc}}$**.** We need one last preliminary material before we can prove Theorem 4.2.6. Recall that if X is an S-scheme, then there is a presheaf of sets

$$\mathrm{Div}_{X/S} : \mathrm{Aff}_S^{\mathrm{op}} \to \mathrm{Set}; \mathrm{Div}_{X/S}(T) = \{D \subset X_T : D \text{ is a relative effective divisor over } T\}. \tag{4.47}$$

Grothendieck proved that presheaf $\mathrm{Div}_{X/S}$ is in fact representable by a scheme and is an open subscheme of the Hilbert scheme of X [**37**]. Whenever $X \to S$ is a relative curve (so pure relative dimension 1) $\mathrm{Div}_{X/S}$ is in fact a smooth scheme [**12**]. We denote the universal relative effective divisor by $Z_{X/S} \subset \mathrm{Div}_{X/S} \times_S X$.

4.2.0.12. There is a morphism of prestacks that relates $\mathrm{Ran}^{\mathrm{cyc}}(X)$ and $\mathrm{Div}_{X/S}$

Proposition 4.2.14. *Let S be a base scheme and X and S-scheme, there is a morphism of prestacks,*

$$\gamma : \mathrm{Div}_{X/S} \to \mathrm{Ran}^{\mathrm{cyc}}(X). \tag{4.48}$$

Suppose that $S = \operatorname{Spec} k$ is the spectrum of a field k and X is a curve, the map f is surjective on all field extensions of k.

Proof. Taking the support of the relative effective divisor defines the map γ; alternatively $Z_{X/S}$ itself defines a $\mathrm{Div}_{X/S}$-point of $\mathrm{Ran}^{\mathrm{cyc}}(X)$[2]. For the second statement, let L/k be a finitely generated extension of k. Then any L point of $\mathrm{Ran}^{\mathrm{cyc}}(X)$ classifies a closed subset $Z \subset X_L$, which determines an effective Cartier divisor on X_L which is flat over the field L. □

4.2.0.13. From our point of view, $\mathrm{Div}_{X/S}$ is useful for "drawing" $\mathbf{A}^1$-paths in the Ran space.

Proposition 4.2.15. *Let S be a base scheme and X an S-scheme, then the relative effective divisor D is linearly equivalent to E if and only if there a morphism*

$$f : \mathbf{A}^1_S \to \mathrm{Div}_{X/S} \tag{4.49}$$

such that $i_0 \circ f = D$ and $i_1 \circ f = E$.

Proof. Being linearly equivalent is equivalent to saying (or by definition) that there exists an effective relative cycle $Z \subset \mathbf{A}^1_S \times_S X$ such that if $\pi : Z \to \mathbf{A}^1_k$ is the first projection, then $\pi^{-1}(0) = D$ and $\pi^{-1}(1) = E$. Hence this data is the same as a morphism $\mathbf{A}^1_S \to \mathrm{Div}_{X/k}$ satisfying the conditions above. □

We call a map as in (4.49) an $\mathbf{A}^1$-*path*.

[2]So we get a map of prestacks defined as functors out of $\mathrm{Sch}_S^{\mathrm{op}}$.

4.2.0.14. We will now provide a

PROOF OF THEOREM 4.2.6. First we let $X = C$ a curve. By Theorem 4.2.10 we need only show $\mathrm{Ran}^{\mathrm{cyc}}(C)$ is $\mathbf{A}^1$-chain connected. Let L be a finitely generated separable extension of k and take $x, y \in \mathrm{Ran}^{\mathrm{cyc}}(C)(L)$. By the last statement of Proposition 4.2.14 there exists divisors which maps onto x and y. For any $m \geq 0$. x and mx have the same support in $L \times_k X$, so they determine the same point in $\mathrm{Ran}(X)^{\mathrm{cyc}}(X)(L)$. Hence we claim that there exists an $\mathbf{A}^1$-path connecting mx to my for m large enough. Choose n such that $nx - y$ and $ny - x$ are linearly equivalent to effective divisors on X, call them D and E respectively.

Now up to linear equivalence we have the following equalities:

(1) $(n^2 + n)x = (n+1)nx = n + 1(y + D) = x + E + y + (n+1)D$
(2) Similarly, $(n^2 + n)y = x + D + y + (n+1)E$

By Proposition 4.2.15, we conclude that there exists an $\mathbf{A}^1$-path connecting $(n^2 + n)x$ to $x + E + y + (n+1)D$ and $(n^2 + n)y$ to $x + D + y + (n+1)E$. However in $\mathrm{Ran}^{\mathrm{cyc}}(X)(L)$ we have the identifications

(1) $x = (n^2 + n)x$,
(2) $y = (n^2 + n)y$ and,
(3) $x + D + y + (n+1)E = x + D + y + E = x + E + y + (n+1)D$.

Therefore we conclude that there exists an $\mathbf{A}^1$-path connecting $(n^2 + n)x$ to $(n^2 + n)y$ and thus an $\mathbf{A}^1$ connecting x and y in $\mathrm{Ran}^{\mathrm{cyc}}(X)(L)$.

For the general case, let X be quasiprojective over k and L a finitely generated separable extension of k. Suppose that $x \in \mathrm{Ran}^{\mathrm{cyc}}(X)(L)$ and $y \in \mathrm{Ran}^{\mathrm{cyc}}(X)(L)$ are two L-points of $\mathrm{Ran}^{\mathrm{cyc}}(X)$. By Lemma 4.2.16, for any two L-points of X, a, b, there exists a smooth connected curve C and a map $\theta : C \to X$ such that a, b are in the image of θ. Hence, we can construct a map $\theta_{\mathrm{Ran}} : \mathrm{Ran}^{\mathrm{cyc}}(C) \to \mathrm{Ran}^{\mathrm{cyc}}(X)$ such that a and b are in the image of θ_{Ran}. We then done using the $\mathbf{A}^1$-connectedness of $\mathrm{Ran}^{\mathrm{cyc}}(C)$.

$\square$

Lemma 4.2.16. *Let X be a connected quasi-projective L-variety, then given any two L-points, x, y of X, there exists a smooth connected curve C over L and a morphism $f : C \to X$ such that $x, y \in f(C)$*

Proof. By Chow's lemma, we may find a projective L-scheme X' such that $f : X' \to X$ is surjective and X' is proper. Therefore we may assume that X is proper. Consider the blow-up $\mathrm{Bl}_{x,y}X \to X$ with two exceptional divisors E_x and E_y. We may choose a closed embedding $\mathrm{Bl}_{x,y}X \hookrightarrow \mathbf{P}^N_k$ since X, and thus $\mathrm{Bl}_{x,y}X$, is projective. In this scenario, we can use Bertini's theorem (use [**58**] in the finite field case) to pick a hyperplane section that meets E_x and E_y properly. Blowing E_x and E_y back down gives us a codimension 1 subscheme of X on which x, y lies. Therefore the problem reduces to a dimension lower and we may eventually assume that X is a projective curve in which the claim is true by taking a normalization $\tilde{X} \to X$ of X, whence $\tilde{X}$ is smooth and connected. $\square$

4.2.0.15. Let us obtain some consequences of Theorem 4.2.6. Suppose that X, Y are k-schemes. Let $U \subset Y$ be a dense open subscheme.

Proposition 4.2.17. *The* $\mathrm{RatMaps}(X, U) \to \mathrm{RatMaps}(X, U \subset Y)$ *is an* $\mathrm{L_{mot}L_h}$*-equivalence. In particular, we have an* $\mathrm{L_{mot}L_h}$*-equivalence*

$$\mathrm{M_E}(\mathrm{RatMaps}(X, U)) \to \mathrm{M_E}(\mathrm{RatMaps}(X, U \subset Y)) \tag{4.50}$$

Proof. We claim that the map $\mathrm{RatMaps}(X, U) \to \mathrm{RatMaps}(X, U \subset Y)$ is an $\mathrm{L_{mot}L_h}$-equivalence in $\mathrm{P}(\mathrm{Aff}_k)$. The proof follows [**32**, Proposition 3.5.3] closely. By universality of colimits (Proposition B.0.3) it suffices to prove the following: for any T an affine k-scheme and any map $\alpha : T \to \mathrm{RatMaps}(X, U \subset Y)$, classifying $(I \subset X(T), f : X_R \setminus \Gamma_I \to U)$, the map $\mathscr{X} := \mathrm{RatMaps}(X, U) \times_{\mathrm{RatMaps}(X, U \subset Y)} \mathrm{RatMaps}(X, U \subset Y)_{\alpha/} \to T$ is an $\mathrm{L_{mot}L_h}$-equivalence. Let $K := X_R \setminus f^{-1}(U)$, which is a closed subset of K. We observe that

$$\mathscr{X} \hookrightarrow \mathrm{Ran}(X)_T \tag{4.51}$$

where, as a Cartesian fibration over Aff_T, $\mathscr{X}$ classifies $(T', I' \subset X(T'))$ such that I' gets mapped to I and that I' contains the inverse image of K. We claim that $\mathscr{X} \hookrightarrow \mathrm{Ran}(X)_T$ is an L_h-equivalence on Aff_T. Indeed, h-locally, we may find a finite subset $J \subset X(T)$ containing I and the graph of J contains the closed subset K. This lets us write down an adjoint equivalence

$$\mathscr{X} \rightleftarrows \mathrm{Ran}(X)_T \tag{4.52}$$

where the right adjoint is given by sending (T', I') to $(T', I' \cup p^{-1}J)$ where $p : T' \to T$ is the structure map. Hence, the classifying ∞-groupoids of $\mathscr{X}$ and $\mathrm{Ran}(X)_T$ are h-locally equivalent. We thus conclude by Theorem 4.2.6 and Propostion 4.2.5. $\square$

4.2.0.16. Finally we have that

Theorem 4.2.18. *Let C be a smooth complete curve over a field k, and let Y be a connected affine scheme which can be covered by open subsets of $\mathbf{A}^n$ (i.e. quasi-affine). Then the map*

$$\mathrm{M_E} f_Y : \mathrm{M_E}(\mathrm{L_{mot}L_h RatMaps}(C, Y)) \to \mathrm{E} \tag{4.53}$$

is an equivalence.

Proof. After Proposition 3.2.10 we are reduced to the case that $Y \subset \mathbf{A}^n$. This case follows from a string of equivalences

$$\begin{aligned} \mathrm{M_E}(\mathrm{L_{mot}L_h RatMaps}(C, Y)) &\simeq \mathrm{M_E}(\mathrm{L_{mot}L_h RatMaps}(C, Y \subset \mathbf{A}^n)) \\ &\simeq \mathrm{M_E}(\mathrm{L_{mot}L_h Ran}(X)) \\ &\simeq \mathrm{M_E}(\mathrm{Ran^{cyc}}(X)) \\ &\simeq \mathrm{E}. \end{aligned}$$

Here, the first equivalence is Proposition 4.2.17, the second equivalence is Theorem 4.1.11, the third is Proposition 4.2.5 and the last is Corollary 4.2.7.

$\square$

CHAPTER 5

Consequences and realizations

5.1. Rational versus generic maps

We now compare rational and generic maps. Let $X \in \mathrm{Sch}$ and $Y \in \mathrm{P(Sch)}$. We have a map of prestacks from (4.10)

$$\alpha : \mathrm{RatMaps}(X, Y) \to \mathrm{GenMaps}(X, Y) \tag{5.1}$$

Proposition 5.1.1. *The map α is an $\mathrm{L}_{\mathrm{fppf}}$-equivalence.*

Proof. This follows from [**8**, Proposition 5.2.2]. □

5.1.0.1. Hence, using Theorem 3.3.8 we can give a more general version of Theorem 4.2.18

Theorem 5.1.2. *Let C be a curve over a field k, and let Y be a connected, separated scheme which has a Zariski cover $\{U_\alpha\}$ where U_α is a dense open subset of $\mathbf{A}^{n_\alpha}$. Then the map*

$$\Sigma^\infty_{\mathbb{T}} f_{Y+} : \Sigma^\infty_{\mathbb{T}} \mathrm{L_h RatMaps}(C, Y)_+ \to \Sigma^\infty_{\mathbb{T}} \operatorname{Spec} k_+ = \mathbf{1} \tag{5.2}$$

is an equivalence. In particular, for any $\mathrm{E} \in \mathrm{CAlg(SH}(k))$, the map

$$\mathrm{M_E} f_Y : \mathrm{M_E}(\mathrm{L_{mot} L_h RatMaps}(C, Y)) \to \mathrm{E} \tag{5.3}$$

is an equivalence.

Proof. Follows from the string of equivalences

$$\begin{aligned}\Sigma^\infty_{\mathbb{T}}\mathrm{L_h}\mathrm{RatMaps}(C,Y)_+ &\simeq \Sigma^\infty_{\mathbb{T}}\mathrm{L_h}\mathrm{GenMaps}(C,Y)_+ \\ &\simeq \Sigma^\infty_{\mathbb{T}}\,\mathrm{Spec}\ k_+\end{aligned}$$

Where the first equivalence is Proposition 5.1.1 and the fact that the h-topology is finer than the fppf topology (any faithfully flat map of finite presentation is a universal topological epimorphism), the second follows from Theorem 3.3.8. □

Remark 5.1.3. Theorem 5.1.2 is indeed stronger than Theorem 4.2.18. However our intention in giving an independent proof of Theorem 4.2.18 is to show explain the fact that there is a motivic relationship between $\mathrm{RatMaps}(C, Y)$ and the Ran space, which is of independent interest, and show the proofs of the corresponding statements in [**31**] and [**32**] are indeed motivic in nature. Indeed, this motivic relationship shows that an "on-the-nose" contractibility of $\mathrm{RatMaps}(C, Y)$ is not true exactly because the Ran space is not "on-the-nose" contractible — it is only so after h-sheafification. What makes Theorem 5.1.2 stronger is that contractibility is proved without recourse to dimension estimates but instead relies on the unstable statement Theorme 3.3.8 which in turn relies on Suslin's Theorem 3.3.4.

5.2. Étale and Betti realizations

5.2.0.1. Betti realization. We let k be a field of characteristic zero and choose an embedding $k \subset \mathbf{C}$. Recall that the Betti realization functor is defined by first consider the functor

$$\mathrm{Sm}_k \to \mathrm{Sm}_{\mathbf{C}} \to \mathrm{Spt}; X \mapsto X_{\mathbf{C}} \mapsto \Sigma^\infty X_{\mathbf{C}}(\mathbf{C})_+. \tag{5.4}$$

where, for a $\mathbf{C}$-scheme Y, $Y(\mathbf{C})$ is the $\mathbf{C}$-points of Y endowed with the analytic topology, i.e. its analytification. By left Kan extension we have a functor $\mathrm{P}_{\mathrm{Spt}}(\mathrm{Sm}_k) \to \mathrm{Spt}$. This functor factors through Nisnevich sheaves of spectra (since it takes Nisnevich covers to covers of the analytification) which are $\mathbf{A}^1$-invariant (since $\mathbf{A}^1$ gets sent to the contractible space $\mathbf{C}$) and also factors through $\mathbb{T}$-inversion (since $\mathbb{T}$ gets sent to the invertible object S^2 and the functor is monoidal). As a net effect we get a functor

$$\mathrm{Betti}_k : \mathrm{SH}(k) \to \mathrm{Spt}. \tag{5.5}$$

In fact, since étale covers of schemes are sent to covers in the analytic topology as well, the functor Betti_k factors through the étale localization functor $\pi^* : \mathrm{SH}(k) \to \mathrm{SH}_{\text{ét}}(k)$ as

$$\mathrm{Betti}_k^{\text{ét}} : \mathrm{SH}_{\text{ét}} \to \mathrm{Spt}. \tag{5.6}$$

5.2.0.2. Since the h-topology is finer than the étale topology, it is not clear that the functor Betti_k will factor through the stable motivic homotopy ∞-category constructed from the h-topology $\mathrm{SH}_{\mathrm{h}}(k)$. Instead we work with a linearized version of the above functors. Consider the motivic spectrum $\mathrm{M}\mathbf{Z}$ representing motivic cohomology with coefficients in $\mathbf{Z}$. Using the fact that the constituent motivic spaces $\mathrm{M}\mathbf{Z}$ can be represented by symmetric powers of schemes [**76**] and resolution of singularities over characteristic zero, we deduce that $\mathrm{Betti}_k\mathrm{M}\mathbf{Z} \simeq \mathrm{H}\mathbf{Z}$. As a result we have a Betti realization functor on the level of $\mathrm{M}\mathbf{Z}$-modules

$$\mathrm{Betti}_{k,\mathbf{Z}} : \mathrm{Mod}_{\mathrm{M}\mathbf{Z}}(k) \to \mathrm{Mod}_{\mathrm{H}\mathbf{Z}}, \tag{5.7}$$

compatible with the functor 5.5 in the sense that the following diagram of left adjoints commute

$$\begin{tikzcd} \mathrm{SH}(k) \arrow[r, "\mathrm{Betti}_k"] \arrow[d, "\mathrm{M}_{\mathrm{M}\mathbf{Z}}(-)"'] & \mathrm{Spt} \arrow[d, "\mathrm{H}\mathbf{Z}\wedge -"] \\ \mathrm{Mod}_{\mathrm{M}\mathbf{Z}}(k) \arrow[r, "\mathrm{Betti}_{k,\mathbf{Z}}"] & \mathrm{Mod}_{\mathrm{H}\mathbf{Z}}. \end{tikzcd} \tag{5.8}$$

5.2.0.3. Now, recall that there is an equivalence of ∞-categories

$$\mathrm{DM}(k,R) \simeq \mathrm{Mod}_{\mathrm{M}R}(k) \tag{5.9}$$

whenever k is a field with a characteristic invertible in R (see [**61**] for the original theorem proved in characteristic zero, see [**27**] for the general statement over perfect fields and [**17**] to obtain the result over general fields). So we have a functor $\mathrm{DM}(k,\mathbf{Z}) \to \mathrm{Mod}_{\mathrm{H}\mathbf{Z}}$ which factors through the étale localization $\pi^* : \mathrm{DM}(k;\mathbf{Z}) \to \mathrm{DM}_{\text{ét}}(k;\mathbf{Z})$ as the functor

$$\mathrm{Betti}^{\text{ét}}_{k,\mathbf{Z}} : \mathrm{DM}_{\text{ét}}(k,R) \to \mathrm{D}(\mathbf{Z}). \tag{5.10}$$

The functor $\mathrm{Betti}^{\text{ét}}_{k,\mathbf{Z}}$ is also compatible with the functor 5.6 in the obvious way.

5.2.0.4. According to [**18**, Corollary 5.5.5] we have a further equivalence

$$\lambda^* : \mathrm{DM}_{\text{ét}}(k,R) \overset{\simeq}{\rightleftarrows} \mathrm{DM}_{\mathrm{h}}(k,R) : \lambda_*, \tag{5.11}$$

between étale motives and h-motives (true for any coefficient R).

5.2.0.5. Here's a consequence of the equivalence (5.11). We have a functor $R^{\mathrm{tr}}_k : \mathrm{Sch}_k \to \mathrm{Shv}^{\mathrm{tr}}_{\text{ét}}(\mathrm{Sm}_k;R)$ [**18**, 2.1.3] sending a scheme X to the free presheaf of R-modules with transfers:

$$R^{\mathrm{tr}}_k(X) : U \mapsto c_k(U,X) \otimes_{\mathbf{Z}} R, \tag{5.12}$$

where $c_k(U,X)$ are finite correspondences from U to X in the sense of [**16**, 9.1.2]. This is an étale sheaf (by [**18**, Proposition 2.1.4]). Under the various localizations and $\mathbb{T}$-inversion

[**18**, 2.2.4] we have a functor

$$\Sigma^{\infty}_{\mathrm{tr,\acute{e}t}} : \mathrm{Shv}^{\mathrm{tr}}_{\mathrm{\acute{e}t}}(\mathrm{Sm}_k; R) \to \mathrm{DM}_{\mathrm{\acute{e}t}}(k, R). \tag{5.13}$$

We have a composite of functors $\mathrm{M}_{\mathrm{\acute{e}t}}(-; R) := \Sigma^{\infty}_{\mathrm{tr,\acute{e}t}} \circ R^{\mathrm{tr}} : \mathrm{Sch}_k \to \mathrm{DM}_{\mathrm{\acute{e}t}}(k, R)$ which we Kan extend to a functor

$$\mathrm{M}_{\mathrm{\acute{e}t}}(-; R) : \mathrm{P}(\mathrm{Sch}_k) \to \mathrm{DM}_{\mathrm{\acute{e}t}}(k, R). \tag{5.14}$$

With this set-up, the equivalence (5.11) gives us an equivalence for any $X \in \mathrm{P}(\mathrm{Sch}_S)$.

$$\lambda_* R_{\mathrm{h}} \mathrm{M}_{\mathrm{M}R} \mathrm{L}_{\mathrm{mot}} \mathrm{L}_{\mathrm{h}} X \simeq \mathrm{M}_{\mathrm{\acute{e}t}}(X; R). \tag{5.15}$$

In other words, the h-sheafified motivic homotopy type realizes in $\mathrm{DM}_{\mathrm{\acute{e}t}}(k, R)$ to its étale motive. Thus, in the situation of Theorem 5.1.2 we get that

$$\mathrm{M}_{\mathrm{\acute{e}t}}(\mathrm{RatMaps}(C, Y); R) \simeq \mathrm{M}_{\mathrm{\acute{e}t}}(\mathrm{Spec}\ k; R). \tag{5.16}$$

From this we deduce immediately that

Theorem 5.2.1. *Let C be a curve over a field of characteristic zero k, and let Y be a connected, separated scheme which has a Zariski cover $\{U_\alpha\}$ where U_α is a dense open subset of $\mathbf{A}^{n_\alpha}$. Then the homology groups $\mathrm{H}_{*,\mathrm{Sing}}(\mathrm{RatMaps}(C, Y); \mathbf{Z})$ are concetrated in degree 0 and $H^0_{\mathrm{Sing}}(\mathrm{RatMaps}(C, Y); \mathbf{Z}) = \mathbf{Z}$.*

This reproves the main theorem of [**8**] which is, in turn, a more general version of [**31**]

5.2.0.6. Étale realization. Now we let k be an arbitrary field. Let us turn to the étale realizations of Theorem 5.1.2 which is just a matter of plugging in the right coefficients. Indeed, in the discussion of (5.2.0.3) just plug in $R = \mathbf{Z}_\ell$, the ℓ-adic integers, and ℓ is prime to the characteristic of k then Ayoub [**5**] and Cisinski-Déglise [**18**] proves an equivalence between the triangulated category [1] of h-motives and the unbounded derived category of

[1] It is easy to see that the equivalence is one of stable ∞-categories.

ℓ-adic sheaves, generalizing Suslin rigidity

$$\mathrm{DM}_{\mathrm{h}}(k, \mathbf{Z}_\ell) \xrightarrow{\simeq} \mathrm{D}_{\text{ét}}(k, \mathbf{Z}_\ell). \tag{5.17}$$

As a net result, under the realization functor $\mathrm{Mod}_{\mathrm{M}\mathbf{Z}_\ell} \to \mathrm{DM}_{\mathrm{h}}(k, \mathbf{Z}_\ell) \simeq \mathrm{D}_{\text{ét}}(k, \mathbf{Z}_\ell)$, Theorem 5.1.2 gives us the ℓ-adic contractibility statement of [**32**, Lemma 3.6.1]:

Theorem 5.2.2. *Let C be a curve over a field k and let ℓ be prime to the characteristic of k. Let Y be a connected, separated scheme which has a Zariski cover $\{U_\alpha\}$ where U_α is a dense open subset of $\mathbf{A}^{n_\alpha}$. Then for any $q \in \mathbf{Z}$, the étale homology groups $\mathrm{H}_{*,\text{ét}}(\mathrm{RatMaps}(C, Y); \mathbf{Z}_\ell(q))$ are concentrated in degree 0 and $\mathrm{H}_{0,\text{ét}}(\mathrm{RatMaps}(C, Y); \mathbf{Z}_\ell) \simeq \mathbf{Z}_\ell(q))$.*

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APPENDIX A

Gabber's presentation lemma

In this very brief appendix we recall the statement of Gabber's Lemma, as stated in [**19**].

Theorem A.0.1. [Gabber's presentation lemma] *Suppose that X is an d-dimensional affine scheme over a field k, and Z is a closed subscheme of codimension $n > 0$. Then for any $x \in X$ there exists (U, V, ϕ) where:*

(1) $\phi = (\psi, v) : X \to \mathbf{A}_k^d$
(2) *$V \subset \mathbf{A}_k^{d-1}$ is an open subscheme, and $U \subset \psi^{-1}(V)$ is an open subscheme containing x*

subject to the following conditions:

(1) *$\psi \mid_Z : Z \to \mathbf{A}_k^{d-1}$ is finite.*
(2) *$\phi \mid_U : U \to \mathbf{A}_k^d$ is étale*
(3) *$\phi \mid_{Z\cap U} : Z \cap U \to \mathbf{A}_k^d$ is a closed immersion.*
(4) *$\phi^{-1}(\phi(Z \cap U)) \simeq Z \cap U$*

The point of the Gabber's lemma is that we obtain the following Nisnevich distinguished square. Write $Z \cap U := Z_U$, then

Corollary A.0.2. *With the notation above, we get an Nisnevich distinguished square:*

$$
\begin{tikzcd}
U \setminus Z_U \arrow[r] \arrow[d] & U \arrow[d, "\phi"] \\
\mathbf{A}_V^1 \setminus Z_U \arrow[r] & \mathbf{A}_V^1
\end{tikzcd}
$$

where we have implicitly identified Z_U with its image under ϕ.

APPENDIX B

Foundational aspects of motivic homotopy theory

In this appendix we review some constructions and terminology in motivic homotopy theory. The only notion unfamiliar to practioners in the field is $\mathbb{T}$-prestability in B.0.2. This notion and the notion of motivic module categories will appear in greater detail in a revised version of the author's paper with Kolderup in [**27**] where we used Lurie's Barr-Beck theorem to prove that certain stable ∞-categories defined as "cycle complexes" are actually modules over a motivic $\mathcal{E}_\infty$-ring spectrum.

B.0.1. Motivic homotopy theory in various contexts

Let us recall some properties of functors $F : \mathrm{Sch}^{\mathrm{op}} \to \mathrm{C}$.

Definition B.0.1. Let M be a collection of arrows in Sch which are closed under pullback along arbitrary maps. We say that F is M-*invariant* if for all arrows $X \to Y$ in M, the induced map $F(Y) \to F(X)$ is an equivalence. In the following special cases of M we say that F is

(1) *homotopy invariant* if $\mathrm{M} = \{X \times \mathbf{A}^1 \to X\}_{X \in \mathrm{Sch}}$,
(2) *vector bundle invariant* if $\mathrm{M} = \{p : \mathcal{E} \to X : p \text{ is a vector bundle}\}_{X \in \mathrm{Sch}}$,

Denote by $\mathrm{P}_{\mathrm{M}}(\mathrm{Sch})$ the ∞-category of M-invariant presheaves.

B.0.1.1. Suppose that $\mathrm{C} \subset \mathrm{Sch}^{\mathrm{op}}$ is a full subcategory and suppose that $\mathrm{M} \cap \mathrm{C}$ is still closed under pullbacks. For each of the classes of M in Definition (B.0.1), we would like to construct a "well behaved" localization functor

$$\mathrm{P}(\mathrm{C}) \to \mathrm{P}_{\mathrm{M}}(\mathrm{C}). \tag{B.1}$$

These good behavior should already be exhibited by arguably the first non-formal input of motivic homotopy theory: the *Suslin construction* after [**55**, Section 2]. The generality for which this works is an ∞-category with an interval object; we adopt the definition in [**2**, Definition 4.1.1].

Definition B.0.2. Let C be a small ∞-category with products. A *representable interval object* I is a presheaf on C equipped with a *multiplication map* $m : I \times I \to I$ and *end point maps* $i_0, i_1 : * \to I$ such that

(1) For any $X \in \mathrm{C}$ the presheaf $X \times I$ is representable.
(2) Let $p : I \to *$ be the canonical map, then we have homotopies:
   - $m \circ (i_0 \times \mathrm{id}) \simeq m(\mathrm{id} \times i_0) \simeq i_0 p$,
   - $m \circ (i_1 \times \mathrm{id}) \simeq m(\mathrm{id} \times i_1) \simeq \mathrm{id}$

A presheaf F on C is I-invariant if for all $X \in \mathrm{C}$ the natural map $F(X) \to F(X \times I)$ is an equivalence. The *Suslin construction* applied to $F \in \mathrm{C}$ is the functor

$$\mathrm{P(C)} \to \mathrm{P(C)}^{\Delta^{\mathrm{op}}}; F \mapsto \mathrm{Sing}^I F(X) := F(X \times I^{\bullet}) \tag{B.2}$$

where the maps $F(X \times I^n) \to F(X \times I^m)$ is induced by the endpoint maps in Definition B.0.2. The geometric realization of $\mathrm{Sing}^I(\mathrm{F})$ will be denoted, as usual, by $|\mathrm{Sing}^I(F)|$.
B.0.1.2. Indeed, the basic properties of the Suslin construction are summarized in the next proposition.

Proposition B.0.3. *Let* C *be a small* ∞*-category with a representable interval object* I.

(1) *The functor* $|\mathrm{Sing}^I| : \mathrm{P(C)} \to \mathrm{P(C)}$ *is a localization at* $\mathrm{M}_I = \{X \times I \to X\}_{X \in \mathrm{C}}$.
(2) *Suppose that* C *has finite coproducts and finite coproducts distributes over products, then the functor* Sing^I *preserves coproduct-preserving presheaves so it descends to*

a localization

$$|\mathrm{Sing}^I| : \mathrm{P}_\Sigma(\mathrm{C}) \to \mathrm{P}_\Sigma(\mathrm{C}).$$

(3) *The functor* $|\mathrm{Sing}^I|$ *preserves finite products.*

(4) *The map* $F \to |\mathrm{Sing}^I(F)|$ *induces an epimorphism* $\pi_0(F) \to \pi_0(|\mathrm{Sing}^I(F)|)$.

(5) *The functor* Sing^I *is locally Cartesian and hence the localization at* M_I *has universal colimits.*

Proof. The fact that Sing^I is localization is [**55**, Section 2 Corollaries 3.5, 3.8]. The second claim follows from

$$\begin{aligned}
|\mathrm{Sing}^I F(X \textstyle\coprod Y)| &= \operatorname*{colim}_{\Delta^{op}} F((X \textstyle\coprod Y) \times I^n) \\
&\simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}} F(X \times I^n \textstyle\coprod Y \times I^n) \\
&\simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}} F(X \times I^n) \times F(Y \times I^n) \\
&\simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}} F(X \times I^n) \times \operatorname*{colim}_{\Delta^{\mathrm{op}}} F(Y \times I^n) \\
&= |\mathrm{Sing}^I F(X)| \times |\mathrm{Sing}^I F(Y)|.
\end{aligned}$$

Here, the second equivalence follows from the assumption on C, the second follows from the assumption on F, the third follows because sifted colimits commutes with finite products in Spc. This last fact also leads to the fact that Sing^I preserves finite products. The fourth claim follows from the fact that the diagram

$$\pi_0(F(X \times I)) \rightrightarrows \pi_0(F(X)) \to \pi_0(|\mathrm{Sing}^I(F)(X)|). \tag{B.3}$$

is a coequalizer for all $X \in \mathrm{C}$. For the last claim, recall that an endofunctor $F : \mathrm{D} \to \mathrm{D}$ is locally Cartesian if the map $L(X \times_Y Z) \to X \times_Y L(Z)$ is an equivalence for any $X, Y \in F(\mathrm{D})$; if F is a localization it is easy to see that the essential image of F ends up having universal colimits if D does. To check this for Sing, suppose that F, G, H are presheaves on C where F, G are I-invariant, and $X \in \mathrm{C}$ is an object then

$$\begin{aligned} |\mathrm{Sing}^I F \times_{\mathrm{Sing}^I G} \mathrm{Sing}^I H(X)| &= \operatorname*{colim}_{\Delta^{\mathrm{op}}} F(X \times I^\bullet) \times_{G(X \times I^\bullet)} H(X \times I^\bullet) \\ &\simeq \operatorname*{colim}_{\Delta^{\mathrm{op}}} F(X) \times_{G(X)} H(X \times I^\bullet) \\ &\simeq F(X) \times_{G(X)} \operatorname*{colim}_{\Delta^{\mathrm{op}}} H(X \times I^\bullet) \\ &\simeq F(X) \times_{G(X)} |\mathrm{Sing}^I(H)(X)| \end{aligned}$$

Here, the equivalence in the third line is justified by the fact that we are taking a colimit of a contractible diagram taking a constant value and universality of colimits in $\mathrm{P(C)}$. □

Remark B.0.4. We remark that Hoyois has a less explicit but equally useful localization functor that inverts a collection of arrows M which are stable under pullback — these include the other examples in B.0.1 — in [**42**]. It satisfies all the properties of Propostion B.0.3.

B.0.1.3. From the above discussion, we see that a "homotopy theory of schemes where $\mathbf{A}^1$ is treated as the unit interval" can be carried out for any subcategory $\mathrm{C} \subset \mathrm{Sch}$ such that for any $X \in \mathrm{C}$, the object $X \times \mathbf{A}^1 \in \mathrm{C}$. In fact Voevodsky has discussed axioms for C where motivic homotopy theory can be suitably carried out; for example in [**76**, Appendix A]. These axioms are designed such that that we may contemplate the motive/motivic homotopy type of n-th symmetric powers $\mathrm{Sym}^n(X) := X^n/\Sigma_n$ for all $n \geq 1$. These are, in general, singular schemes which are the scheme-theoretic quotients of a finite group acting on a scheme.

Definition B.0.5. A subcategory $\mathrm{C} \subset \mathrm{Sch}_S$ is admissible if

(1) The terminal object and $\mathbf{A}^1$ are in C,
(2) the category C is closed under finite products and coproducts and,

(3) closed under étale extensions: if $U \to Y$ is an étale morphism and $Y \in \mathrm{C}$ then $U \in \mathrm{C}$.

Furthermore we say that C is f-admissible if it closed under the formation of (scheme-theoretic) quotients with respect to actions of finite groups.

B.0.1.4. After [**7**], it is best to consider C satisfying the following condition

Definition B.0.6. An ∞-category C is *extensive* if

(1) has finite coproducts,
(2) binary coproducts are disjoint and,
(3) coproduct decompositions are stable under pullbacks.

If this is the case then [**7**, Lemma 2.4] says that $\mathrm{P}_\Sigma(\mathrm{C})$ is an ∞-topos.; in fact $\mathrm{P}_\Sigma(\mathrm{C})$ is the ∞-category of sheaves on C for the topology generated by coproduct decompositions. In particular $\mathrm{P}_\Sigma(\mathrm{C})$ has universal colimits. With these remarks in mind, any kind of motivic homotopy theory that we will construct will always start with the following assumptions on $\mathrm{C} \subset \mathrm{Sch}_S$:

(1) C contains the terminal object S and $\mathbf{A}^1$,
(2) is closed under coproducts and products,
(3) is extensive,
(4) is closed under étale extensions,
(5) C is closed under formation of scheme-theoretic quotients with respect to actions of finite groups.

We call subcategories of Sch_S satisfying (1)-(4) *geometrically admissible* and subcategories satisfying (1)-(5) *geometrically admissible with quotients.*

Example B.0.7. Examples of geometrically admissible categories with quotients include quasi-affine and quasi-projective schemes over Sch_S . If we were to drop assumption (5) then the usual category Sm_S used to build motivic homotopy theory is an example.

B.0.1.5. Now we discuss topologies. Many topologies in motivic homotopy theory come from a cd-structure:

Definition B.0.8. Suppose that C is a small discrete category with an initial object $\emptyset$. A cd-*structure* on C is a collection of squares P which is stable under isomorphisms. This generates a topology τ_P which is the coasrest topology C in which the empty sieve covers $\emptyset$ and for any square

$$\begin{array}{ccc} W & \longrightarrow & V \\ \downarrow & & \downarrow \\ U & \longrightarrow & X \end{array} \tag{B.4}$$

the sieve on X generated by $\{V \to X, U \to X\}$ is a τ_P covering sieve of X.

Example B.0.9. Many examples are listed at the beginning of [**2**, Example 2.1.2]. Most prominently we have the Nisnevich topology where the squares are Nisnevich distinguished squares [**2**, Example 2.1.2.2] and the Zariski topology where the squares are open covers. There is also the cdh-squares which are of the form

$$\begin{array}{ccc} W & \longrightarrow & X_Z \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array} \tag{B.5}$$

where $Z \to X$ is a closed immersion, $X_Z \to X$ is proper and the map $(X \setminus Z) \times_X X_Z \to X \setminus Z$ is an isomorphism. Another one are the square of the form

$$\begin{array}{ccc} \emptyset & \longrightarrow & U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array} \tag{B.6}$$

where $X = U \coprod V$. This is also cd-structure and generates the topology of coproduct decomposiiton — on an extensive ∞-category C, the ∞-category of sheaves with respect to this topology is equivalent to the nonabelian derived ∞-category.

B.0.1.6. The main theorem about cd-structures is due to Voevodsky. Suppose that C is a small discrete category and P is a cd-structure. We say that the cd-structure is *excisive* if

(1) every square in P is Cartesian,
(2) pullbacks in P exists and are in P,
(3) For each square (B.7) the bottom horizontal map $U \to X$ is monic and,
(4) The squares in P are closed under diagonals: given a square as in (B.7) the square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & V \arrow[d] \\ W \times_U W \arrow[r] & V \times_X V \end{tikzcd} \tag{B.7}$$

is also in P.

Theorem B.0.10 (Voevodsky, [**2**, Theorem 3.2.5], [**75**, Corollary 3.2.5])**.** *Let* C *be a small discrete category equipped with an excisive* cd*-structure on* C*. Then* $F \in \mathrm{Shv}_{\tau_P}(\mathrm{C})$ *if and only if* F *takes squares in* P *to pullbacks.*

We remark, however, that the h-topology that has featured in the main body of the paper is *not* an example of a topology generated by a cd-structure.

B.0.1.7. Thus motivic homotopy theory can be performed efficiently with any $\mathrm{C} \subset \mathrm{Sch}_S$ which is geometrically admissible and a τ_P a topology which is assumed to be finer than the topology generated by coproduct decompositions. We define

$$\mathrm{H}_{\tau_P}(\mathrm{C}) := \mathrm{Shv}_{\tau_P}(\mathrm{C}) \cap \mathrm{P}_{\mathbf{A}^1}(\mathrm{C}) \subset \mathrm{P}_{\Sigma}(\mathrm{C}) \tag{B.8}$$

where $\mathrm{P}_{\mathbf{A}^1}$ denote $\mathbf{A}^1$-invariant sheaves. We denote the localization functor endofunctor as $\mathrm{L}_{\mathrm{mot}} : \mathrm{P}_{\Sigma}(\mathrm{C}) \to \mathrm{P}_{\Sigma}(\mathrm{C})$ and the context will always be clear. As prevalent throughout the main text we've also used $\mathrm{L}_{\mathbf{A}^1} : \mathrm{P}_{\Sigma}(\mathrm{C}) \to \mathrm{P}_{\Sigma}(\mathrm{C})$ as the $\mathbf{A}^1$-localization endofunctor. Morel and Voevodsky's motivic homotopy category was originally constructed where $\mathrm{C} = \mathrm{Sm}_S$ and $\tau_P = \mathrm{Nis}$. We drop just write $\mathrm{H}(S)$ in this case as we did throughout the text.

Remark B.0.11. We make the following simple observation: as soon as τ_p is at least as fine as the Zariski topology if we let M to be the class of vector bundles, then $\mathrm{P}_{\mathrm{M}}(\mathrm{C}) \cap \mathrm{Shv}_{\tau_P}(\mathrm{C}) \simeq \mathrm{H}(C)$.

Remark B.0.12. Suppose that C is a geometrically admissible category with quotients. Suppose that G is a finite group and let C^{BG} denote the category of G-objects in C (simply $\mathrm{Fun}(\mathrm{BG}, \mathrm{C})$ and hence the notation), then Voevodsky in [**22**] has defined in the equivariant Nisnevich topology then we have a functor $(-)/G : \mathrm{C}^{\mathrm{BG}} \to \mathrm{C}$, taking a scheme to its quotients. In our language, Voevodsky proves that this functor defines a functor, via sifted-colimit preserving extensions $(-)/G : \mathrm{P}_{\Sigma}(\mathrm{C}^{\mathrm{BG}}) \to \mathrm{P}_{\Sigma}(\mathrm{C})$ which descends to a functor $\mathrm{Shv}_{\mathrm{Nis}}(\mathrm{C}^{\mathrm{BG}}) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathrm{C})$. This functor is important in defining the symmetric powers of a motivic space in [**76**]. This is one value for considering $\mathrm{P}_{\Sigma}(\mathrm{C})$ and working with sifted colimits — we have more control over the kinds of colimits we adjoin in the first place.

B.0.1.8. The key property that one uses all the time when there is a cd-structure around is the following

Proposition B.0.13. *Suppose that* C *is a geometrically admissible subcategory of* Sch_S *and suppose that* τ_P *is an excisive* cd*-structure then the functor* $\mathrm{L}_{\mathrm{mot}} : \mathrm{P}_{\tau_P}(\mathrm{C}) \to \mathrm{P}_{\tau_P}(\mathrm{C})$ *preserves compact objects.*

Proof. We claim that the inclusion $\mathrm{H}_{\tau}(\mathrm{C}) \subset \mathrm{P}_{\Sigma}(\mathrm{C})$ preserves filtered colimits. This is obvious for $\mathbf{A}^1$-invariant presheaves. Now since colimits commute with finite limits in any ∞-topoi, the claim follows from the fact being a sheaf can be checked using the equivalent condition of Theorem B.0.10. $\square$

B.0.1.9. Stabilization, in the sense of [**49**, Section 1.4] is easy enough to define using, for example, the technology of spectrum objects as in [**49**, Section 1.4.2]. Alternatively, since $\mathrm{P}_{\Sigma}(\mathrm{C})$ is a presentable ∞-category the stabilization of $\mathrm{P}_{\Sigma}(\mathrm{C})$ can be calculated as the colimit

$$\mathrm{P}_{\Sigma}(\mathrm{C})_{*} \xrightarrow{\Sigma} \mathrm{P}_{\Sigma}(\mathrm{C})_{*} \xrightarrow{\Sigma} \cdots . \tag{B.9}$$

as explained in [**49**, Proposition 1.4.4.4]. The universal property of $\mathrm{Stab}(\mathrm{P}_{\Sigma}(\mathrm{C}))$ is then given in the following way: there is a functor $\Sigma^{\infty}_{S^1,+} : \mathrm{P}_{\Sigma}(\mathrm{C}) \to \mathrm{Stab}(\mathrm{P}_{\Sigma}(\mathrm{C}))$ such that given any other presentable stable ∞-category D the induced map

$$\mathrm{Maps}_{\mathrm{Pr}^L}(\mathrm{Stab}(\mathrm{P}_{\Sigma}(\mathrm{C})), \mathrm{D}) \xrightarrow{(\Sigma^{\infty}_{S^1,+})^{*}, \simeq} \mathrm{Maps}_{\mathrm{Pr}^L}(\mathrm{C}, \mathrm{D}) \tag{B.10}$$

is an equivalence. According to [**35**, Theorem 5.1], the the functor $\Sigma^{\infty}_{S^1,+} : \mathrm{P}_{\Sigma}(\mathrm{C}) \to \mathrm{Stab}(\mathrm{P}_{\Sigma}(\mathrm{C}))$ is symmetric monoidal and the universal property discussed above can be enhanced to one that takes into account symmetric monoidal structures by [**35**, Proposition 5.4]. We also note that $\mathrm{Stab}(\mathrm{P}_{\Sigma}(\mathrm{C}))$ can be concretely modeled as a presentably symmetric monoidal ∞-category by $\mathrm{Fun}^{\times}(\mathrm{C}^{\mathrm{op}}, \mathrm{Spt}) =: \mathrm{P}_{\Sigma,\mathrm{Spt}}(\mathrm{C})$ or the full subcategory presheaves of spectra on C which takes coproducts to products (equivalently coproducts as Spt is stable). As a sum total of this discussion, we have the stabilization of $\mathrm{H}_{\tau}(\mathrm{C}), \mathrm{Stab}(\mathrm{H}(C))$ is the full symmetric monoidal subcategory of $\mathrm{P}_{\Sigma,\mathrm{Spt}}(\mathrm{C})$ which are τ-sheaves of spectra and are $\mathbf{A}^1$-invariant.

B.0.2. Digression: the notion of $\mathbb{T}$-prestability

We introduce a notion which captures many phenomena in motivic homotopy theory — this notion will be elaborated further in [**27**] where we will prove that this notion generalizes the "effective motives" used to define the slice filtration, as introduced by Voevodsky in [**73**]. To motivate this notion, let us recall the notion of prestability introduced by Lurie in [**50**, Appendix C].

Definition B.0.14. Suppose that C is an ∞-category which is pointed and has finite colimits. Then the *Spanier-Whitehead ∞-category* of C, which we denote by $\mathrm{SW}(\mathrm{C})$, is the colimit

$$\mathrm{C} \xrightarrow{\Sigma} \mathrm{C} \xrightarrow{\Sigma} \mathrm{C} \cdots \tag{B.11}$$

taken in Cat_∞.

B.0.2.1. The following properties about the colimits of the form (B.11) will be repeatedly used in the sequel

Proposition B.0.15. *Suppose that $\mathrm{C} : I \to \mathrm{Cat}_\infty, i \mapsto \mathrm{C}_i$ is a diagram such that*

- *Each C_i is a pointed, small ∞-category.*
- *Each C_i has finite colimits, is idempotent complete and the transition functors are right exact.*

Then:

(1) *The colimit $\mathrm{C} := \operatorname{colim} C_i$ exists and the colimit can be calculated in $\mathrm{Cat}_\infty^{\mathrm{idem,rex}}$, the ∞-category of idempotent complete ∞-categories with finite colimits and right exact functors so that, in particular, C is idempotent complete and has finite colimits.*

(2) *Suppose that κ is a regular cardinal, then we have an equivalence*

$$\operatorname*{colim}_I \mathrm{Ind}_\kappa(\mathrm{C}_i) \simeq \mathrm{Ind}_\kappa(\mathrm{C}). \tag{B.12}$$

(3) *We also have an equivalence*

$$\mathrm{Ind}_\kappa(\mathrm{C})^\kappa \simeq \mathrm{C}. \tag{B.13}$$

Proof. The first statement follows from [**49**, Proposition 7.3.5.10]. The next two statements are immmediate consequences of [**48**, Proposition 5.5.7.10]. $\square$

B.0.2.2. A *prestable* ∞*-category* C can then be characterized in the following way

Definition B.0.16. A pointed ∞-category C with finite colimits is *prestable* if

(1) The suspension functor C is fully faithful.
(2) The essential image of the canonical functor $\mathrm{C} \to \mathrm{SW}(\mathrm{C})$ is closed under extensions.

Example B.0.17. Let $\mathrm{C} = \mathrm{Spt}_{\geq 0}$ the ∞-category of finite spectra. Then C satisfies condition (1) of Definition B.0.16. Now $\mathrm{SW}(\mathrm{C}) \simeq \mathrm{Spt}$ and the essential image of the canonical functor $\mathrm{Spt}_{\geq 0} \hookrightarrow \mathrm{Spt}$ is closed under extensions. Indeed, this is the prototypical example of a prestable ∞-category which is also closed under finite limits by [**50**, Proposition C.1.2.9] and so we may think of a prestable ∞-category (at least those with finite limits) as the nonnegative part of a t-structure on a stable ∞-category.

B.0.2.3. We will now give motivic analogs of the above definitions; the point is that we want to axiomatize what it means to the "effective part" of a "motivic" category. To begin, let S be a scheme, then the ∞-category $\mathrm{H}(S)_*^\omega$ is the full subcategory of $\mathrm{H}(S)_*$ spanned by the compact objects of $\mathrm{H}(S)_*$. Here is a more concrete description.

Proposition B.0.18. *Suppose that S is quasiseparated. The ∞-category $\mathrm{H}(S)_*^\omega$ is the idempotent completion of the ∞-category generated under finite colimits by $\mathrm{L}_{\mathrm{mot}}(X_+)$ where X is a smooth S-scheme which is affine.*

Proof. The ∞-category $\mathrm{H}(S)_*$ is generated under sifted colimits by $\mathrm{L}_{\mathrm{mot}}(X_+)$ where X is smooth and affine: from the fact that $\mathrm{P}_\Sigma(\mathrm{Sm}_{S+}) \simeq \mathrm{P}_\Sigma(\mathrm{Sm}_S)_*$, we get that it is generated by $\mathrm{L}_{\mathrm{mot}}(X_+)$ where X is a smooth S-scheme and using Zariski descent we may resolve any such X using a simplicial resolution by smooth affine schemes under the quasiseparatedness hypothesis. Since reflection onto Nisnevich sheaves preserves filtered colimits, the object $\mathrm{L}_{\mathrm{mot}}(X_+)$ is compact and hence the result follows.

$\square$

B.0.2.4. We also claim that $\mathrm{H}(S)_*^\omega$ is a symmetric monoidal ∞-category.

Proposition B.0.19. *Let $X, Y \in \mathrm{H}(S)_*^\omega$ then $X \otimes Y \in \mathrm{H}(S)_*^\omega$ so that $\mathrm{H}(S)_*^\omega$ is a symmetric monoidal ∞-category and the inclusion $\mathrm{H}(S)_*^\omega \subset \mathrm{H}(S)_*$ is symmetric monoidal.*

Proof. Suppose that $\{Z_i\}$ is a filtered diagram in $\mathrm{H}(S)_*$. The claim follows by:

$$\begin{aligned}
\mathrm{Maps}(X \otimes Y, \operatorname{colim} Z_i) &\simeq \mathrm{Maps}(X, \underline{\mathrm{Maps}}(Y, \operatorname{colim} Z_i)) \\
&\simeq \mathrm{Maps}(X, \operatorname{colim} \underline{\mathrm{Maps}}(Y, Z_i)) \\
&\simeq \lim \mathrm{Maps}(X, \underline{\mathrm{Maps}}(Y, Z_i)) \\
&\simeq \lim \mathrm{Maps}(X \otimes Y, Z_i).
\end{aligned}$$

□

The ∞-category $\mathrm{H}(S)_*^\omega$ is then a symmetric monoidal ∞-category which is idempotent complete [**48**, Proposition 5.3.4.16] and has finite colimits. We let $\mathrm{Cat}_\infty^{\mathrm{idem,rex}}$ be the ∞-category whose objects are idempotent complete ∞-categories with small colimits and whose functors are finite-colimit preserving functors. This ∞-category inherits a symmetric monoidal structure [**49**, 4.8.1] so we may speak of algebras and modules in this ∞-category. We note that $\mathrm{H}(S)_*^\omega$ is a $\mathcal{E}_\infty$-algebra object and write $\mathrm{Mod}_{\mathrm{H}(S)_*^\omega}$ as the ∞-category of modules over this $\mathcal{E}_\infty$-algebras in $\mathrm{Cat}_\infty^{\mathrm{idem,rex}}$

B.0.2.5. Definitions B.0.16 and B.0.14 then motivates the following definition in motivic homotopy theory.

Definition B.0.20. Let $\mathrm{C} \in \mathrm{Mod}_{\mathrm{H}(S)_*^\omega}$. Then the *Spanier-Whitehead motivic ∞-category* of C is the colimit

$$\mathrm{C} \xrightarrow{\mathbb{T}\otimes -} \mathrm{C} \xrightarrow{\mathbb{T}\otimes -} \mathrm{C} \cdots \tag{B.14}$$

taken in $\mathrm{Mod}_{\mathrm{H}(S)_*^\omega}$. We denote this ∞-category by $\mathrm{SW}_{\mathrm{mot}}(\mathrm{C})$.

Proposition B.0.21. *There is an equivalence of* $\mathrm{H}(S)_*$*-modules*

$$\mathrm{C}[\mathbb{T}^{-1}] \xrightarrow{\simeq} \mathrm{SW}_{\mathrm{mot}}(\mathrm{C}). \tag{B.15}$$

In particular, the ∞*-category* $\mathrm{SW}_{\mathrm{mot}}(\mathrm{C})$ *is a stable* ∞*-category and satisfies the following universal property: suppose that* D *is an* $\mathrm{H}(S)_*$*-module such that the action of* $\mathbb{T}$ *is invertible, then the canonical functor* $\mathrm{C} \to \mathrm{SW}_{\mathrm{mot}}(\mathrm{C})$ *defines a fully faithful embedding*

$$\mathrm{Maps}_{\mathrm{Mod}_{\mathrm{H}(S)_*}}(\mathrm{SW}_{\mathrm{mot}}(\mathrm{C}), \mathrm{D}) \hookrightarrow \mathrm{Maps}_{\mathrm{H}(S)_*}(\mathrm{C}, \mathrm{D}) \tag{B.16}$$

where the essential image is spanned by $\mathrm{H}(S)_*$*-linear functors* F *such that the action of* $F(\mathbb{T})$ *is invertible in* D.

Proof. According to Robalo [**60**, Proposition 2.19], since $\mathbb{T}$ is 3-symmetric, the $\mathbb{T}$-inversion of C is calculated as the colimit of the diagram (B.14); this verifies the desired universal property. Using the equivalence $S^1 \wedge (\mathbf{G}_m, 1) \simeq \mathbb{T}$ in $\mathrm{H}(S)_*^{\omega}$ we deduce the stability of C. □

Example B.0.22. Using Proposition B.0.15, we obtain that $\mathrm{SW}_{\mathrm{mot}}(\mathrm{Ind}(\mathrm{C}))^{\omega} \simeq (\mathrm{Ind}(\mathrm{C})[\mathbb{T}^{-1}])^{\omega}$. In particular if $\mathrm{C} = \mathrm{H}(S)_*^{\omega}$. Suppose that $\mathrm{C} = \mathrm{H}(S)_*^{\omega}$ itself, then we get that $\mathrm{SW}_{\mathrm{mot}}(\mathrm{H}(S)_*^{\omega})$ is just, $\mathrm{SH}(S)^{\omega}$, the ∞-category of compact objects in $\mathrm{SH}(S)$.

B.0.2.6. We now define

Definition B.0.23. An $\mathrm{H}(S)_*$-module C is *motivically prestable* or $\mathbb{T}$-*prestable* if

(1) The endofunctor $\mathbb{T} \otimes - : \mathrm{C} \hookrightarrow \mathrm{C}$ is fully faithful.
(2) The essential image of the canonical functor $\mathrm{C} \to \mathrm{SW}_{\mathrm{mot}}(\mathrm{C})$ is closed under extensions.

A motivically prestable $\mathrm{H}(S)_*$-module C is furthermore *motivically stable* or $\mathbb{T}$-*stable* one where the the endofunctor in (1) acts invertibly.

B.0.2.7. We now present some examples

Example B.0.24. The first example of motivically prestable ∞-categories is the derived ∞-category of étale sheaves on a scheme S with coefficients in R-modules where the residue characteristics of S are invertible in R (e.g. ℓ-adic sheaves where ℓ is prime to the residue characteristics). One way to obtain the $\mathrm{H}(S)_*^\omega$-module structure uses the general version of Suslin rigidity already used in §5.2, to obtain a composite of symmetric monoidal functors

$$\mathrm{H}(S)_*^\omega \hookrightarrow \mathrm{H}(S)_* \xrightarrow{\Sigma^\infty} \mathrm{SH}(S) \xrightarrow{\mathrm{L}_{\text{ét}}(-)^{\mathrm{tr}}} \mathrm{DM}_{\text{ét},\mathrm{tr}}(S,R) \simeq \mathrm{D}(S_{\text{ét}},R) \tag{B.17}$$

witnessing the latter as an $\mathrm{H}(S)_*^\omega$-algebra. To see that the action of $\mathbb{T}$ is fully faithful, we note that for any sheaf $\mathscr{F} \in \mathrm{D}(S_{\text{ét}},R)$, the object $\mathbb{T} \otimes \mathscr{F}$ is computed as the Tate twist shifted by 2, $\mathbb{T} \otimes \mathscr{F} = \mathscr{F}(1)[2]$, which is an invertible operation in the étale topology.

Example B.0.25. Suppose that k is a perfect field and R is a commutative ring of coefficients, then the ∞-category of effective Voevodsky motives, $\mathrm{DM}^{\mathrm{eff}}(k,R)$, is $\mathbb{T}$-prestable since the endofunctor given by tensoring with $R_{\mathrm{tr}}(\mathbb{T})$ is fully faithful by [**74**, Corollary 4.10].

Example B.0.26. Suppose k is a perfect field and that $c(k) \neq 2$. Then the ∞-category of effective Milnor-Witt motives, $\widetilde{\mathrm{DM}^{\mathrm{eff}}}(k,R)$, is $\mathbb{T}$-prestable since the endofunctor given by tensoring with $\widetilde{R_{\mathrm{tr}}}(\mathbb{T})$ is fully faithful by the main theorem of [**28**].

Example B.0.27. Suppose k is a perfect field and that $c(k) \neq 2$. Then the ∞-category of framed motivic spectra $\mathrm{SH}(k)^{\mathrm{fr}}$ as in [**26**] is $\mathbb{T}$-prestable by [**26**, Theorem 3.5.8]. In fact $\mathrm{SH}(k)^{\mathrm{fr}}$ recovers $\mathrm{SH}(k)^{\mathrm{eff}}$ the ∞-category of effective motivic spectra, which is obviously $\mathbb{T}$-prestable.

Example B.0.28. We also remark that the compact objects in all the examples above are also $\mathbb{T}$-prestable ∞-categories.

B.0.2.8. In [**27**] we will discuss the following generalization of the slice filtration:

Definition B.0.29. Let (C, c) be a pair where C is an $\mathrm{H}(S)^{\omega}_{*}$-module and $c \in \mathrm{C}$. We define the ∞-category of *c-effective objects of* C, denoted by $\mathrm{C}^{\mathrm{eff},c}$, to be the localizing ∞-category in C generated by $X_{+} \otimes c$ where $X \in \mathrm{Sm}_S$.

Out of this notion we can construct slice filtrations on C.

Construction B.0.30. Define $\Sigma^n_{\mathbb{T}}\mathrm{C}^{\mathrm{eff},c}$ to be the localizing subcategory generated by $\mathbb{T}^{\otimes n} \otimes X_{+} \otimes c$. Then the categories $\Sigma^q_{\mathbb{T}}\mathrm{C}^{\mathrm{eff},c}$ assemble into a filtration:

$$\cdots \subset \Sigma^q_T \mathrm{C}^{\mathrm{eff},c} \subset \Sigma^{q-1}_T \mathrm{C}^{\mathrm{eff},c} \subset \cdots \subset \mathrm{C}^{\mathrm{eff},c} \subset \Sigma^{-1}_T \mathrm{C}^{\mathrm{eff},c} \subset \cdots \subset \mathrm{C}. \tag{B.18}$$

The fully faithful embedding $i_q^{(\mathrm{C},c)} : \Sigma^q_T \mathrm{C}^{\mathrm{eff},c} \hookrightarrow \mathrm{C}$ admits a right adjoint $i_q^{(\mathrm{C},c)} : \mathrm{C} \to \Sigma^q_T \mathrm{C}^{\mathrm{eff},c}$

Setting $f_q^{(\mathrm{C},c)} := i_q^{(\mathrm{C},c)} i_q^{(\mathrm{C},c)}$ we obtain for every $\mathrm{M} \in \mathrm{C}$ a (C, c)-*slice tower*:

$$\cdots \to f_{q+1}^{(\mathrm{C},c)}\mathrm{M} \to f_q^{(\mathrm{C},c)}\mathrm{M} \to \cdots \to f_0^{(\mathrm{C},c)}\mathrm{M} \to f_{-1}^{(\mathrm{C},c)}\mathrm{M} \to \cdots \to \mathrm{M}. \tag{B.19}$$

We refer to $f_q^{(\mathrm{C},c)}\mathrm{M}$ as the *q-th* (C, c)-*effective cover of* M, and the cofiber $s_q^{(\mathrm{C},c)}\mathrm{M}$ of $f_{q+1}^{(\mathrm{C},c)}\mathrm{M} \to f_q^{(\mathrm{C},c)}\mathrm{M}$ as the *q-th* (C, c)-*slice of* M.

We will prove the following classifcation theorem for $\mathbb{T}$-prestable ∞-categories in [**27**]

Theorem B.0.31. *Let* C *be an* $\mathrm{H}(S)^{\omega}_{*}$*-module then the following are equivalent:*

(1) C *is a* $\mathbb{T}$*-prestable* ∞*-category with finite limits*
(2) *There exists a* $\mathbb{T}$*-stable* ∞*-category* C' *an object* $c \in \mathrm{C}'$ *and a fully faithful embedding* $i : \mathrm{C} \hookrightarrow \mathrm{C}'$ *such that the essential image of* i *is the subcategory of* c*-effective objects of* C.

Indeed there is a canonical choice of C' by simply forming the Spanier-Whitehead motivic ∞-category of C by taking the colimit of (B.0.14) which is $\mathbb{T}$-stable by Proposition B.0.21.

B.0.2.9. From this perspective, the ∞-category of motivic spectra, as a symmetric monoidal ∞-category, is constructed by first considering $\mathrm{H}(S)^\omega$ and taking its Spanier-Whitehead motivic ∞-category which identifies with $\mathrm{SH}(S)^\omega$ and then taking Ind. The important properties of $\mathrm{SH}(S)$ and its universal property is then summarized in the following way

Proposition B.0.32. *Let S be a base scheme and suppose that D is an $\mathrm{H}(S)^\omega_*$-module which is $\mathbb{T}$-stable, then the functor $\Sigma^\infty_{\mathbb{T},+} : \mathrm{H}(S)^\omega \to \mathrm{SH}(S)^\omega$ induces an equivalence*

$$\mathrm{Fun}^{\mathrm{rex}}_{\mathrm{Mod}_{\mathrm{H}(S)^\omega_*}}(\mathrm{SH}(S)^\omega, \mathrm{D}) \to \mathrm{Fun}^{\mathrm{rex}}_{\mathrm{Mod}_{\mathrm{H}(S)^\omega_*}}(\mathrm{SH}(S)^\omega, \mathrm{D}). \tag{B.20}$$

Suppose that D is furthermore an $\mathrm{H}(S)^\omega_*$-algebra, then the equivalence in (B.20) *can be promoted to an equivalence of functors which are symmetric monoidal, i.e., maps of $\mathrm{H}(S)^\omega_*$-algebra. The presentably symmetric monoidal ∞-category of motivic spectra is then obtained by taking* Ind *of* $\mathrm{SH}(S)^\omega$.