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ABSTRACT

Essays in Economic Theory

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The main theme of this dissertation are departures from standard assumptions in economic theory, specifically, departures from the model of subjective expected utility in decison theory.

Part 1 axiomatizes the robust control criterion of multiplier preferences introduced by Hansen and Sargent (2001). The axiomatization shows that the class of multiplier preferences is precisely the intersection of the class of variational preferences, of Maccheroni, Marinacci and Rustichini (2006), and the class of second order expected utility preferences, of Ergin and Gul (2004) and Neilson (1993).

The main contributions of Part 2 are an axiomatization of dynamic multiplier preferences and a characterization of preference for earlier resolution of uncertainty in the class of variational preferences. The latter result says that in the class of variational preferences the only preferences that satisfy indifference to timing are maxmin expected utility preferences of Gilboa and Schmeilder (1989). These questions are studied in a recursive setting, where time is infinite.

Part 3 studies the same questions in a different setting—that of Maccheroni, Marinacci, and Rustichini (2006b) and Epstein and Schneider (2003b). In this setting time is finite, but there is more flexibility to model the state space, which does not have to be an infinite product of identical sets, as is the case in the setting of Part 2.

The main result of Part 4 shows that probabilistic sophistication implies expected utility under an assumption that there exists a nontrivial unambiguous event. This means that although variational preferences are an excellent tool for studying behavior exemplified by the Ellsberg paradox, their ability to account for the Allais paradox is limited.

Part 5 studies a definition of subjective beliefs for general ambiguity averse preferences. This definition leads to a characterization of the efficiency of ex ante trade: when aggregate uncertainty is absent, full insurance is efficient if and only if agents share some common subjective beliefs. This part of my dissertation was written jointly with Luca Rigotti and Chris Shannon and is forthcoming in *Econometrica*.

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CHAPTER 1

Introduction

The concept of uncertainty has been studied by economists since the work of Keynes (1921) and Knight (1921). As opposed to risk, where probability is well specified, uncertainty, or ambiguity, is characterized by the decision maker's inability to formulate a single probability or by his lack of trust in any unique probability.

Indeed, as demonstrated by Ellsberg (1961), people often make choices that cannot be justified by a unique probability, thereby exhibiting a preference for risky choices over those involving ambiguity. Such *ambiguity aversion* has been one of the central issues in decision theory, motivating the development of axiomatic models of such behavior.[1]

The lack of trust in a single probability has also been a source of concern in macroeconomics. In order to capture concern about model misspecification, Hansen and Sargent (2001) formulated an important model of *multiplier preferences*. Thanks to their great tractability, multiplier preferences are now being adopted in applications.[2]

Despite their importance in macroeconomics, multiplier preferences have not been fully understood at the level of individual decision making. Although Maccheroni et al. (2006a) showed that they are a special case of the variational preferences that they axiomatized, an axiomatization of multiplier preferences has so far been elusive. Indeed, some authors even

[1]See Gilboa and Schmeidler (1989); Schmeidler (1989); Ergin and Gul (2004); Klibanoff, Marinacci, and Mukerji (2005); Maccheroni, Marinacci, and Rustichini (2006a) among others.
[2]See, e.g. Woodford (2006); Barillas, Hansen, and Sargent (2007); Karantounias, Hansen, and Sargent (2007); Kleshchelski and Vincent (2007).

doubted the existence of behaviorally meaningful axioms that would pin down multiplier preferences within the broad class of variational preferences.

The main contribution of this paper is precisely a set of axioms satisfying this property. The proposed axiomatic characterization is important for three reasons. First, it provides a set of testable predictions of the model that allow for its empirical verification. This will help evaluate whether multiplier preferences, which are useful in modeling behavior at the macro level, are an accurate model of individual behavior. Second, the axiomatization establishes a link between the parameters of the multiplier criterion and the observable behavior of the agent. This link enables measurement of the parameters on the basis of observable choice data alone, without relying on unverifiable assumptions. Finally, the axiomatization is helpful in understanding the relation between multiplier preferences and other axiomatic models of preferences and ways in which they can and cannot be used for modeling Ellsberg-type behavior.

1.1. Background and Overview of Results

The Expected Utility criterion ranks payoff profiles f according to

$$V(f) = \int u(f)\,\mathrm{d}q, \tag{1.1}$$

where u is a utility function and q is a subjective probability distribution on the states of the world. A decision maker with such preferences is considered ambiguity neutral, because he is able to formulate a single probability that governs his choices.

In order to capture lack of trust in a single probability, Hansen and Sargent (2001) formulated the following criterion

$$V(f) = \min_p \int u(f)\,\mathrm{d}p + \theta R(p\|\,q), \tag{1.2}$$

where $\theta \in (0, \infty]$ is a parameter and function $R(p\,\|\,q)$ is the *relative entropy* of p with respect to q. Relative entropy, otherwise known as Kullback-Leibler divergence, is a measure of “distance” between two probability distributions. An interpretation of (1.2) is that the decision maker has some best guess q of the true probability distribution, but does not fully trust it. Instead, he considers other probabilities p to be plausible, with plausibility diminishing proportionally to their “distance” from q. The role of the proportionality parameter θ is to measure the degree of trust of the decision maker in the reference probability q. Higher values of θ correspond to more trust; in the limit, when $\theta = \infty$, the decision maker fully trusts his reference probability and uses the expected utility criterion (1.1).

Multiplier preferences also belong to the more general class of variational preferences studied by Maccheroni et al. (2006a); those preferences have the following representation:

$$V(f) = \min_p \int u(f)\,\mathrm{d}p + c(p), \tag{1.3}$$

where $c(p)$ is a “cost function”. The interpretation of (1.3) is like that of (1.2), and multiplier preferences are a special case of variational preferences with $c(p) = \theta R(p\,\|\,q)$. In general, the conditions that the function $c(p)$ in (1.3) has to satisfy are very weak, which makes variational preferences a very broad class. In addition to expected utility preferences and multiplier preferences, this class also nests the maxmin expected utility preferences of Gilboa and Schmeidler (1989), as well as the mean-variance preferences of Markowitz (1952) and Tobin (1958).

An important contribution of Maccheroni et al. (2006a) was to provide an axiomatic characterization of variational preferences. However, because variational preferences are a very broad class of preferences, it would be desirable to establish an observable distinction between multiplier preferences and other subclasses of variational preferences. Ideally, an axiom, or set of axioms, would exist that, when added to the list of axioms of Maccheroni et al. (2006a), would deliver multiplier preferences. This is, for example, the case with the maxmin expected utility preferences of Gilboa and Schmeidler (1989): a strengthening of one of the Maccheroni et al.'s (2006a) axioms restricts the general cost function $c(p)$ to be in the class used in Gilboa and Schmeidler's (1989) model. The reason for skepticism about the existence of an analogous strengthening in the case of multiplier preferences has been that the relative entropy $R(p\|q)$ is a very specific functional-form assumption, which does not seem to have any behaviorally significant consequences. The main finding of this paper is that these consequences *are* behaviorally significant. The main theorem shows that standard axioms characterize the class of multiplier preferences within the class of variational preferences. This is possible because, as the main theorem shows, the class of multiplier preferences is precisely the intersection of the class of variational preferences of Maccheroni et al. (2006a), and the class of second order expected utility preferences of Ergin and Gul (2004) and Neilson (1993). Figure 1.1 depicts the relationships between those classes.

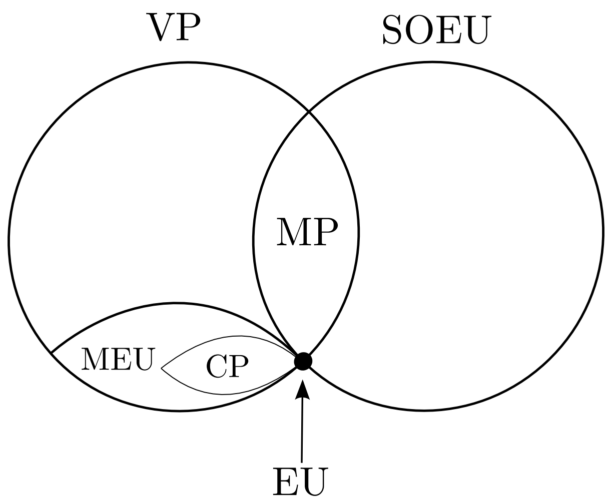

Figure 1.1. Relations between classes of preferences: VP—variational preferences, MP—multiplier preferences, SOEU—second order expected utility preferences, EU—expected utility preferences, MEU—maxmin expected utility preferences, CP—constraint preferences.

1.2. Ellsberg's Paradox and Measurement of Parameters

Ellsberg's (1961) experiment demonstrates that most people prefer choices involving risk (i.e., situations in which the probability is well specified) to choices involving ambiguity (where the probability is not specified). Consider two urns containing colored balls. The decision maker can bet on the color of the ball drawn from each urn. Urn I contains 100 red and black balls in unknown proportion, while Urn II contains 50 red and 50 black balls.

In this situation, most people are indifferent between betting on red from Urn I and on black from Urn I. This reveals that, in the absence of evidence against symmetry, they view those two contingencies as interchangeable. Moreover, most people are indifferent between betting on red from Urn II and on black from Urn II. This preference is justified by their knowledge of the composition of Urn II. However, most people strictly prefer betting on red from Urn II to betting on red from Urn I, thereby displaying ambiguity aversion.

Ambiguity aversion cannot be reconciled with a single probability governing the distribution of draws from Urn I. For this reason, expected utility preferences are incapable of explaining the pattern of choices revealed by Ellsberg's experiment. Such pattern can, however, be explained by multiplier preferences. Recall that

$$V(f) = \min_{p} \int u(f)\,\mathrm{d}p + \theta R(p\|q). \tag{1.2}$$

The curvature of the utility function u measures the decision maker's risk aversion and governs his choices when probabilities are well specified—for example, choices between bets on red and black from Urn II. In contrast, the parameter θ measures the decision maker's attitude towards ambiguity, and influences his choices when probabilities are not well specified—for example, choices between bets on red and black from Urn I.

Formally, betting \$100 on red from Urn II corresponds to an objective lottery r_{II} paying \$100 with probability $\frac{1}{2}$ and \$0 with probability $\frac{1}{2}$. Betting \$100 on black from Urn II corresponds to lottery b_{II}, which is equivalent to r_{II}. The decision maker values r_{II} and b_{II} at

$$V(r_{II}) = V(b_{II}) = \frac{1}{2}u(100) + \frac{1}{2}u(0).$$

Moreover, let x denote the certainty equivalent of r_{II} and b_{II}, i.e., the amount of money that, when received for sure, would be indifferent to r_{II} and b_{II}. Formally

$$V(x) = u(x) = V(r_{II}) = V(b_{II}). \tag{1.4}$$

On the other hand, betting \$100 on red from Urn I corresponds to r_I, which pays \$100 when a red ball is drawn and \$0 otherwise. Similarly, betting \$100 on black from Urn I corresponds to b_I, which pays \$100 when a black ball is drawn and \$0 otherwise. The decision maker values r_I and b_I at

$$V(r_I) = V(b_I) = \min_{p\in[0,1]} pu(100) + (1-p)u(0) + \theta R(p\| q)$$

where q is the reference measure, assumed to put equal weights on red and black. Moreover, let y be the certainty equivalent of r_I and b_I, i.e., the amount of money that, when received for sure, would be indifferent to r_I and b_I. Formally

$$V(y) = u(y) = V(r_I) = V(b_I). \tag{1.5}$$

In Ellsberg's experiments most people prefer objective risk to subjective uncertainty, implying that $y < x$. This pattern of choices is implied by multiplier preferences with $\theta < \infty$. The equality $y = x$ holds only when $\theta = \infty$, i.e., when preferences are expected utility and there is no ambiguity aversion.

Ellsberg's paradox provides a natural setting for experimental measurement of parameters of the model. The observable choice data reveals the decision maker's preferences over objective lotteries, and hence his aversion toward pure risk embodied in the utility function u. The observed value of certainty equivalent x allows to infer the curvature of u.[3] Similarly, decision maker's choices between uncertain gambles reveal his attitude

[3]For example, let $u(z) = (w+z)^{1-\gamma}$, where w is the initial level of wealth. Then (1.4) establishes a 1-1 relationship between x and γ. The value of γ can be derived from observed values of x and w.

toward subjective uncertainty, represented by parameter θ. The observed "ambiguity premium" $x - y$ enables inferences about the value of θ: a big difference $x - y$ reveals that the decision maker has low trust in his reference probability, i.e., θ is low.[4]

The procedure described above suggests that simple choice experiments could be used for empirical measurement of both u and θ. Such revealed-preference measurement of parameters would be a useful tool in applied settings, where it is important to know the numerical values of parameters, and would be complementary to the heuristic method of detection error probabilities developed by Anderson, Hansen, and Sargent (2000) and Hansen and Sargent (2007).

1.3. Outline

The paper is organized as follows. After introducing some notation and basic concepts in Section 2.1, Section 2.2 defines static multiplier preferences, discusses their properties in the classic setting of Savage, and indicates that richer choice domains are needed for axiomatization. Chapter 3 uses one of such richer domains, introduced by Anscombe–Aumann, and discusses the class of variational preferences, which nests multiplier preferences. Chapter 3 presents axioms that characterize the class of multiplier preferences within the class of variational preferences. Additionally, extending a result of Marinacci (2002), Chapter 3 discusses the extent to which variational preferences can be used for modelling the Allais paradox. Chapter 4 studies a different enrichment of choice domain and presents an axiomatization of multiplier preferences in a setting introduced by

[4]Continuing the example from footnote 3, holding γ and w fixed, (1.5) establishes a 1-1 relationship between y and θ. Thus, the value of θ can be derived from observed values of y, x, and w.

Ergin and Gul (2004), thereby obtaining a fully subjective axiomatization of multiplier preferences.

CHAPTER 2

Multiplier Preferences

2.1. Preliminaries

Decision problems considered in this paper involve a set S of *states of the world*, which represents all possible contingencies that may occur. One of the states, $s \in S$, will be realized, but the decision maker has to choose the course of action before learning s. His possible choices, called acts, are mappings from S to Z, the set of consequences. Each act is a complete description of consequences, contingent on states.

Formally, let Σ be a sigma-algebra of subsets of S. An act is a finite-valued, Σ-measurable function $f : S \to Z$; the set of all such acts is denoted $\mathcal{F}(Z)$. If $f, g \in \mathcal{F}(Z)$ and $E \in \Sigma$, then fEg denotes an act with $fEg(s) = f(s)$ if $s \in E$ and $fEg(s) = g(s)$ if $s \notin E$. The set of all finitely additive probability measures on (S, Σ) is denoted $\Delta(S)$; the set of all countably additive probability measures is denoted $\Delta^\sigma(S)$; its subset consisting of all measures absolutely continuous with respect to $q \in \Delta^\sigma(S)$ is denoted $\Delta^\sigma(q)$.

The choices of the decision maker are represented by a preference relation $\succsim$, where $f \succsim g$ means that the act f is weakly preferred to the act g. A functional $V : \mathcal{F}(Z) \to \mathbb{R}$ *represents* $\succsim$ if for all $f, g \in \mathcal{F}(Z)$ $f \succsim g$ if and only if $V(f) \geq V(g)$.

An important class of preferences are Expected Utility (EU) preferences, where the decision maker has a probability distribution $q \in \Delta(S)$ and a utility function which evaluates each consequence $u : Z \to \mathbb{R}$. A preference relation $\succsim$ has an *EU representation* (u, q) if there exists a functional $V : \mathcal{F}(Z) \to \mathbb{R}$ that represents $\succsim$ with $V(f) = \int_S (u \circ f)\, \mathrm{d}q$.

Let $Z = \mathbb{R}$, i.e., acts have monetary payoffs. Risk aversion is the phenomenon where sure payoffs are preferred to ones that are stochastic but have the same expected monetary value. Risk averse EU preferences have concave utility functions u. Likewise, one preference relation is more risk averse than another if it has a "more concave" utility function. More formally, a preference relation represented by (u_1, q_1) is *more risk averse* than one represented by (u_2, q_2) if and only if $q_1 = q_2$ and $u_1 = \phi \circ u_2$, where $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing concave transformation.

A special role will be played by the class of transformations ϕ_θ, indexed by $\theta \in (0, \infty]$

$$\phi_\theta(u) = \begin{cases} -\exp\left(-\frac{u}{\theta}\right) & \text{for } \theta < \infty, \\ u & \text{for } \theta = \infty. \end{cases} \tag{2.1}$$

Lower values of θ correspond to "more concave" transformations, i.e., more risk aversion.

2.2. Concern about model misspecification

2.3. Model Uncertainty

A decision maker with expected utility preferences formulates a probabilistic model of the world, embodied by the subjective distribution $q \in \Delta(S)$. However, in many situations, a single probability cannot explain people's choices, as illustrated by the Ellsberg paradox.

Example 2.1: (Ellsberg Paradox). Consider two urns containing colored balls; the decision maker can bet on the color of the ball drawn from each urn. Urn I contains 100 red and black balls in unknown proportion, while Urn II contains 50 red and 50 black balls.

In this situation, most people are indifferent between betting on red from Urn I and on black from Urn I. This reveals that they view those two contingencies as interchangeable. Moreover, most people are indifferent between betting on red from Urn II and on black from Urn II. This preference is justified by their knowledge of the composition of Urn II. However, most people strictly prefer betting on red from Urn II to betting on red from Urn I, thereby avoiding decisions based on imprecise information. Such a pattern of preferences cannot be reconciled with a single probability distribution, hence the paradox. ▲

In addition to this descriptive failure, a single probabilistic model of the world may also be too strong an assumption from a normative, or frequentist point of view. In many situations the decision maker may not have enough information to formulate a single probabilistic model. For example, it may be hard to statistically distinguish between similar probabilistic models, and thus hard to select one model and have full confidence in it. Hansen, Sargent, and coauthors (Hansen and Sargent, 2001; Hansen, Sargent, Turmuhambetova, and Williams, 2006) introduced a way of modelling such situations. In their model the decision maker does not know the true probabilistic model p, but has a "best guess", or *approximating model* q, also called a *reference probability*. The decision maker thinks that the true probability p is somewhere near to the approximating probability q. The notion of distance used by Hansen and Sargent is relative entropy.

Definition 2.1: Let a reference measure $q \in \Delta^\sigma(S)$ be fixed. The *relative entropy* $R(\cdot \| q)$ is a mapping from $\Delta(S)$ into $[0, \infty]$ defined by

$$R(p \| q) = \begin{cases} \int_S (\log \frac{\mathrm{d}p}{\mathrm{d}q}) \, \mathrm{d}p & \text{if } p \in \Delta^\sigma(q), \\ \infty & \text{otherwise.} \end{cases}$$

A decision maker who is concerned with model misspecification computes his expected utility according to all probabilities p, but he does not treat them equally. Probabilities closer to his "best guess" have more weight in his decision.

Definition 2.2: A relation $\succsim$ has a *multiplier representation* if it is represented by

$$V(f) = \min_{p \in \Delta(S)} \int_S (u \circ f) \, \mathrm{d}p + \theta R(p \| q),$$

where $u : Z \to \mathbb{R}$, $q \in \Delta^\sigma(S)$ is nonatomic, and $\theta \in (0, \infty]$. In this case, $\succsim$ is called a *multiplier preference*.

The multiplier representation of $\succsim$ may suggest the following interpretation. First, the decision maker chooses an act without knowing the true distribution p. Second, "Nature" chooses the probability p in order to minimize the decision maker's expected utility. Nature is not free to choose, but rather it incurs a "cost" for using each p. Probabilities p that are farther from the reference measure q have a larger potential for lowering the decision maker's expected utility, but Nature has to incur a larger cost in order to select them.

This interpretation suggests that a decision maker with such preferences is concerned with model misspecification and makes decisions that are robust to such misspecification.

He is pessimistic about the outcome of his decision which leads him to exercise caution in choosing the course of action.[1] Such cautious behavior is reminiscent of Ellsberg's paradox from Example 2.1. However, the following theorem shows that such caution is equivalent to increased risk aversion.

2.4. Link to Increased Risk Aversion

The following variational formula (see, e.g., Proposition 1.4.2 of Dupuis and Ellis, 1997) plays a critical role in the analysis and applications of multiplier preferences.

$$\min_{p \in \Delta S} \int_S (u \circ f)\, \mathrm{d}p + \theta R(p \| q) = -\theta \log \left(\int_S \exp\left(-\frac{u \circ f}{\theta} \right) \mathrm{d}q \right). \tag{2.2}$$

This formula links model uncertainty, as represented by the left hand side of formula (2.2), to increased risk aversion, as represented by the right hand side of formula (2.2). Jacobson (1973), Whittle (1981), Skiadas (2003), and Maccheroni et al. (2006b) showed that in dynamic settings this link manifests itself as an observational equivalence between dynamic multiplier preferences and a (subjective analogue of) Kreps and Porteus (1978) preferences. As a consequence, in a static Savage setting multiplier preferences become expected utility preferences.

[1]Hansen and Sargent also study a closely related class of *constraint preferences*, represented by $V(f) = \min_{\{p | R(p\|q) \leq \eta\}} \int_S (u \circ f)\, \mathrm{d}p$, which are a special case of Gilboa and Schmeidler's (1989) maxmin expected utility preferences. Due to their greater analytical tractability, multiplier, rather than constraint, preferences are used in the analysis of economic models (see, e.g., Woodford, 2006; Barillas et al., 2007; Karantounias et al., 2007; Kleshchelski and Vincent, 2007).

Observation 2.1: *The relation $\succsim$ has a multiplier representation (θ, u, q) if and only if $\succsim$ has an EU representation V with*

$$V(f) = \int_S (\phi_\theta \circ u \circ f)\,\mathrm{d}q, \tag{2.3}$$

where the transformation ϕ_θ is defined by (2.1).

Corollary 2.1: *If $\succsim$ has a multiplier representation, then it has an EU representation with utility bounded from above. Conversely, if $\succsim$ has an EU representation with utility bounded from above, then for any $\theta \in (0, \infty]$ preference $\succsim$ has a multiplier representation with that θ.*[2]

This observation suggests that multiplier preferences do not reflect model uncertainty, because the decision maker bases his decisions on a well specified probability distribution. For the same reason such preferences cannot be used for modeling Ellsberg's paradox in the Savage setting.

More importantly, given a multiplier preference $\succsim$, only the function $\phi_\theta \circ u$ is identified in absence of additional assumptions. Because of this lack of identification, there is no way of disentangling risk aversion (curvature of u) from concern about model misspecification (value of θ).

Example 2.2: (Lack of Identification). Consider a multiplier preference $\succsim_1$ with $u_1(x) = -\exp(-x)$ and $\theta_1 = \infty$. This representation suggests that the decision maker

[2] It can be verified that $\succsim$ has an EU representation with utility bounded from above if and only if $\succsim$ has an EU representation and the following axiom is satisfied: There exist $z \prec z'$ in Z and a non-null event E, such that $wEz \prec z'$ for all $w \in Z$. According to Corollary 2.1, in the Savage setting this axiom is the only behavioral consequence of multiplier preferences beyond expected utility.

$\succsim_1$ is risk averse, while not being concerned about model misspecification or ambiguity. In contrast, consider a multiplier preference $\succsim_2$ with $u_2(x) = x$ and $\theta_2 = 1$. This representation suggests that the decision maker with $\succsim_2$ is risk neutral, while being concerned about model misspecification or ambiguity.

Despite the apparent differences between $\succsim_1$ and $\succsim_2$, it is true that $\phi_{\theta_1} \circ u_1 = \phi_{\theta_2} \circ u_2$, so, by 2.1, the two preference relations are identical. Hence, the two decision makers behave in exactly the same way and there are no observable differences between them. ▲

This lack of identification means that, within this class of models, choice data alone is not sufficient to distinguish between risk aversion and ambiguity. As a consequence, any econometric estimation of a model involving such decision makers would not be possible without additional ad-hoc assumptions about parameters. Likewise, policy recommendations based on such a model would depend on a somewhat arbitrary choice of the representation. Different representations of the same preferences could lead to different welfare assessments and policy choices, but such choices would not be based on observable data.[3]

Sections 3 and 4 present two ways of enriching the domain of choice and thereby making the distinction between model uncertainty and risk aversion based on observable choice data. In both axiomatizations the main idea is to introduce a subdomain of choices where, either by construction or by revealed preference, the decision maker is not concerned about model misspecification. This subdomain serves as a point of reference and makes

[3]See, e.g., Barillas et al. (2007), who study welfare consequences of eliminating model uncertainty. The evaluation of such consequences depends on the value of parameter θ.

it possible to distinguish between concern for model misspecification (and related to it Ellsberg-type behavior) and Expected Utility maximization.

CHAPTER 3

Axiomatization with Objective Risk

This section discusses an extension of the domain of choice to the Anscombe-Aumann setting, where objective risk coexists with subjective uncertainty. In this setting a recent model of *variational preferences* (introduced and axiomatized by Maccheroni et al., 2006a) nests multiplier preferences as a special case. Despite this classification, additional axioms that, together with the axioms of Maccheroni et al. (2006a), would deliver multiplier preferences have so far been elusive. This section presents such axioms. It is also shown that in the Anscombe–Aumann setting multiplier preferences can be distinguished from expected utility on the basis of Ellsberg-type experiments.

3.1. Introducing Objective Risk

One way of introducing objective risk into the present model is to replace the set Z of consequences with (simple) probability distributions on Z, denoted $\Delta(Z)$.[1] An element of $\Delta(Z)$ is called a lottery. A lottery paying off $z \in Z$ for sure is denoted δ_z. For any two lotteries $\pi, \pi' \in \Delta(Z)$ and a number $\alpha \in (0,1)$ the lottery $\alpha\pi + (1-\alpha)\pi'$ assigns probability $\alpha\pi(z) + (1-\alpha)\pi'(z)$ to each prize $z \in Z$.

Given this specification, preferences are defined on acts in $\mathcal{F}(\Delta(Z))$. Every such act $f : S \to \Delta(Z)$ involves two sources of uncertainty: first, the payoff of f is contingent on

[1]This particular setting was introduced by Fishburn (1970); settings of this type are usually named after Anscombe and Aumann (1963), who were the first to work with them.

the state of the world, for which there is no objective probability given; second, given the state, f_s is an objective lottery.

The original axioms of Anscombe and Aumann (1963) and Fishburn (1970) impose the same attitude towards those two sources. They imply the existence of a utility function $u : Z \to \mathbb{R}$ and a subjective probability distribution $q \in \Delta(S)$ such that each act is evaluated by

$$V(f) = \int_S \left(\sum_{z \in Z} u(z) f_s(z) \right) \mathrm{d}q(s). \tag{3.1}$$

Thus, in each state of the world s the decision maker computes the expected utility of the lottery f_s and then averages those values across states. By slightly abusing notation, define $u : \Delta(Z) \to \mathbb{R}$ by $u(\pi) = \sum_{z \in Z} u(z)\pi(z)$. Using this definition, the Anscombe-Aumann Expected Utility criterion can be written as

$$V(f) = \int_S u(f_s) \, \mathrm{d}q(s).$$

3.2. Multiplier Preferences

In this environment, the multiplier preferences take the following form

$$V(f) = \min_{p \in \Delta S} \int_S u(f_s) \, \mathrm{d}p + \theta R(p \| q), \tag{3.2}$$

The decision maker with such preferences makes a distinction between objective risk and subjective uncertainty: he uses the expected utility criterion to evaluate lotteries, while using the multiplier criterion to evaluate acts.

3.3. Variational Preferences

To capture ambiguity aversion, Maccheroni et al. (2006a) introduce a class of variational preferences, with representation

$$V(f) = \min_{p \in \Delta S} \int_S u(f_s)\,\mathrm{d}p + c(p), \tag{3.3}$$

where $c : \Delta S \to [0, \infty]$ is a *cost function.*

Multiplier preferences are a special case of variational preferences where $c(p) = \theta R(p \| q)$. The variational criterion (3.3) can be given the same interpretation as the multiplier criterion (3.2): Nature wants to reduce the decision maker's expected utility by choosing a probability distribution p, but she is not entirely free to choose. Using different p's leads to different values of the decision maker's expected utility $\int_S u(f_s)\,\mathrm{d}p$, but comes at a cost $c(p)$.

In order to characterize variational preferences behaviorally, Maccheroni et al. (2006a) use the following axioms.

Axiom A1—Weak Order: The relation $\succsim$ is transitive and complete.

Axiom A2—Weak Certainty Independence: For all $f, g \in \mathcal{F}(\Delta(Z))$, $\pi, \pi' \in \Delta(Z)$, and $\alpha \in (0, 1)$,

$$\alpha f + (1-\alpha)\pi \succsim \alpha g + (1-\alpha)\pi \Rightarrow \alpha f + (1-\alpha)\pi' \succsim \alpha g + (1-\alpha)\pi'.$$

Axiom A3—Continuity: For any $f, g, h \in \mathcal{F}(\Delta(Z))$ the sets $\{\alpha \in [0,1] \mid \alpha f + (1-\alpha)g \succsim h\}$ and $\{\alpha \in [0,1] \mid h \succsim \alpha f + (1-\alpha)g\}$ are closed.

Axiom A4—Monotonicity: If $f, g \in \mathcal{F}(\Delta(Z))$ and $f(s) \succsim g(s)$ for all $s \in S$, then $f \succsim g$.

Axiom A5—Uncertainty Aversion: If $f, g \in \mathcal{F}(\Delta(Z))$ and $\alpha \in (0,1)$, then

$$f \sim g \Rightarrow \alpha f + (1-\alpha)g \succsim f.$$

Axiom A6—Nondegeneracy: $f \succ g$ for some $f, g \in \mathcal{F}(\Delta(Z))$.

Axiom A7—Unboundedness: There exist $\pi' \succ \pi$ in $\Delta(Z)$ such that, for all $\alpha \in (0,1)$, there exists $\rho \in \Delta(Z)$ that satisfies either $\pi \succ \alpha\rho + (1-\alpha)\pi'$ or $\alpha\rho + (1-\alpha)\pi \succ \pi'$.

Axiom A8—Weak Monotone Continuity: If $f, g \in \mathcal{F}(\Delta(Z))$, $\pi \in \Delta(Z)$, $\{E_n\}_{n \geq 1} \in \Sigma$ with $E_1 \supseteq E_2 \supseteq \cdots$ and $\bigcap_{n \geq 1} E_n = \emptyset$, then $f \succ g$ implies that there exists $n_0 \geq 1$ such that $\pi E_{n_0} f \succ g$.

Maccheroni et al. (2006a) show that preference $\succsim$ satisfies Axioms A1-A6 if and only if $\succsim$ is represented by (3.3) with a non-constant $u : \Delta(Z) \to \mathbb{R}$ and $c : \Delta S \to [0, \infty]$ that is convex, lower semicontinuous, and grounded (achieves value zero). Moreover, Axiom A7 implies unboundedness of the utility function u, which guarantees uniqueness of the cost function c, while Axiom A8 guarantees that function c is concentrated only on countably additive measures.

The conditions that the cost function c satisfies are very general. For example, if $c(p) = \infty$ for all measures $p \neq q$, then (3.3) reduces to (3.1), i.e., preferences are expected utility. Axiomatically, this can be obtained by strengthening Axiom A2 to

Axiom A2'—Independence: For all $f, g, h \in \mathcal{F}(\Delta(Z))$ and $\alpha \in (0,1)$,

$$f \succsim g \Leftrightarrow \alpha f + (1-\alpha)h \succsim \alpha g + (1-\alpha)h.$$

Similarly, setting $c(p) = 0$ for all measures p in a closed and convex set C and $c(p) = \infty$ otherwise, denoted $c = \delta_C$, reduces (3.3) to

$$V(f) = \min_{p \in C} \int_S \left(\sum_{z \in Z} u(z) f_s(z) \right) \mathrm{d}p,$$

which is a representation of the Maxmin Expected Utility preferences introduced by Gilboa and Schmeidler (1989). Axiomatically, this can be obtained by strengthening Axiom A2 to

Axiom A2"—Certainty Independence: For all $f, g \in \mathcal{F}(\Delta(Z))$, $\pi \in \Delta(Z)$ and $\alpha \in (0,1)$,

$$f \succsim g \Leftrightarrow \alpha f + (1-\alpha)\pi \succsim \alpha g + (1-\alpha)\pi.$$

As mentioned before, multiplier preferences also are a special case of variational preferences. They can be obtained by setting $c(p) = \theta R(p \,\|\, q)$. However, because relative entropy is a specific functional form assumption, Maccheroni et al. (2006a) were skeptical that a counterpart of Axiom A2' or Axiom A2" exists that would deliver multiplier preferences:

> [...] we view entropic preferences as essentially an analytically convenient specification of variational preferences, much in the same way as, for example, Cobb-Douglas preferences are an analytically convenient specification of homothetic preferences. As a result, in our setting there

might not exist behaviorally significant axioms that would characterize entropic preferences (as we are not aware of any behaviorally significant axiom that characterizes Cobb-Douglas preferences).

Despite this seeming impasse, the next section shows that pinning down the functional form is possible with behaviorally significant axioms. In fact, somewhat unexpectedly, they are the well known Savage's P2 and P4 axioms (together with his technical axiom of continuity—P6).[2]

3.4. Axiomatization of Multiplier Preferences

Axiom P2—Savage's Sure-Thing Principle: For all $E \in \Sigma$ and $f, g, h, h' \in \mathcal{F}(\Delta(Z))$

$$fEh \succsim gEh \Rightarrow fEh' \succsim gEh'.$$

Axiom P4—Savage's Weak Comparative Probability: For all $E, F \in \Sigma$ and $\pi, \pi', \rho, \rho' \in \Delta(Z)$ such that $\pi \succ \rho$ and $\pi' \succ \rho'$

$$\pi E\rho \succsim \pi F\rho \Rightarrow \pi' E\rho' \succsim \pi' F\rho'.$$

Axiom P6—Savage's Small Event Continuity: For all acts $f \succ g$ and $\pi \in \Delta(Z)$, there exists a finite partition $\{E_1, \ldots, E_n\}$ of S such that for all $i \in \{1, \ldots, n\}$

$$f \succ \pi E_i g \quad \text{and} \quad \pi E_i f \succ g.$$

Theorem 3.1: *Suppose $\succsim$ is a variational preference. Then Axioms P2, P4, and P6, are necessary and sufficient for $\succsim$ to have a multiplier representation (3.2). Moreover,*

[2]Those axioms, together with axioms A1-A8, imply other Savage axioms.

two triples (θ', u', q') and (θ'', u'', q'') represent the same multiplier preference $\succsim$ if and only if q' and q'' are identical and there exist $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $u' = \alpha u'' + \beta$ and $\theta' = \alpha\theta''$.

The two cases: $\theta = \infty$ (lack of concern for model misspecification) and $\theta < \infty$ (concern for model misspecification) can be distinguished on the basis of the Independence Axiom (Axiom A2').[3] In the case when θ is finite, its numerical value is uniquely determined, given u. A positive affine transformation of u changes the scale on which θ operates, so θ has to change accordingly. This is reminiscent of the necessary adjustments of the CARA coefficient when units of account are changed.

Alternative axiomatizations are presented in Appendix 3.6.2.9. It is shown there that Axiom A7 can be dispensed with in the presence of another of Savage's axioms—P3. Also, Savage's axiom P6 can be be dispensed with if Axiom A8 is strengthened to Arrow's (1970) Monotone-Continuity axiom and an additional axiom of nonatomicity is assumed.

3.5. Discussion

Any Anscombe-Aumann act can be viewed as a Savage act where prizes have an internal structure: they are lotteries. Because of this, an Anscombe-Aumann setting with the set of prizes Z can be viewed as a Savage setting with the set of prizes $\Delta(Z)$. Compared to a Savage setting with the set of prizes Z, more choice-observations are available in the Anscombe-Aumann setting. This additional information makes it possible to distinguish EU preferences from multiplier preferences.

[3]The weaker Certainty Independence Axiom (Axiom A2") is also sufficient for making such a distinction. Alternatively, Machina and Schmeidler's (1995) axiom of Horse/Roulette Replacement could be used.

To understand this distinction, observe that by 2.1, multiplier preferences have the following representation.

$$V(f) = \int_S \phi_\theta\left(\sum_{z\in Z} u(z) f_s(z)\right) \mathrm{d}q(s), \tag{3.4}$$

Because of the introduction of objective lotteries, this equation does not reduce to (2.3). The existence of two sources of uncertainty enables a distinction between *purely objective lotteries*, i.e., acts which pay the same lottery $\pi \in \Delta(Z)$ irrespectively of the state of the world and *purely subjective acts*, i.e., acts that in each state of the world pay off δ_z for some $z \in Z$.

From representation (3.4) it follows that for any two purely objective lotteries $\pi' \succsim \pi$ if and only if

$$\sum_{z\in Z} u(z)\pi'(z) \succsim \sum_{z\in Z} u(z)\pi(z).$$

On the other hand, each purely subjective act f induces a lottery $\pi_f(z) = q(f^{-1}(z))$. However, for any two such acts $f' \succsim f$ if and only if

$$\sum_{z\in Z} \phi_\theta(u(z))\pi_{f'}(z) \succsim \sum_{z\in Z} \phi_\theta(u(z))\pi_f(z).$$

What is crucial here is that the decision maker has a different attitude towards objective lotteries and subjective acts. In particular, if $\theta < \infty$ he is more averse towards subjective uncertainty than objective risk. The coexistence of those two sources in one model permits a joint measurement of those two attitudes.

It has been observed in the past that differences in attitudes towards risk and uncertainty lead to Ellsberg-type behavior. Neilson (1993) showed that the following *Second-Order Expected Utility* representation

$$V(f) = \int_S \phi\Bigg(\sum_{z\in Z} u(z) f_s(z)\Bigg)\,\mathrm{d}q(s), \tag{3.5}$$

can be obtained by a combination of von Neumann-Morgenstern axioms on lotteries and Savage axioms on acts.[4] A similar model was studied by Ergin and Gul (2004), see Chapter 4. From this perspective, multiplier preferences are a special case of (3.5) where $\phi = \phi_\theta$. Theorem 3.1 shows that this specific functional form of the function ϕ is implied by Weak Certainty Independence (Axiom A2) and by Uncertainty Aversion (Axiom A5).[5] Thus, the class of multiplier preferences is the intersection of the class of variational preferences and the class of second-order expected utility preferences. The following example shows that, because of this property, multiplier preferences can be used for modelling Ellsberg-type behavior.

Example 3.1: (Ellsberg's Paradox revisited). Suppose Urn I contains 100 red and black balls in unknown proportion, while Urn II contains 50 red and 50 black balls. Let the state space $S = \{R, B\}$ represent the possible draws from Urn I. Betting \$100 on red from Urn I corresponds to an act $f_R = (\delta_{100}, \delta_0)$ while betting \$100 on black from Urn I corresponds to an act $f_B = (\delta_0, \delta_{100})$. On the other hand, betting \$100 on red from Urn II corresponds to a lottery $\pi_R = \frac{1}{2}\delta_{100} + \frac{1}{2}\delta_0$, while betting \$100 on black from Urn II corresponds to a lottery $\pi_B = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_{100}$. These correspondences reflect the fact

[4] I am grateful to Peter Klibanoff for this reference.

[5] This stems from the fact that, as elucidated by Grant and Polak (2007), variational preferences display constant absolute ambiguity aversion,

that betting on Urn I involves subjective uncertainty, while betting on Urn II involves objective risks. Note in particular, that $\pi_R = \pi_B$.

Consider the two multiplier preferences from Example 2.2: $\succsim_1$ with $u_1(x) = -\exp(-x)$ and $\theta_1 = \infty$, and $\succsim_2$ with $u_2(x) = x$ and $\theta_2 = 1$. Suppose also, that they both share the probability assessment $q(B) = q(R) = \frac{1}{2}$.

As explained in Example 2.2, the representation of $\succsim_1$ suggests that the decision maker is not concerned about model misspecification or ambiguity. Indeed, his choices reveal that $\pi_B \sim \pi_R \sim f_R \sim f_B$. This decision maker is indifferent between objective risk and subjective uncertainty, avoiding the Ellsberg paradox.

In contrast, the representation of $\succsim_2$ suggests that the decision maker is concerned about model misspecification or ambiguity. And indeed, his choices reveal that $\pi_B \sim \pi_R \succ f_R \sim f_B$. This decision maker prefers objective risk to probabilistically equivalent subjective uncertainty, displaying behavior typical in Ellsberg's experiments.

This means that introducing objective uncertainty makes it possible to disentangle risk aversion from concern about model misspecification and thus escape the consequences of 2.1. As a consequence, the interpretations of representations of $\succsim_1$ and $\succsim_2$ become behaviorally meaningful. ▲

It is worthwile to notice that for $\theta < \infty$ the decision maker behaves according to EU on the subdomain of objective lotteries and also on the subdomain of purely subjective acts. What leads to Ellsberg-type behavior are violations of EU across those domains: the decision maker's aversion towards objective risk (captured by u) is lower than his aversion towards objective risk (captured by $\phi_\theta \circ u$). This phenomenon is called *Second Order Risk Aversion*.[6]

3.6. Proofs

Let $B_0(\Sigma)$ denote the set of all real-valued Σ-measurable simple functions and let $B_0(\Sigma, K)$ be the set of all functions in $B_0(\Sigma)$ that take values in a convex set $K \subseteq \mathbb{R}$.

3.6.1. Proof of Observation 2.1

Because $\theta^{-1} \cdot (u \circ f)$ is a bounded measurable function on (S, Σ), from Proposition 1.4.2 of Dupuis and Ellis (1997) it follows that

$$\min_{p \in \Delta S} \int_S (u \circ f)\, \mathrm{d}p + \theta R(p \| q) = -\theta \log \left(\int_S \exp\left(-\frac{u \circ f}{\theta} \right) \mathrm{d}q \right).$$

Thus, $\succsim$ is a multiplier preference with θ, u, and q iff it is represented by U with

$$U(f) = -\theta \log \left(\int_S \exp\left(-\frac{u \circ f}{\theta} \right) \mathrm{d}q \right).$$

Rewrite using the definition of ϕ_θ:

$$U(f) = \phi_\theta^{-1}\left(\int_S (\phi_\theta \circ u \circ f)\, \mathrm{d}q \right).$$

Since ϕ_θ is a monotone transformation, $\succsim$ is also represented by $V := \phi_\theta \circ U$, i.e.,

$$V(f) = \int_S (\phi_\theta \circ u \circ f)\, \mathrm{d}q.$$

[6] This notion was introduced by Ergin and Gul (2004) in a setting with two subjective sources of uncertainty (see Section 5).

3.6.2. Proof of Theorem 3.1

3.6.2.1. Niveloidal Representation. By Lemmas 25 and 28 of Maccheroni et al. (2006a), Axioms A1-A7 imply that there exists an unbounded affine function $u : \Delta(Z) \to \mathbb{R}$ and a normalized concave niveloid $I : B_0(\Sigma, u(\Delta(Z))) \to \mathbb{R}$ such that for all $f \succsim g$ iff $I(u \circ f) \geq I(u \circ g)$. Moreover, within this class, u is unique up to positive affine transformations. Define $\mathcal{U} := u(\Delta(Z))$. After normalization, there are three possible cases: $\mathcal{U} \in \{\mathbb{R}_+, \mathbb{R}_-, \mathbb{R}\}$.

3.6.2.2. Utility Acts. For each act f, define the *utility act* associated with f as $u \circ f \in B_0(\Sigma, \mathcal{U})$. The preference on acts induces a preference on utility acts: for any $\xi', \xi'' \in B_0(\Sigma, \mathcal{U})$ define $\xi' \succsim_u \xi''$ iff $f' \succsim f''$, for some $\xi' = u \circ f'$ and $\xi'' = u \circ f''$. The choice of particular versions of f' and f'' is irrelevant, because $\xi' \succsim_u \xi''$ iff $I(\xi') \geq I(\xi'')$.

By Lemma 22 in Maccheroni, Marinacci, and Rustichini (2004), for all $k \in \mathcal{U}$ and $\xi \in B_0(\Sigma, \mathcal{U})$ we have $I(\xi + k) = I(\xi) + k$. Thus, $\xi' \succsim_u \xi''$ iff $I(\xi') \geq I(\xi'')$ iff $I(\xi' + k) \geq I(\xi'' + k)$ iff $\xi' + k \succsim_u \xi'' + k$ for all $k \in \mathcal{U}$ and $\xi', \xi'' \in B_0(\Sigma, \mathcal{U})$.

3.6.2.3. Savage's P3. In order to show that $\succsim$ have an additive representation (3.4), Savage's theorem will be used in 3.6.2.4. To do this, it is necessary to show that his P3 axiom holds.

Definition 3.1: An event $E \in \Sigma$ is *non-null* if there exist $f, g, h \in \mathcal{F}$ such that $fEh \succ gEh$.

Axiom P3—Savage's Eventwise Monotonicity: For all $x, y \in Z$, $h \in \mathcal{F}$, and non-null $E \in \Sigma$

$$x \succsim y \Leftrightarrow xEh \succsim yEh.$$

Lemma 3.1: *Axioms A1-A7, together with Axiom P2 imply axiom P3.*

Proof. First, suppose that $x \succsim y$. It follows from Axiom A4 (Monotonicity) that $xEh \succsim yEh$ for any $h \in \mathcal{F}$ and any E. Second, suppose that $y \succ x$. It follows from Monotonicity that $yEh \succsim xEh$ for any $h \in \mathcal{F}$ and any E. Towards contradiction, suppose that $yEh \sim xEh$ for a non-null $E \in \Sigma$ and some $h \in \mathcal{F}$.

Because E is non-null, there exist $f, g \in \mathcal{F}$ such that $fEh \succ gEh$. Let $\{E_1, \ldots, E_n, E\}$ be a partition of S with respect to which both fEh and gEh are measurable. Let y' be the most preferred element among $\{f(E_i) \mid i = 1, \ldots, n\}$ and let x' be the least preferred element among $\{g(E_i) \mid i = 1, \ldots, n\}$. By Monotonicity, $y'Eh \succsim fEh$ and $gEh \succsim x'Eh$. Thus $y'Eh \succ x'Eh$.

Observe that there exist $a, a' \in \mathcal{U}$ and $k, k' > 0$, such that $a = u(x), a + k = u(y), a' = u(x')$ and $a' + k' = u(y')$. Thus there exists $\xi \in B_0(\Sigma, \mathcal{U})$, such that $aE\xi = u \circ (xEh)$, $(a + k)E\xi = u \circ (yEh)$, $a'E\xi = u \circ (x'Eh)$, and $(a' + k')E\xi = u \circ (y'Eh)$. It follows that

$$I((a + k)E\xi) = I(aE\xi) \tag{3.6}$$

$$I((a' + k')E\xi) > I(a'E\xi). \tag{3.7}$$

Suppose that $\mathcal{U} = \mathbb{R}_+$. By translation invariance, it follows from (3.6) that $I((a + 2k)E(\xi + k)) = I((a + k)E(\xi + k))$ and by P2, that $I((a + 2k)E\xi) = I((a + k)E\xi)$. Hence, $I((a + 2k)E\xi) = I(aE\xi)$. By induction $I((a + nk)E\xi) = I(aE\xi)$ for all $n \in \mathbb{N}$, and by Monotonicity $I((a + r)E\xi) = I(aE\xi)$ for all $r \in \mathbb{R}_+$. In particular, letting $r = k'$, we have

$$I((a + k')E\xi) = I(aE\xi). \tag{3.8}$$

Suppose that $a' \geq a$. By translation invariance, $I((a'+k')E(\xi+a'-a) = I(a'E(\xi+a'-a))$ and by P2, $I((a'+k')E\xi) = I(a'E\xi)$. Contradiction with (3.8). Thus, it must be that $a > a'$. By translation invariance, it follows from (3.7), that $I((a+k')E(\xi+a-a')) > I(aE(\xi+a-a'))$ and by P2, $I((a+k')E\xi) > I(aE\xi)$. Contradiction with (3.8). The proof is analogous in case when $\mathcal{U} = \mathbb{R}_-$ or $\mathcal{U} = \mathbb{R}$. $\square$

3.6.2.4. Application of Savage's Theorem. It follows from Chapters 1-5 of Savage (1972) that there exists a (not necessarily affine) function $\psi : \Delta(Z) \to \mathbb{R}$ and a measure $q \in \Delta S$, such that for any $f, g \in \mathcal{F}$, $f \succsim g$ iff $\int_S (\psi \circ f)\, \mathrm{d}q \geq \int_S (\psi \circ g)\, \mathrm{d}q$. Moreover, ψ is unique up to positive affine transformations. From Theorem 1 in Section 1 of Villegas (1964) it follows that Axiom A8 implies that $q \in \Delta^\sigma(S)$.

3.6.2.5. Proof of representation (3.5). By 3.6.2.2, $f \succsim g$ iff $\int_S (\psi\circ f)\, \mathrm{d}q \geq \int_S (\psi\circ g)\, \mathrm{d}q$. In particular, $x \succsim y$ iff $\psi(x) \geq \psi(y)$. From axioms A1-A6 it follows that $x \succsim y$ iff $u(x) \geq u(y)$. Thus, there exists a unique strictly increasing function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\psi = \phi \circ u$. Thus, $f \succsim g$ iff $\int_S (\phi \circ u \circ f)\, \mathrm{d}q \geq \int_S (\phi \circ u \circ g)\, \mathrm{d}q$. This leads to the following representation of $\succsim_u$: $\xi' \succsim_u \xi''$ iff $\int_S (\phi \circ \xi')\, \mathrm{d}q \geq \int_S (\phi \circ \xi'')\, \mathrm{d}q$.

3.6.2.6. Concavity of ϕ. Let $a, b \in \mathcal{U}$. Let $\pi, \rho \in \Delta(Z)$ be such that $a = u(\pi)$ and $b = u(\rho)$. Because q is range convex, there exists a set E with $q(E) = \frac{1}{2}$. Let $f = \pi E \rho$ and $g = \rho E \pi$ and observe that $V(f) = \frac{1}{2}\phi(a) + \frac{1}{2}\phi(b) = V(g)$; thus, $f \sim g$. By Axiom A5, $\frac{1}{2}f + \frac{1}{2}g \succsim f$, i.e., $\phi\big(\frac{1}{2}a + \frac{1}{2}b\big) = V\big(\frac{1}{2}f + \frac{1}{2}g\big) \geq V(f) = \frac{1}{2}\phi(a) + \frac{1}{2}\phi(b)$. Thus,

$$\phi\left(\frac{1}{2}a + \frac{1}{2}b\right) \geq \frac{1}{2}\phi(a) + \frac{1}{2}\phi(b). \tag{3.9}$$

Let $\alpha \in (0,1)$. Let the sequence $\{\alpha_n\}$ be a dyadic approximation of α. By induction, inequality (3.9) implies that $\phi\big(\alpha_n a + (1-\alpha_n)b\big) \geq \alpha_n \phi(a) + (1-\alpha_n)\phi(b)$ for all n. By continuity of ϕ, $\lim_{n\to\infty} \phi\big(\alpha_n a + (1-\alpha_n)b\big) = \phi\big(\alpha a + (1-\alpha)b\big)$. Thus, $\phi\big(\alpha a + (1-\alpha)b\big) \geq \alpha\phi(a) + (1-\alpha)\phi(b)$.

3.6.2.7. Proof that $\phi = \phi_\theta$. By defining $\phi^k(x) := \phi(x+k)$ for all $k, x \in \mathcal{U}$, it follows from 3.6.2.2 and 3.6.2.5 that $\int_S \phi^k \circ \xi' \,\mathrm{d}q \geq \int_S \phi^k \circ \xi'' \,\mathrm{d}q$ iff $\int_S \phi \circ \xi' \,\mathrm{d}q \geq \int_S \phi \circ \xi'' \,\mathrm{d}q$. Thus, (ϕ, q) and (ϕ^k, q) are EU representations of the same preference on $B_0(\Sigma, \mathcal{U})$. By uniqueness, $\phi(x+k) = \alpha(k)\phi(x) + \beta(k)$ for all $k, x \in \mathcal{U}$. This is a generalization of Pexider's equation (see equation (3) of Section 3.1.3, p. 148 of Aczél, 1966). If $\mathcal{U} \in \{\mathbb{R}, \mathbb{R}_+\}$, then by Corollary 1 in Section 3.1.3 of Aczél (1966), up to positive affine transformations, the only strictly increasing concave solutions are of the form ϕ_θ, for $\theta \in (0,\infty]$. It is easy to prove that the same is true for $\mathcal{U} = \mathbb{R}_-$.

3.6.2.8. Conclusion of the Proof. Combining Steps 4 and 5, $f \succsim g$ iff $\int_S (\phi_\theta \circ u \circ f)\,\mathrm{d}q \geq \int_S (\phi_\theta \circ u \circ g)\,\mathrm{d}q$. Because $q \in \Delta^\sigma$, by 2.1, it follows that $f \succsim g$ iff $\min_{p\in\Delta S} \int_S (u\circ f)\,\mathrm{d}p + \theta R(p\|\, q) \geq \min_{p\in\Delta S} \int_S (u \circ g)\,\mathrm{d}p + \theta R(p\|\, q)$. $\square$

3.6.2.9. Alternative Axiomatizations.

Removing P6. Instead of Axiom P6, the following two axioms could be assumed:

Axiom A8"—Arrow's Monotone Continuity: If $f, g \in \mathcal{F}$, $x \in Z$, $\{E_n\}_{n\geq 1} \in \Sigma$ with $E_1 \supseteq E_2 \supseteq \cdots$ and $\bigcap_{n\geq 1} E_n = \emptyset$, then $f \succ g$ implies that there exists $n_0 \geq 1$ such that $xE_{n_0}f \succ g$ and $f \succ xE_{n_0}g$.

Axiom A9—Nonatomicity: Every nonnull event can be partitioned into two nonnull events.

Axiom A8" is stronger than Axiom A8 and is necessary to obtain a countably additive probability. Axiom A9 (see Villegas, 1964) is needed to obtain fineness and tightness of the qualitative probability.

This leads to the following theorem: Axioms A1-A7, A8", together with P2, P4, and A9 are necessary and sufficient for $\succsim$ to have a multiplier representation. The proof is analogous, but instead of Savage's Theorem, as in 3.6.2.4, Arrow's (1970) theorem is used (cf. Chapter 2 of his book).

Removing Unboundedness. Instead of Axiom A7, Savage's axiom P3 could be assumed. as verified by Klibanoff et al. (2005) in the proof of their Proposition 2, the family of functions ϕ_θ remains to be the only solution of Pexider's functional equation when domain is restricted to an interval.

Savage Axioms Only on Purely Objective Acts. If the existence of certainty equivalents for lotteries is assumed, i.e., for any $\pi \in \Delta(Z)$ there exists $z \in Z$ with $z \sim \pi$, then the Savage axioms can be weakened in the following sense. In Theorem 3.1 Axioms P2, P4, and P6 were assumed to hold on all (Anscombe–Aumann) acts. Assuming the existence of certainty equivalents makes it possible to impose Savage axioms only on Savage acts, i.e., acts paying out a degenerate lottery in each state.

CHAPTER 4

Axiomatization within Ergin-Gul's model

This section discusses another enrichment of the domain of choice, which does not rely on the assumption of objective risk. Instead, it is assumed that there are two sources of subjective uncertainty, towards which the decision maker may have different attitudes. This type of environment was discussed by Chew and Sagi (2007), Ergin and Gul (2004), and Nau (2001, 2006); for an empirical application see Abdellaoui, Baillon, and Wakker (2007).

4.1. Subjective Sources of Uncertainty

Assume that the state space has a product structure $S = S_a \times S_b$, where a and b are two separate *issues*, or sources of uncertainty, towards which the decision maker may have different attitudes. In comparison with the Anscombe–Aumann framework, where objective risk is one of the sources, here both sources are subjective. Let $\mathcal{A}_a$ be a sigma algebra of subsets of S_a and $\mathcal{A}_b$ be a sigma algebra of subsets of S_b. Let Σ_a be the sigma algebra of sets of the form $A \times S_b$ for all $A \in \mathcal{A}_a$, Σ_b be the sigma algebra of sets of the form $S_a \times B$ for all $B \in \mathcal{A}_b$, and Σ be the sigma algebra generated by $\Sigma_a \cup \Sigma_b$. As before, $\mathcal{F}(Z)$ is the set of all simple acts $f : S \to Z$. In order to facilitate the presentation, it will be assumed that certainty equivalents exist, i.e., for any $f \in \mathcal{F}(Z)$ there exists $z \in Z$ with $z \sim f$. The full analysis without this assumption is contained in Sections 4.4.1 and 4.4.2.

Ergin and Gul (2004) axiomatized preferences which are general enough to accommodate probabilistic sophistication and even second-order probabilistic sophistication. An important subclass of those preferences are second-order expected utility preferences.

$$V(f) = \int_{S_b} \phi\biggl(\int_{S_a} u(f(s_a, s_b))\, \mathrm{d}q_a(s_a) \biggr)\, \mathrm{d}q_b(s_b) \tag{4.1}$$

where $u : Z \to \mathbb{R}$, $\phi : \mathbb{R} \to \mathbb{R}$ is a strictly increasing and continuous function, and the measures $q_a \in \Delta(S_a)$ and $q_b \in \Delta(S_b)$ are nonatomic.

To characterize preferences represented by (4.1), Ergin and Gul (2004) assume Axioms A1, A6, and P3, together with weakenings of P2 and P4 and a strengthening of P6. There is a close relationship between (4.1) and Neilson's (1993) representation (3.5). The role of objective risk is now taken by a subjective source: issue a. For each s_b, the decision maker computes the expected utility of $f(\cdot, s_b)$ and then averages those values using function ϕ.

4.2. Second-Order Risk Aversion

In the Anscombe–Aumann framework, concavity of the function ϕ is responsible for second-order risk aversion, i.e., higher aversion towards subjective uncertainty than towards objective risk. This property is a consequence of the axiom of Uncertainty Aversion (Axiom A5).[1] Similarly, in the present setup, concavity of function ϕ is responsible for higher aversion towards issue b than towards issue a. This property was introduced by Ergin and Gul (2004) who formally defined it in terms of mean-preserving spreads. However, this definition refers to the probability measures obtained from the representation and hence is not directly based on preferences. Theorems 2 and 5 of Ergin and Gul

[1]This follows from the proof of Theorem 3.1, see section 3.6.2.6 in Appendix 3.6.2.

(2004) characterize second-order risk aversion in terms of induced preferences over induced Anscombe-Aumann acts and an analogue of Axiom A5 in that induced setting. However, just as with mean-preserving spreads, those induced Anscombe-Aumann acts are constructed using the subjective probability measure derived from the representation. As a consequence, the definition is not expressed directly in terms of observables.

In the presence of other axioms, the following purely behavioral axiom is equivalent to Ergin and Gul's (2004) definition.

Axiom A5'—Second Order Risk Aversion: For any $f, g \in \mathcal{F}_b$ and any $E \in \Sigma_a$ if $f \sim g$, then $fEg \succsim f$.

This axiom is a direct subjective analogue of Schmeidler's (1989) axiom of Uncertainty Aversion (Axiom A5).

Theorem 4.1: *Suppose $\succsim$ has representation (4.1). Then Axiom A5' is satisfied if and only if the function ϕ in (4.1) is concave.*

4.3. Axiomatization of Multiplier Preferences

The additional axiom that delivers multiplier preferences in this framework is Constant Absolute Second Order Risk Aversion.

Axiom A2'''—Constant Absolute Second Order Risk Aversion: There exists a non-null event $E \in \Sigma_a$ such that for all $f, g \in \mathcal{F}_b(Z)$, $x, y \in Z$

$$fEx \succsim gEx \Rightarrow fEy \succsim gEy.$$

In addition, two technical axioms, similar to Axioms 7 and 8, are needed.

Axiom A7'—$\mathcal{F}_a$-Unboundedness: There exist $x \succ y$ in Z such that, for all non-null $E_a \in \Sigma_a$ there exist $z \in Z$ that satisfies either $y \succ zE_a x$ or $zE_a y \succ x$.

Axiom A8'—$\mathcal{F}_b$-Monotone Continuity: If $f, g \in \mathcal{F}(Z)$, $x \in Z$, $\{E_n\}_{n\geq 1} \in \Sigma_b$ with $E_1 \supseteq E_2 \supseteq \cdots$ and $\bigcap_{n\geq 1} E_n = \emptyset$, then $f \succ g$ implies that there exists $n_0 \geq 1$ such that $xE_{n_0} f \succ g$.

Theorem 4.2: *Suppose $\succsim$ has representation (4.1). Then Axioms A2''', A5', A7, and A8 are necessary and sufficient for $\succsim$ to be represented by V, where*

$$V(f) = \min_{p_b \in \Delta S_b} \int_{S_b} \left(\int_{S_a} u(f(s_a, s_b))\, \mathrm{d}q_a(s_a) \right) \mathrm{d}p_b(s_b) + \theta R(p_b \| q_b)$$

and $u : Z \to \mathbb{R}$, $\theta \in (0, \infty]$, and q_a, q_b are nonatomic measures.

4.4. Proofs

4.4.1. Proof of Theorem 4.1

In order to relax the assumption of existence of certainty equivalents, the following definition will be used.

Definition 4.1: Act $f \in \mathcal{F}_a(Z)$ is *symmetric with respect to* $E \in \Sigma_a$ if for all $z \in Z$

$$fEz \sim zEf.$$

Symmetric acts have the same expected utility on each "half" of the state space.[2]

[2]Symmetric acts are acts that can be "subjectively mixed". Such subjective mixtures are different from subjective mixtures studied by Ghirardato, Maccheroni, Marinacci, and Siniscalchi (2003), whose construction relies on range-convexity of u. In the present setting, subjective mixtures are not needed under range-convexity of u.

Axiom A5"—Second Order Risk Aversion: If acts $f, g \in \mathcal{F}_a$ are symmetric with respect to $E \in \Sigma_a$, then for all $F \in \Sigma_b$

$$fFg \sim gFf \Rightarrow (fFg)E(gFf) \succsim fFg.$$

The proof of Theorem 4.1 follows from the proof of the following stronger theorem

Theorem 4.3: *Suppose $\succsim$ has representation* (4.1)*. Then Axiom A5" is satisfied if and only if the function ϕ in (4.1) is concave.*

Proof.

4.4.1.1. Necessity. Suppose $f \in \mathcal{F}_a(Z)$ is symmetric with respect to $E \in \Sigma_a$. Let $\alpha = q_a(E)$. Axiom A6 and representation (4.1) imply that there exist $z', z'' \in Z$ with $z' \succ z''$. Thus, $fEz' \sim z'Ef$ and $fEz'' \sim z''Ef$ imply that

$$\int_E (u \circ f)\, \mathrm{d}q_a + (1-\alpha)u(z') = \alpha u(z') + \int_{E^c} (u \circ f)\, \mathrm{d}q_a, \tag{4.2}$$

$$\int_E (u \circ f)\, \mathrm{d}q_a + (1-\alpha)u(z'') = \alpha u(z'') + \int_{E^c} (u \circ f)\, \mathrm{d}q_a. \tag{4.3}$$

By subtracting (4.3) from (4.2)

$$(1-\alpha)[u(z') - u(z'')] = \alpha[u(z') - u(z'')];$$

thus, $\alpha = \frac{1}{2}$ and therefore

$$\int_E (u \circ f)\, \mathrm{d}q_a = \int_{E^c} (u \circ f)\, \mathrm{d}q_a.$$

Let $f, g \in \mathcal{F}_a(Z)$. Denote $U(f) = \int_{S_a} (u \circ f)\,\mathrm{d}q_a$ and $U(g) = \int_{S_a} (u \circ g)\,\mathrm{d}q_a$. Because f and g are symmetric with respect to $E \in \Sigma_a$,

$$\int_E (u \circ f)\,\mathrm{d}q_a = \int_{E^c} (u \circ f)\,\mathrm{d}q_a = \frac{1}{2}U(f)$$
$$\int_E (u \circ g)\,\mathrm{d}q_a = \int_{E^c} (u \circ g)\,\mathrm{d}q_a = \frac{1}{2}U(g).$$

Let $F \in \Sigma_b$ and $\beta = q_b(F)$. If $fFg \sim gFf$, then

$$\beta\phi\big(U(f)\big) + (1-\beta)\phi\big(U(g)\big) = \beta\phi\big(U(g)\big) + (1-\beta)\phi\big(U(f)\big).$$

Thus,

$$(2\beta - 1)\phi\big(U(f)\big) = (2\beta - 1)\phi\big(U(g)\big).$$

If $\beta \neq \frac{1}{2}$, then $U(f) = U(g)$ and trivially

$$\begin{aligned} V\big((fFg)E(gFf)\big) &= \beta\phi\left(\frac{1}{2}U(f) + \frac{1}{2}U(g)\right) + (1-\beta)\phi\left(\frac{1}{2}U(g) + \frac{1}{2}U(f)\right) \\ &= \beta\phi\big(U(f)\big) + (1-\beta)\phi\big(U(g)\big) = V(fFg). \end{aligned}$$

If $\beta = \frac{1}{2}$, then

$$\begin{aligned} V\big((fFg)E(gFf)\big) &= \frac{1}{2}\phi\left(\frac{1}{2}U(f) + \frac{1}{2}U(g)\right) + \frac{1}{2}\phi\left(\frac{1}{2}U(g) + \frac{1}{2}U(f)\right) \\ &= \phi\left(\frac{1}{2}U(f) + \frac{1}{2}U(g)\right) \geq \frac{1}{2}\phi\big(U(f)\big) + \frac{1}{2}\phi\big(U(g)\big) = V(fFg), \end{aligned}$$

where the inequality follows from concavity of ϕ.

4.4.1.2. Sufficiency.

Convexity of Domain of ϕ. Let D_ϕ be the domain of function ϕ, i.e., $D_\phi = \{U(f) \mid f \in \mathcal{F}_a\}$. Suppose $k, l \in D_\phi$ and $\alpha \in (0,1)$. Wlog $k < l$. Let $f, g \in \mathcal{F}_a$ be such that $k = U(f)$ and $l = U(g)$. Define $A = \min_{s \in S} f(s)$ and $B = \max_{s \in S} g(s)$ and let $x, y \in Z$ be such that $u(x) = A$ and $u(y) = B$. By nonatomicity of q_a, there exists $E \in \Sigma_a$ with $q_a(E) = \big(B - [\alpha k + (1-\alpha)l]\big)(B-A)^{-1}$. Verify, that $U(xEy) = \alpha k + (1-\alpha)l$. Hence, D_ϕ is a convex set.

Dyadic Convexity of ϕ. Suppose $k, l \in D_\phi$ and let $f, g \in \mathcal{F}_a$ be such that $k = U(f)$ and $l = U(g)$. Define $\underline{k} = \min_{s \in S} f(s)$, $\bar{k} = \max_{s \in S} f(s)$, $\underline{l} = \min_{s \in S} g(s)$, and $\bar{l} = \max_{s \in S} g(s)$. Let $\underline{x}, \bar{x}, \underline{y}, \bar{y}$ be such that $u(\underline{x}) = \underline{k}$, $u(\bar{x}) = \bar{k}$, $u(\underline{y}) = \underline{l}$, $u(\bar{y}) = \bar{l}$. Also, define $\kappa = \frac{\bar{k} - k}{\bar{k} - \underline{k}}$ and $\lambda = \frac{\bar{l} - l}{\bar{l} - \underline{l}}$. By nonatomicity of q_a there exist partitions $\{E_1^\kappa, E_2^\kappa, E_3^\kappa, E_4^\kappa\}$ and $\{E_1^\lambda, E_2^\lambda, E_3^\lambda, E_4^\lambda\}$ of S_a such that $E_1^\kappa \cup E_2^\kappa = E_1^\lambda \cup E_2^\lambda$, $q_a(E_1^\kappa \cup E_2^\kappa) = q_a(E_1^\lambda \cup E_2^\lambda) = \frac{1}{2}$, $q_a(E_1^\kappa \cup E_3^\kappa) = \frac{\kappa}{2}$, and $q_a(E_1^\lambda \cup E_3^\lambda) = \frac{\lambda}{2}$.

Define acts $f = \underline{x} E_1^\kappa\, \bar{x} E_2^\kappa\, \underline{x} E_3^\kappa\, \bar{x} E_4^\kappa$ and $g = \underline{y} E_1^\lambda\, \bar{y} E_2^\lambda\, \underline{y} E_3^\lambda\, \bar{y} E_4^\lambda$. Verify that f and g are symmetric with respect to $E = E_1^\kappa \cup E_2^\kappa = E_1^\lambda \cup E_2^\lambda$ and satisfy $U(f) = k$ and $U(g) = l$. By nonatomicity of q_b, there exists $F \in \Sigma_b$ with $q_b(F) = \frac{1}{2}$. Verify that $V(fFg) = \frac{1}{2}\phi(k) + \frac{1}{2}\phi(l) = V(gFf)$. Hence, by Axiom A5',

$$\begin{aligned}\phi\left(\frac{1}{2}k + \frac{1}{2}l\right) = \frac{1}{2}\phi\left(\frac{1}{2}k + \frac{1}{2}l\right) + \frac{1}{2}\phi\left(\frac{1}{2}l + \frac{1}{2}k\right) &= V\big((fFg)E(gFf)\big) \\ &\geq V(fFg) = \frac{1}{2}\phi(k) + \frac{1}{2}\phi(l).\end{aligned}$$

As a consequence,

$$\phi\left(\frac{1}{2}k + \frac{1}{2}l\right) \geq \frac{1}{2}\phi(k) + \frac{1}{2}\phi(l) \tag{4.4}$$

for all $k, l \in D_\phi$.

Limiting argument. Let $\alpha \in [0,1]$. From 4.4.1.2 it follows that $\alpha k + (1-\alpha)l \in D_\phi$. Let the sequence $\{\alpha_n\}$ be a dyadic approximation of α. By induction, inequality (4.4) implies that $\phi(\alpha_n k + (1-\alpha_n)l) \geq \alpha_n \phi(k) + (1-\alpha_n)\phi(l)$ for all n. By continuity of ϕ, $\lim_{n\to\infty} \phi(\alpha_n k + (1-\alpha_n)l) = \phi(\alpha k + (1-\alpha)l)$. Thus, $\phi(\alpha k + (1-\alpha)l) \geq \alpha\phi(k) + (1-\alpha)\phi(l)$. □

4.4.2. Proof of Theorem 4.2

By Theorem 3 of Ergin and Gul (2004), Axioms A1, A6, P2', P3, P4', and P6' guarantee the existence of nonatomic measures $q_a \in \Delta S_a$ and $q_b \in \Delta S_b$, function $u : Z \to \mathbb{R}$, and a continuous and strictly increasing $\phi : \mathbb{R} \to \mathbb{R}$ such that $\succsim$ is represented by V with

$$V(f) = \int_{S_b} \phi\biggl(\int_{S_a} u(f(s_a, s_b))\, \mathrm{d}q_a(s_a) \biggr)\, \mathrm{d}q_b(s_b). \tag{4.5}$$

Let x, y be as in Axiom A7'. Wlog $u(y) = 0$, thus $u(x) > 0$. Nonatomicity of q_a guarantees that there exists a sequence of events $\{E_n\}_{n\geq 1}$ in Σ_a with $q_a(E_n) = \frac{1}{n}$. Axiom A7' guarantees that there exist a sequence $\{z'_n\}_{n\geq 1}$ with $\phi\bigl(0\bigr) > \phi\bigl(\frac{1}{n}u(z'_n) + \frac{n-1}{n}u(x)\bigr)$ or a sequence $\{z''_n\}_{n\geq 1}$ with $\phi\bigl(\frac{1}{n}u(z''_n)\bigr) > \phi\bigl(u(x)\bigr)$ (or both such sequences exist). By strict monotonicity of ϕ if follows that, in the first case, $-(n-1)u(x) > u(z'_n)$; thus $u(z'_n) \to -\infty$; hence, u is unbounded from below. In the second case, $u(z''_n) > nu(x)$; thus, $u(z''_n) \to +\infty$; hence, in this case u is unbounded from above. Define $\mathcal{U} := u(Z)$. After normalization, there are three possible cases: $\mathcal{U} \in \{\mathbb{R}+, \mathbb{R}_-, \mathbb{R}\}$.

Let $E \in \Sigma_a$ be as in Axiom A2''' and let $p := q_a(E)$. For any $k \in \mathcal{U}$ define a preference $\succsim^k$ on $\mathcal{F}_b$ as follows. Let $z \in Z$ be such that $u(z) = k$ and for any $f, g \in \mathcal{F}_b(Z)$ define

$f \succsim^k g$ iff $fEz \succsim gEz$. (Because of Axiom A2''', the choice of particular z does not matter.) Define $\phi^k(u) := \phi(u + (1-p)k)$. From representation (4.5), it follows that $\succsim^k$ is represented by V^k with

$$V^k(f) = \int_{S_b} \phi^k\bigg(\int_E u(f(s_a, s_b))\,\mathrm{d}q_a(s_a)\bigg)\,\mathrm{d}q_b(s_b).$$

By Axiom A2''', $\succsim^k = \succsim^0$ for all $k \in \mathcal{U}$. Hence, ϕ^k and ϕ^0 are equal up to positive affine transformations, i.e., $\phi(u + (1-p)k) = \alpha(k)\phi(u) + \beta(k)$ for all $u, k \in \mathcal{U}$. By changing variables: $k' := (1-p)k$, $\alpha'(k') = \alpha(\frac{k'}{p})$, and $\beta'(k') = \beta(\frac{k'}{p})$, it follows that $\phi(k' + u) = \alpha'(k')\phi(u) + \beta'(k')$ for all $u, k' \in \mathcal{U}$, which is is a generalization of Pexider's equation (see equation (3) of Section 3.1.3, p. 148 of Aczél, 1966). By Theorem 4.1, ϕ is concave. By Corollary 1 in Section 3.1.3 of Aczél (1966), up to positive affine transformations, the only strictly increasing quasiconcave solutions are of the form ϕ_θ, for $\theta \in (0, \infty]$.

It follows from Theorem 1 in Section 1 of Villegas (1964) that Axiom A8' delivers countable additivity of q_b. A reasoning similar to 2.1 of this paper concludes the proof. $\square$

Part 2

Ambiguity and Timing—First Setting

This part of the dissertation presents an axiomatization of dynamic multiplier preferences, which are central to applications of robustness to macroeconomics and finance. This class of preferences is related to a special case of the model of Kreps and Porteus (1978) and they both exhibit a preference for earlier resolution of uncertainty. The main result establishes that preference for earlier resolution of uncertainty is exhibited by all stationary variational preferences, except for the subclass of maxmin expected utility preferences. Thus, the fact that dynamic multiplier preferences display such preference should not be attributed to their relation to the Kreps and Porteus (1978) model but rather is a "generic" feature of variational preferences.

The material is organized as follows. Chapter 5 introduces the domain of dynamic choice and defines and axiomatizes dynamic variational preferences. These constructions and results rely on the work of Hayashi (2005) on recursive maxmin expected utility preferences. Chapter 6 defines and axiomatizes dynamic multiplier preferences. The main challenge is to make sure that the penalization parameter θ is constant over time and history-independent. This is achieved by applying a version of Wakker's tradeoff consistency (see, e.g., Köbberling and Wakker, 2003). Chapter 7 extends the notion of IID ambiguity of studied by Chen and Epstein (2002) and Epstein and Schneider (2003a) in the context of maxmin expected utility to the class of variational preferences. This assumption makes it possible to define preference for earlier resolution of uncertainty. Finally a characterization of indifference to timing of resolution of uncertainty in the class of variational preferences is obtained: the only preferences satisfying indifference are the maxmin expected utility preferences.

CHAPTER 5

Axiomatic Foundations of Dynamic Variational Preferences

5.1. Domain of Dynamic Choice

Hayashi (2005), who studied a dynamic model of stationary maxmin expected utility preferences, used a domain of choice $\mathcal{H}$, which proves useful also for studying variational preferences. In each period the state space S is finite and the set of outcomes is a compact set Z.[1] The domain of *temporal Anscombe–Aumann acts*, $\mathcal{H}$, is constructed inductively

$$\mathcal{H}_0 = \mathcal{F}(\Delta(Z))$$

and

$$\mathcal{H}_t = \mathcal{F}(\Delta(Z \times \mathcal{H}_{t-1}))$$

for each $t \geq 1$.[2] Define $f \in \prod_{t=0}^{\infty} \mathcal{H}_t$ to be coherent if for any t the act f_{t+1} induces the same consumption process as f_t. As asserted by Theorem 1 of Hayashi (2005), the set $\mathcal{H}$ of such coherent acts satisfies the following homeomorphism

$$\mathcal{H} \simeq \mathcal{F}(\Delta(Z \times \mathcal{H})).$$

[1]Finiteness of S and compactness of Z can be relaxed at the expense of modyfying the construction of space $\mathcal{H}$ and of additional notation related to conditioning on measure zero events.

[2]For any compact metric space X, the set of Borel probability measures $\Delta(X)$ is a compact metric space with the Prohorov metric and the set $\mathcal{F}(X) = X^S$ is a compact metric space under the product metric.

This recursive property facilitates axiomatizations of stationary preferences, because $\mathcal{H}$ is a mixture space under the usual state-by-state mixing of Anscombe–Aumann acts. An important subdomain of $\mathcal{H}$ is the space $\mathcal{D}$ of *temporal lotteries* of Kreps and Porteus (1978) and Epstein and Zin (1989)

$$\mathcal{D} \simeq \mathcal{F}(\Delta(Z \times \mathcal{D})).$$

Another important subdomain consists of *one-step-ahead acts* $\mathcal{H}_{+1}$ where all subjective uncertainty resolves in the first period.

$$\mathcal{H}_{+1} = \{h_{+1} \in \mathcal{F}(\Delta(Z \times \mathcal{H})) \mid h_{+1}(s) \in \mathcal{D} \text{ for all } s \in S\}.$$

5.2. Axiomatization of Variational Preferences

Following Hayashi (2005), for each $t \geq 0$ and history $s^t = (s_1, \ldots, s_t) \in S^t$ the decision maker's preference $\succsim_{s^t}$ over $\Delta(Z \times \mathcal{H})$ is observed. For any $z \in Z$ and $h \in \mathcal{H}$ the degenerate lottery $\delta_{(z,h)}$ will, with a slight abuse of notation, be denoted (z, h).

Definition 5.1: Family $\{\succsim_{s^t}\}$ is a Dynamic Variational Preference if it is represented by a family of continuous, nonconstant functions $U_{s^t} : \Delta(Z \times \mathcal{H}) \to \mathbb{R}$ such that

$$U_{s^t}(\mu) = \int_{Z \times \mathcal{H}} \left\{ u(z) + \beta \left[\min_{p \in \Delta(S)} \int_S U_{(s^t, s)}\big(h(s)\big)\, \mathrm{d}p(s) + c_{s^t}(p) \right] \right\} \mathrm{d}\mu(z, h)$$

for any $p \in \Delta(Z \times \mathcal{H})$, where $u : Z \to \mathbb{R}$ is continuous and nonconstant and $\beta \in (0, 1)$ and cost functions c_{s^t} are grounded, convex, and lower semicontinous. Moreover, β is unique and the function u is unique up to positive affine transformations.

An axiomatization of dynamic variational preferences is obtained by modifying Hayashi's (2005) axiomatization of dynamic maxmin expected utility preferences, in particular by relaxing certainty independence to weak certainty independence. Sections 5.2.1 and 5.2.2 present this axiomatization.

5.2.1. Axiomatization of Recursive Variational Preferences

Definition 5.2: Family $\{\succsim_{s^t}\}$ is a Recursive Variational Preference if it is represented by a family of continuous, nonconstant functions $U_{s^t} : \Delta(Z \times \mathcal{H}) \to \mathbb{R}$ such that

$$U_{s^t}(\mu) = \int_{Z\times\mathcal{H}} W\Big(z,\ \min_{p\in\Delta S} \int_S U_{(s^t,s)}\big(h(s)\big)\,\mathrm{d}p(s) + c_{s^t}(p)\Big)\,\mathrm{d}\mu(z,h) \tag{5.1}$$

for any $p \in \Delta(Z \times \mathcal{H})$, where the aggregator $W : Z \times R_U \to R_U$ is continuous and strictly increasing in the second argument and cost functions c_{s^t} are grounded, convex, and lower semicontinous. Here, $R_U = \bigcup_{t\geq 1} \bigcup_{s^t\in S^t} \bigcup_{p\in\Delta(Z\times\mathcal{H})} U_{s^t}(p)$.

The axiomatization of recursive variational preferences combines Hayashi's (2005) axiomatization of recursive maxmin expected utility preferences with the axiomatization of Maccheroni et al. (2006a). All of Hayashi's (2005) axioms are retained, except that certainty independence is relaxed to weak certainty independence.

Axiom D1—Order: For any $s^t \in S^t$ relation $\succsim_{s^t}$ is a continuous, complete, transitive, and there exist $y, y' \in Z^\infty$ such that $y \succ_{s^t} y'$.

Axiom D2—Consumption Separability: For any $s^t \in S^t$, $z, z' \in Z$, and $h, h' \in \mathcal{H}$

$$(z, h) \succsim_{s^t} (z, h') \text{ if and only if } (z', h) \succsim_{s^t} (z', h').$$

Axiom D3—Risk Preference: For any $s^t, \hat{s}^t \in S^t$, $z \in Z$, and $d, d' \in \mathcal{D}$

(i) (History-Independence)

$$d \succsim_{s^t} d' \text{ if and only if } d \succsim_{\hat{s}^t} d',$$

(ii) (Stationarity)

$$(z, d) \succsim_{s^t} (z, d') \text{ if and only if } d \succsim_{s^t} d'.$$

Axiom D4—Risk Equivalence Preservation: For any $s^t \in S^t$, $p, p' \in \Delta(Z \times \mathcal{H})$, $d, d' \in \mathcal{D}$, and $\alpha \in (0,1)$

$$[p \sim_{s^t} d \text{ and } p' \sim_{s^t} d'] \Longrightarrow [\alpha p + (1-\alpha)p' \sim_{s^t} \alpha d + (1-\alpha)d'].$$

By Axiom D2, for each $s^t \in S^t$ the preference $\succsim_{s^t}$ over degenerate lotteries of the form (z, h_{+1}) induces a preference over one-step-ahead acts. By a slight abuse of notation this induced preference will also be denoted $\succsim_{s^t}$.

Axiom D5—One-Step-Ahead Variational Preference: For any $s^t \in S^t$, $h, h' \in \mathcal{H}_{+1}$, $d, d' \in \mathcal{D}$, and $\alpha \in (0,1)$

(i) (Weak Certainty Independence)

$$\begin{aligned} &\alpha h_{+1} + (1-\alpha)d \succsim_{s^t} \alpha h'_{+1} + (1-\alpha)d \\ &\qquad\qquad \Longrightarrow \alpha h_{+1} + (1-\alpha)d' \succsim_{s^t} \alpha h'_{+1} + (1-\alpha)d', \end{aligned}$$

(ii) (Uncertainty Aversion)

$$h_{+1} \sim_{s^t} h'_{+1}$$
$$\implies \alpha h_{+1} + (1-\alpha) h'_{+1} \succsim_{s^t} h_{+1}.$$

Axiom D6—Dynamic Consistency: For any $s^t \in S^t$ and $h, h' \in \mathcal{H}_{+1}$

$$\left[h(s) \succsim_{s^t,s} h'(s) \text{ for all } s \in S\right] \implies h \succsim_{s^t} h'.$$

Theorem 5.1: *Family $\{\succsim_{s^t}\}$ satisfies Axioms D1-D6 if and only if it has a variational representation* (5.1).

Proof. This proof adapts the proof of Hayashi's (2005) Theorem 1.

5.2.1.1. Lemmas. The following Lemmas of Hayashi (2005) hold for $\{\succsim_{s^t}\}$

Lemma H8. For any $s^t \in S^t$, $z \in Z$, and $h_{+1}, h'_{+1} \in \mathcal{H}_{+1}$ if $(c, h_{+1}(s)) \succsim_{s^t} (c, h'_{+1}(s))$ for every $s \in S$, then $(c, h_{+1}) \succsim_{s^t} (c, h'_{+1})$.

Lemma H9. For any $s \in S$, $h \in \mathcal{H}$, and $\mu \in \Delta(Z \times \mathcal{H})$ there exist risk equivalents $d, d' \in D$ such that $(z, h) \sim_{s^t} (z, d)$ and $\mu \sim_{s^t} d'$.

Lemma H10. For any $h \in \mathcal{H}$ there exists $h_{+1} \in \mathcal{H}_{+1}$ such that (i) $h(s) \sim_{s^t,s} h_{+1}(s)$ for all $s \in S$, (ii) $(z, h) \sim_{s^t} (z, h_{+1})$.

Hayashi (2005) Lemma 11 relies on C-independence and has to be weakended.

Lemma H11'. For any $s^t \in S^t$, $d, d', d'' \in \mathcal{D}$, and $\alpha \in (0,1)$ if $d \sim_{s^t} d'$ then $\alpha d + (1-\alpha) d'' \sim_{s^t} \alpha d' + (1-\alpha) d''$.

Proof. First show that $\frac{1}{2}d + \frac{1}{2}d'' \sim_{s^t} \frac{1}{2}d' + \frac{1}{2}d''$. This modifies part of the proof of Lemma 28 of Maccheroni et al. (2006a). Towards contradiction, suppose wlog $\frac{1}{2}d + \frac{1}{2}d'' \succ_{s^t}$

$\frac{1}{2}d' + \frac{1}{2}d''$. By Axiom B5($i$), $\frac{1}{2}d + \frac{1}{2}d \succ_{s^t} \frac{1}{2}d' + \frac{1}{2}d$ and, by Axiom B5(i) again, $\frac{1}{2}d + \frac{1}{2}d' \succ_{s^t} \frac{1}{2}d' + \frac{1}{2}d'$; thus $d \succ_{s^t} d'$; contradiction. Second, because continuity implies mixture continuity, the conclusion follows from Theorem 2 of Herstein and Milnor (1953). □

From Axiom B4 and Lemma H11' follows

Lemma H12'. For any $s^t \in S^t$, $\mu, \mu', \mu'' \in \Delta(Z \times \mathcal{H})$, and $\alpha \in (0,1)$ if $\mu \sim_{s^t} \mu'$ then $\alpha\mu + (1-\alpha)\mu'' \sim_{s^t} \alpha\mu' + (1-\alpha)\mu''$.

Following Hayashi (2005), risk preference is uniquely determined by a history-independent preference $\succsim$ over $\mathcal{D}$. By Theorem 2 of Grandmont (1972), $\succsim$ is represented by $U : \mathcal{D} \to \mathbb{R}$ where $U(d) = \int u(z, d')\, \mathrm{d}d(z, d')$. By continuity and compactness, U can be chosen so that $U(\mathcal{D}) = [-M, M]$.

By continuity and Lemma H12', Theorem 2 of Grandmont (1972) implies that $\{\succsim_{s^t}\}$ is represented by a family $\{U_{s^t}\}$ where $U_{s^t}(\mu) : \Delta(Z \times \mathcal{H}) \to \mathbb{R}$ has $U_{s^t} = \int u_{s^t}(z, h)\, \mathrm{d}\mu(z, h)$ with $u_{s^t} : Z \times \mathcal{H} \to \mathbb{R}$ continuous.

By Axiom B2, $u_{s^t} = W_{s^t}(z, u_{s^t}(\hat{z}, h))$ for some fixed $\hat{z} \in Z$. Moreover, as argued by Hayashi (2005), W_{s^t} can be chosen to be independent of history and time. It will be denoted W.

5.2.1.2. Representation over one-step-ahead acts. As before, with a slight abuse of notation let $h_{+1} \succsim_{s^t} h'_{+1}$ iff $(z, h_{+1}) \succsim_{s^t} (z, h'_{+1})$ for some $z \in Z$ (which doesn't matter). By Axiom B1, $\succsim_{s^t}$ is a continuous, non-degenerate preference relation.

Thus, by Axiom B5 and Lemma H8 the assumptions of Maccheroni et al.'s (2006a) Theorem 3 are satisfied. Therefore, there exists a nonconstant affine function $v_{s^t} : \mathcal{D} \to \mathbb{R}$ and a grounded, convex and lower semicontinuous function $c_{s^t} : \Delta S \to [0, \infty]$ such that on H_{+1} preference $\succsim_{s^t}$ is represented by $V_{s^t}(h_{+1}) = \min_{p \in \Delta S} \int v_{s^t} \circ h_{+1}\, \mathrm{d}p + c_{s^t}(p)$ for all

$h_{+1} \in H_{+1}$. By Axiom B3(i), preference $\succsim_{s^t}$ on $\mathcal{D}$ is history independent, so wlog $v_{s^t} = U$. Thus, on H_{+1} preference $\succsim_{s^t}$ is represented by $V_{s^t}(h_{+1}) = \min_{p \in \Delta S} \int U \circ h_{+1}\, \mathrm{d}p + c_{s^t}(p)$ for all $h_{+1} \in H_{+1}$.

Define function $U_{(s^t,h)}$ by $U_{(s^t,h)}(s) = U_{(s^t,s)}\big(h(s)\big)$. The following lemma is proved by Hayashi (2005).

Lemma H13. For any $h \in \mathcal{H}$ there exists $h_{+1} \in \mathcal{H}_{+1}$ such that $U_{(s^t,h)} = U_{(s^t,h_{+1})}$.

Thus, V_{s^t} represents $\succsim_{s^t}$ on the whole of $\mathcal{H}$. The aggregator W and full support of measures obtained as in Hayashi (2005) $\square$

5.2.2. Attitudes Towards the Timing of Objective Risk

As in the model of Kreps and Porteus (1978), the aggregator W in representation (5.1) is responsible for preference for earlier resolution of objective risk. For any $s^t \in S$, $z \in Z$, $d_1, d_2 \in \mathcal{D}$, and $\alpha \in [0,1]$ define $(1, \alpha; z, d_1, d_2)$ to be a temporal lottery where risk is resolved in period 1, i.e., whose chance node for period 0 is degenerate. Formally, define $(1, \alpha; z, d_1, d_2) = (z, \alpha d_1 + (1-\alpha) d_2)$. In contrast, define $(0, \alpha; z, d_1, d_2)$ to be a temporal lottery where risk is resolved already in period 0, i.e., whose chance node for period 0 is not degenerate. Formally , define $(0, \alpha; z, d_1, d_2) = \alpha(z, d_1) + (1-\alpha)(z, d_2)$.

Definition 5.3: Relation $\succsim$ exhibits *preference for [resp., indifference to, preference against] earlier resolution of risk* if

$$(0, \alpha; z, d_1, d_2) \;\succsim_{s^t}\; [\text{resp., } \sim_{s^t}, \precsim_{s^t}]\, (1, \alpha; z, d_1, d_2)$$

for all $t \geq 0$, $s^t \in S^t$, $z \in Z$, $d_1, d_2 \in \mathcal{D}$, and $\alpha \in (0,1)$.

Preference for earlier resolution of purely objective risk is an important feature of preferences studied by Kreps and Porteus (1978), but is conceptually unrelated to uncertainty about subjective states.

Axiom D7—Risk Timing Indifference: Preference $\succsim$ exhibits indifference to earlier resolution of risk.

Another important property of preferences is that tradeoffs between consumption at period t and $t+1$ are independent from consumption at later periods.[3]

Axiom D8—Future Separability: For any $s^t \in S^t$, $d_{0,1}, d'_{0,1} \in \Delta(Z \times \Delta(Z))$, and $y, y' \in Z^\infty$

$$(d_{0,1}, y) \succsim_{s^t} (d'_{0,1}, y) \iff (d_{0,1}, y') \succsim_{s^t} (d'_{0,1}, y').$$

The following theorem extends Hayashi's (2005).

Theorem 5.2: *The family $\{\succsim_{s^t}\}$ satisfies Axioms D1-D8 if and only if the aggregator $W : Z \times R_U \to \mathbb{R}_U$ in (5.1) has the form $W(z,r) = u(z) + \beta r$; thus, the family is represented by*

$$U_{s^t}(\mu) = \int_{Z\times\mathcal{H}} \left\{ u(z) + \beta \left[\min_{p \in \Delta S} \int_S U_{(s^t,s)}\big(h(s)\big)\, \mathrm{d}p(s) + c_{s^t}(p) \right] \right\} \mathrm{d}\mu(z,h)$$

where $u : Z \to \mathbb{R}$ is continuous and nonconstant and $\beta \in (0,1)$. Moreover, β is unique and the function u is unique up to positive affine transformations.

Proof. Follows from the proof of Corollary 1 in Hayashi (2005), which does not rely on certainty independence. □

[3]This is Assumption 5 in Epstein (1983) and Axiom 8 in Hayashi (2005).

CHAPTER 6

Axiomatic Foundations of Dynamic Multiplier Preferences

6.1. Axiomatization of dynamic Multiplier Preferences

Definition 6.1: Family $\{\succsim_{s^t}\}$ is a dynamic multiplier preference if it is represented by

$$U_{s^t}(\mu) = \int_{Z\times\mathcal{H}} \left\{ u(z) + \beta \left[\min_{p\in\Delta(S)} \int_S U_{(s^t,s)}\big(h(s)\big)\, \mathrm{d}p(s) + \theta R(p\|q_{s^t}) \right] \right\} \mathrm{d}\mu(z,h) \tag{6.1}$$

where $u : Z \to \mathbb{R}$ is continuous and nonconstant, $\beta \in (0,1)$, $\theta \in (0,\infty]$ and $q \in \Delta(S)$.

The reference probability q_{s^t} in representation (6.1) can be history dependent, which is natural in non-stationary environments or when learning takes place. However, the parameter θ is history-independent. Thus, a separation is achieved between the attitude towards model uncertainty, which is constant, and the uncertainty itself can depend on the history of shocks, reflecting possible persistence of shocks or learning about the environment.

In the static version of the model, Savage's axioms were used to characterize multiplier preferences. Because those axioms rely on infiniteness of the state space and in the present setting S is finite, a different approach will be used, that of Wakker's *tradeoff consistency* (see, e.g., Köbberling and Wakker, 2003).[1]

[1]Using a construction of the space $\mathcal{H}$ that accommodates infinite S (as described in footnote 1) would allow to replace tradeoff consistency with Savage axioms.

Relation $\sim^*_{s^t}$ introduced below compares tradeoffs between pairs of temporal lotteries. Pair $[d_1, d_2]$ is in relation with pair $[d_3, d_4]$ if the utility difference between d_1 and d_2 is the same as the utility difference between d_3 and d_4.

Definition 6.2: For any $d_1, d_2, d_3, d_4 \in \mathcal{D}$ define $[d_1, d_2] \sim^*_{s^t} [d_3, d_4]$ if there exist acts $h'_{+1}, h''_{+1} \in \mathcal{H}_{+1}$, and a s^t-nonnull state[2] $s \in S$ such that

$$(d_1)s(h'_{+1}) \sim_{s^t} (d_2)s(h''_{+1}) \ \text{ and } \ (d_3)s(h'_{+1}) \sim_{s^t} (d_4)s(h''_{+1}).$$

Axiom B1—Tradeoff Consistency: For any $s^t \in S^t$ and $d_1, d_2, d_3, d_4 \in \mathcal{D}$ if $[d_1, d_2] \sim^*_{s^t} [d_3, d_4]$, then improving any of the outcomes breaks the relation.

Axiom B4 implies multiplier representation of preferences in each period

$$U_{s^t}(\mu) = \int_{Z\times\mathcal{H}} \left\{ u(z) + \beta \left[\min_{p\in\Delta S} \int_S U_{(s^t,s)}\big(h(s)\big)\,\mathrm{d}p(s) + \theta_{s^t} R(p\|\,q_{s^t}) \right] \right\} \mathrm{d}\mu(z,h),$$

but allows the concern for model misspecification to be time- and state- dependent. The following axiom guarantees constant θ.

Axiom B2—Stationary Tradeoff Consistency: Relation $\sim^*_{s^t}$ is independent of s^t.

Theorem 6.1: *Suppose that $\{\succsim_{s^t}\}$ is a dynamic variational preference. Then Axioms B1 and B2 are necessary and sufficient for $\{\succsim_{s^t}\}$ to be a dynamic multiplier preference. Moreover, $\big(\theta, u, \{q_{s^t}\}\big)$ and $\big(\theta', u', \{q'_{s^t}\}\big)$ represent the same dynamic multiplier preference if and only if $q'_{s^t} = q_{s^t}$ for all s^t and there exists $a > 0$ and $b \in \mathbb{R}$ such that $u' = au + b$ and $\theta' = a\theta$.*

[2] A state s is s^t-non-null if there exist $h'_{+1}, h''_{+1}, g_{+1} \in \mathcal{H}_{+1}$ such that $(h'_{+1})s(g_{+1}) \succ_{s^t} (h''_{+1})s(g_{+1})$.

6.2. Proof of Theorem 8.2

By Theorem 5.2, $\succsim_{s^t}$ is represented by

$$U_{s^t}(\mu) = \int_{Z\times\mathcal{H}} \left\{ u(z) + \beta \left[\min_{p\in\Delta S} \int_S U_{(s^t,s)}\big(h(s)\big)\, \mathrm{d}p(s) + c_{s^t}(p) \right] \right\} \mathrm{d}\mu(z,h).$$

Thus, $\succsim_{s^t}$ on $\mathcal{H}_{+1}$ is represented by V_{s^t} where $V_{s^t}(h_{+1}) = I_{s^t}(U_{(s^t,h_{+1})})$, where, as before $U_{(s^t,h_{+1})}$ is defined as $U_{(s^t,h_{+1})}(s) = U_{(s^t,s)}\big(h_{+1}(s)\big)$.

Observe that $\succsim_{s^t}$ on $\mathcal{H}_{+1}$ is continuous, monotone (by Lemma H8), and satisfies tradeoff consistency (by Axiom B4). Moreover, $\mathcal{D}$ is a connected topological space. Thus, by Corollary 10 of Köbberling and Wakker (2003) there exists a unique probability $q_{s^t} \in \Delta(S)$ and a continuous function $\phi : \mathcal{D} \to \mathbb{R}$ that represents $\succsim_{s^t}$. Moreover, function ϕ is unique up to positive affine transformations.

As in other proofs, translation invariance of I_{s^t} leads to the Pexider equation for ϕ. As verified by Klibanoff et al. (2005) in the proof of their Proposition 2, even when the domain of ϕ is a bounded interval, as is the case here because of the compactness of $\Delta(Z)$ and continuity of u, the only solutions of the Pexider equation are ϕ_θ, where $\theta \in (0,\infty]$ is uniquely pinned down.

Because relation $\sim^*_{s^t}$ is constant across s^t, the scalar θ_{s^t} is constant across s^t. To see that, for each θ define $x(\theta) < 0$ which satisfies $\phi_\theta(1) - \phi_\theta(0) = \phi_\theta(0) - \phi_\theta(x)$. Thus, $x(\theta)$ is implicitly defined by $\Phi(\theta, x) = \phi_\theta(1) + \phi_\theta(x) - 2\phi_\theta(0)$. By the implicit function theorem, $\frac{\mathrm{d}x}{\mathrm{d}\theta} = -\frac{\mathrm{d}\Phi}{\mathrm{d}\theta} / \frac{\mathrm{d}\Phi}{\mathrm{d}x} = \frac{\exp(-\theta^{-1}) - x\cdot\exp(-x\theta^{-1})}{\theta\cdot\exp(-x\theta^{-1})} > 0$. Thus, for any two different values of θ, the corresponding values of $x(\theta)$ are different.

Let $s^t, \hat{s}^{\hat{t}}$ be distinct histories of possibly different length and recall that $\succsim$ over $\mathcal{D}$ is history independent. Let $U(d) = 1, U(d^*) = 0$, and assume that $U(d_{s^t}) = x(\theta_{s^t})$ and

$U(d_{\hat{s}\hat{t}}) = x(\theta_{\hat{s}\hat{t}})$. Observe that $[d, d^*] \sim^*_{s^t} [d^*, d_{s^t}]$ and $[d, d^*] \sim^*_{\hat{s}\hat{t}} [d^*, d_{\hat{s}\hat{t}}]$. If $\theta_{\hat{s}\hat{t}} \neq \theta_{s^t}$, then, wlog, $x(\theta_{\hat{s}\hat{t}}) > x(\theta_{s^t})$, so $d_{\hat{s}\hat{t}}$ is an improvement over d_{s^t}. This contradicts the equality $\sim^*_{\hat{s}\hat{t}} = \sim^*_{s^t}$ and tradeoff consistency of both $\sim^*_{\hat{s}\hat{t}}$ and $\sim^*_{s^t}$. $\square$

CHAPTER 7

Preference for Earlier Resolution of Uncertainty and Variational Preferences

7.1. Stationary Variational Preferences and IID Ambiguity

The discussion so far has concentrated on variational preferences where the utility function u and the discount factor β are constant, but the cost function c_{s^t} is allowed to depend on the history s^t. For example, in the case of multiplier preferences, the reference measure q_{s^t} can be history-dependent. This section introduces a class of *stationary* preferences, where the preference on one-step-ahead acts is the same in every time period. This permits writing $\succsim$ instead of $\succsim_{s^t}$.

Definition 7.1: Relation $\succsim$ is a Stationary Variational Preference if it is represented by function $U : \mathcal{H} \to \mathbb{R}$

$$U(\mu) = \int_{Z\times\mathcal{H}} \left\{ u(z) + \beta \left[\min_{p\in\Delta S} \int_S U\big(h(s)\big)\,\mathrm{d}p(s) + c(p) \right] \right\} \mathrm{d}\mu(z,h) \tag{7.1}$$

for $\beta \in (0,1)$, $u : Z \to \mathbb{R}$, and some grounded, convex, and lower semicontinous cost function c.

This definition extends the notion of *IID Ambiguity* studied by Chen and Epstein (2002) and Epstein and Schneider (2003a) in the context of maxmin expected utility to the class of variational preferences. Intuitively, IID ambiguity means that every period the decision maker faces a new Ellsberg urn. His ex-ante beliefs about each urn are identical, but because he observes only one draw from each urn, he cannot make inferences across urns.[1]

Because the uncertainty that the decision maker faces in period t is identical to the uncertainty in period $t+1$, and the only property that distinguishes them is the timing of their resolution, attitudes towards such timing of resolution can be studied.

7.2. Attitudes Towards the Timing of Subjective Uncertainty

The main objective of this section is to determine which of the stationary variational preferences exhibit preference for earlier resolution of uncertainty. In order to do so, some notation will be introduced. Let $h_{+1} \in \mathcal{H}_{+1}$ be a one-step-ahead act and $z \in Z$ be a deterministic payoff. Define $(1; z, h_{+1})$ to be a temporal act where the subjective uncertainty about h_{+1} is resolved in period 1, i.e., whose chance node for period 0 is degenerate. Formally, define $(1; z, h_{+1})(s) = (z, h_{+1})$ for all $s \in S$.

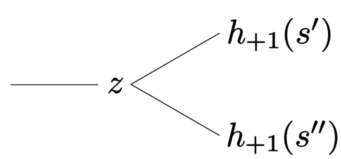

Figure 7.1. Uncertainty resolves tomorrow

On the other hand, define $(0; z, h_{+1})$ to be a one-step-ahead act where the subjective uncertainty about h_{+1} is resolved already in period 0, i.e., whose chance node for period 0 is not degenerate. Formally, define $(0; z, h_{+1})(s) = (z, h_{+1}(s))$ for all $s \in S$.

[1]This failure of inference is known in econometrics as the problem of incidental parameters (see, e.g., Neyman and Scott, 1948).

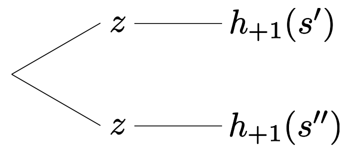

Figure 7.2. Uncertainty resolves today

Note that both in $(0; z, h_{+1})$ and in $(1; z, h_{+1})$ the payoffs of h_{+1} are delivered in period 1. The difference is *when* the decision maker learns about them. Some decision makers may prefer one to the other.

Definition 7.2: Relation $\succsim$ exhibits *preference for [resp., indifference to, preference against] earlier resolution of uncertainty* if

$$(0; z, h_{+1}) \;\succsim\; [\text{resp.,} \;\; \sim, \precsim] \, (1; z, h_{+1})$$

for all $z \in Z$, and $h_{+1} \in \mathcal{H}_{+1}$.

Given this definition, the preference for earlier resolution of uncertainty can be studied in the class of stationary variational preferences. One initial observation is that stationary multiplier preferences, which are represented by

$$U(\mu) = \int_{Z\times\mathcal{H}} \left[u(z) + \beta\phi_\theta^{-1}\Big(\int_S \phi_\theta\big(U\big(h(s)\big)\big)\, \mathrm{d}q(s) \Big) \right] \mathrm{d}\mu(z, h) \tag{7.2}$$

exhibit strict preference for earlier resolution of uncertainty (unless $\theta = \infty$).

Theorem 7.1: *Suppose $\succsim$ is a stationary multiplier preferencewith $\theta < \infty$. Then for all $z \in Z$, and $h_{+1} \in \mathcal{H}_{+1}$ such that $h_{+1}(s) \nsim h_{+1}(s')$ for some $s \neq s'$*

$$(0; z, h_{+1}) \;\succ\; (1; z, h_{+1}).$$

Similarly to stationary multiplier preferences, stationary second-order variational preferences, which are represented by

$$U(\mu) = \int_{Z\times\mathcal{H}} \left[u(z) + \beta\phi_\theta^{-1}\left(\min_{q\in Q}\int_S \phi_\theta\big(U\big(h(s)\big)\big)\,\mathrm{d}q(s)\right)\right]\mathrm{d}\mu(z,h) \tag{7.3}$$

exhibit strict preference for earlier resolution of uncertainty (unless $\theta = \infty$).

Theorem 7.2: *Suppose $\succsim$ is a stationary second-order risk-averse variational preference with $\theta < \infty$. Then for all $z \in Z$, and $h_{+1} \in \mathcal{H}_{+1}$ such that $h_{+1}(s) \nsim h_{+1}(s')$ for some $s \neq s'$*

$$(0; z, h_{+1}) \succ (1; z, h_{+1}).$$

Both in Theorem 8.3 and in Theorem 7.2 the preference for earlier resolution of uncertainty appears to be connected to the function ϕ_θ. Indeed, the strength of the preference depends on the parameter θ; in the extreme case of $\theta = \infty$ the indifference obtains. By Theorem 10.1, the second-order risk-averse variational preferences are the largest subclass of variational preferences with representation

$$U(\mu) = \int_{Z\times\mathcal{H}} \left[u(z) + \beta\phi_\theta^{-1}\left(\min_{p\in\Delta(S)}\int_S \phi_\theta\big(U\big(h(s)\big)\big)\,\mathrm{d}p(s) + c(p)\right)\right]\mathrm{d}\mu(z,h)$$

For this reason, it may be tempting to conclude that all other variational preferences satisfy indifference to the timing of resolution of uncertainty. However, as the next theorem shows, quite the opposite is true.

Theorem 7.3: *Suppose that $\succsim$ is a stationary variational preference. Relation $\succsim$ satisfies indifference to the timing of resolution of uncertainty if and only if it is a stationary* *maxmin expected utility preference, i.e., it is represented by*

$$U(\mu) = \int_{Z\times\mathcal{H}} \left[u(z) + \beta \min_{q\in Q} \int_S U\big(h(s)\big)\,\mathrm{d}q(s) \right] \mathrm{d}\mu(z,h). \tag{7.4}$$

Theorem 8.4 asserts that stationary variational preferences typically exhibit preference for earlier resolution of uncertainty. The only class that satisfies indifference is precisely the class of stationary maxmin expected utility preferences studied by Chen and Epstein (2002) and Epstein and Schneider (2003a).

7.3. Proofs

7.3.1. Proof of Theorem 8.3

Let $u_s = U(h_{+1}(s))$ and let $q_s = q(\{s\})$. Observe that

$$\begin{aligned} U(0; z, h_{+1}) &= \phi_\theta^{-1}\bigg(\sum_{s\in S} \phi_\theta(u(z) + \beta u_s) q_s \bigg) \\ &= \phi_\theta^{-1}\bigg(\sum_{s\in S} \big[-\phi_\theta(u(z)) \cdot \phi_\theta(\beta u_s) \big] q_s \bigg) \\ &= u(z) + \phi_\theta^{-1}\bigg(\sum_{s\in S} \phi_\theta(\beta u_s) q_s \bigg) \end{aligned}$$

and

$$U(1; z, h_{+1}) = u(z) + \beta\phi_\theta^{-1}\bigg(\sum_{s\in S} \phi_\theta(u_s) q_s \bigg).$$

Thus, $U(0; z, h_{+1}) > U(1; z, h_{+1})$ if and only if

$$\frac{1}{\beta}\phi_\theta^{-1}\bigg(\sum_{s\in S} \phi_\theta(\beta u_s) q_s \bigg) > \phi_\theta^{-1}\bigg(\sum_{s\in S} \phi_\theta(u_s) q_s \bigg)$$

if and only if

$$\phi_{\frac{\theta}{\beta}}^{-1}\Big(\sum_{s\in S}\phi_{\frac{\theta}{\beta}}(u_s)q_s\Big) > \phi_{\theta}^{-1}\Big(\sum_{s\in S}\phi_{\theta}(u_s)q_s\Big). \tag{7.5}$$

Because $\beta < 1$, the function ϕ_θ is a strictly concave transformation of $\phi_{\frac{\theta}{\beta}}$. Moreover, $q_s > 0$ for all $s \in S$ and by assumption there exist $s', s'' \in S$ such that $u_{s'} \neq u_{s''}$. Thus, inequality (7.5) follows from Jensen's inequality. □

7.3.2. Proof of Theorem 7.2

Follows from the reasoning in the proof of Theorem 8.3. □

7.3.3. Proof of Theorem 8.4

Let $\succsim$ be a stationary variational preference represented by

$$U(\mu) = \int_{Z\times\mathcal{H}}\Big[u(z) + \beta\min_{p\in\Delta S}\int_S U\big(h(s)\big)\,\mathrm{d}p(s) + c(p)\Big]\,\mathrm{d}\mu(z,h)$$

As before, $U(\mathcal{D}) = [-M, M] =: \mathcal{V}$. Define niveloid $I : B_0(\Sigma,\mathcal{V}) \to \mathbb{R}$ as $I(\xi) = \min_{p\in\Delta(S)}\int \xi\,\mathrm{d}p + c(p)$.

Suppose that $\xi \in B_0(\Sigma,\mathcal{V})$. For each $s \in S$ the value $\xi(s) \in \mathcal{V}$, so there exists $d_s \in \mathcal{D}$ such that $U(d_s) = \xi(s)$. Define $h \in \mathcal{H}_{+1}$ by $h(s) = d_s$ for all $s \in S$. Let $z_0, z_1 \in Z$. Because $\succsim$ satisfies indifference to the timing of resolution of uncertainty, $(z_0,(0;z_1,h)) \sim (z_0,(1;z_1,h))$. Thus, $u(z_0) + \beta I\big(u(z_1) + \beta\xi\big) = u(z_0) + \beta\big(u(z_1) + \beta I(\xi)\big)$. Hence, by translation invariance, $I(\beta\xi) = \beta I(\xi)$ for any $\xi \in B_0(\Sigma,\mathcal{V})$.

Let $0 < b < \beta$ and suppose that there exists $\xi \in B_0(\Sigma,\mathcal{V})$ such that $I(b\xi) \neq bI(\xi)$. Observe that, $I(b\xi) = I(b\xi + (1-b)0) \geq bI(\xi)$, by concavity and because $I(0) = 0$.

Thus, $I(b\xi) > bI(\xi)$. Moreover, $I(\beta^n\xi) = I(\beta\beta^{n-1}\xi) = \beta I(\beta^{n-1}\xi) = \cdots = \beta^n I(\xi)$ for any $n \in \mathbb{N}$. Choose n such that $\beta^n < b$. For this n it follows that $\beta^n I(\xi) = I(\beta^n\xi) = I\big(\frac{\beta^n}{b}b\xi + \frac{b-\beta^n}{b}0\big) \geq \frac{\beta^n}{b}I(b\xi) > \beta^n I(\xi)$. Contradiction.

Let $\beta < b < 1$ and suppose that there exists $\xi \in B_0(\Sigma, \mathcal{V})$ such that $I(b\xi) \neq bI(\xi)$. As above $I(b\xi) > bI(\xi)$ follows. Moreover, $I(b^n\xi) = I(b^{n-1}b\xi) \geq b^{n-1}I(b\xi) > b^n I(\xi)$ for any $n \in \mathbb{N}$. Choose n such that $b^n < \beta$. Contradiction with the case $0 < b < \beta$.

As a consequence, I is a niveloid on $B_0(\Sigma, \mathcal{V})$ that is homogenous of degree one. Extend I to $B_0(\Sigma)$ by homogeneity. Observe that the extension is a normalized niveloid, thus it satisfies the assumptions of Lemma 3.5 of Gilboa and Schmeidler (1989); therefore, there exists a closed and convex set $C \subseteq \Delta(S)$ such that $I(\xi) = \min_{p \in C} \int \xi \, \mathrm{d}p$ for all $\xi \in B_0(\Sigma, \mathcal{V})$. $\square$

Part 3

Ambiguity and Timing—Second Setting

CHAPTER 8

Ambiguity and Timing—Second Setting

8.1. Setup

The setup in this paper follows Maccheroni et al. (2006b) and Epstein and Schneider (2003b). The time is finite $\mathcal{T} = \{0, 1, \ldots, T\}$ and there is a finite state space Ω and a fixed event tree $\{\mathcal{G}_t\}_{t=0}^{T}$ with $\mathcal{G}_0 = \{\Omega\}$, $\mathcal{G}_{t+1}$ finer than $\mathcal{G}_t$, and $\mathcal{G}_T = \{\{\omega\} \mid \omega \in \Omega\}$. For any $\omega \in \Omega$ and $t \in T$ let $G_t(\omega)$ denote the cell of the partition $\mathcal{G}_t$ that contains ω. Let $\Delta(\Omega)$ denote the set of all probability distributions on Ω and let $\Delta^{++}(\Omega)$ denote the set of full-support probability distributions on Ω. For any $p \in \Delta(\Omega)$ let $p_{|\mathcal{G}_t}$ denote the restriction of p to the algebra $\mathcal{A}(\mathcal{G}_t)$ generated by $\mathcal{G}_t$. Let $(\Omega, \mathcal{G}_t)$ denote the set of all probability distributions on $\mathcal{A}(\mathcal{G}_t)$. For any $p \in \Delta(\Omega)$ and any $G \subset \Omega$ such that $p(G) > 0$ let $p_G \in \Delta(\Omega)$ be the conditional distribution

$$p_G(\omega) = \begin{cases} p(\omega)/p(G) & \text{if } \omega \in G, \\ 0 & \text{otherwise.} \end{cases}$$

For any $G \subseteq \Omega$ let $\Delta(\mathcal{G})$ denote the set of all probability distributions on Ω with support contained in G. Finally, for any $p \in \Delta(\Omega)$, any $t \in \mathcal{T}$, and any $\omega \in \Omega$ let $p^{+1}_{G_t(\omega)} \in \Delta(\Omega, \mathcal{G}_{t+1})$ be the one-step-ahead conditional of p at time t, i.e., $p^{+1}_{G_t(\omega)} = \left(p_{G_t(\omega)}\right)_{|\mathcal{G}_{t+1}}$.

The object of choice are acts $h = (h_0, h_1, \ldots, h_T)$ which are lottery valued adapted processes, i.e., each $h_t : \Omega \to \Delta(Z)$ is G_t-measurable. Let $\mathcal{H}_t$ denote the set of all $\mathcal{G}_t$-measurable acts $h_t : \Omega \to \Delta(Z)$ and let $\mathcal{H} = \prod_{t \in T} \mathcal{H}_t$ denote the set of all acts. The decision maker is endowed with a family of conditional preferences $\succsim_{t,\omega}$ over $\mathcal{H}$.

8.2. Dynamic Variational Preferences

Maccheroni et al. (2006b) introduced and axiomatized the class of dynamic variational preferences. In order to do so, they introduced the notion of a dynamic ambiguity index.

Definition 8.1: A *dynamic ambiguity index* is a family $\{c_t\}_{t \in \mathcal{T}}$ of functions $c_t : \Omega \times \Delta(\Omega) \to [0, \infty]$ such that for all $t \in \mathcal{T}$:

(i) $c_t(\cdot, p) : \Omega \to [0, \infty]$ is $\mathcal{G}_t$-measurable for all $p \in \Delta(\Omega)$,

(ii) $c_t(\omega, \cdot) : \Delta(\Omega) \to [0, \infty]$ is grounded, closed, and convex, with dom $c_t(\omega, \cdot) \subseteq \Delta(G_t(\omega))$ and dom $c_t(\omega, \cdot) \cap \Delta^{++}(G_t(\omega)) \neq \emptyset$ for all $\omega \in \Omega$.

Definition 8.2: The family $\succsim_t, \omega$ has a *dynamic variational representation* if and only if it is represented by

$$V_t(\omega, h) = \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau \geq t} \beta^{\tau - t} u(h_\tau) \, \mathrm{d}p_{G_t(\omega)} + c_t(\omega, p_{G_t(\omega)}) \right), \tag{8.1}$$

where $u : \Delta(Z) \to \mathbb{R}$ is unbounded and affine, $\beta > 0$, and $\{c_t\}$ is a dynamic ambiguity index.

An additional assumption made in this paper will be that the discount factor β is less than one, which is a natural requirement in settings where these dynamic models are applied. The following simple axiom is a behavioral counterpart of this assumption.

Axiom B3—Impatience: For any $\omega \in \Omega$, any $t < T$, any $\rho_0, \rho_1, \ldots, \rho_T \in \Delta(Z)$, and any $\pi, \sigma \in \Delta(Z)$ if $(\pi, \ldots, \pi) \succ_{t,\omega} (\sigma, \ldots, \sigma)$ then

$$(\rho_0, \rho_1, \ldots, \rho_{t-1}, \pi, \sigma, \rho_{t+2}, \ldots, \rho_T) \succ_{t,\omega} (\rho_0, \rho_1, \ldots, \rho_{t-1}, \sigma, \pi, \rho_{t+2}, \ldots, \rho_T).$$

This axiom means that if π is preferred to σ then the decision maker would rather receive π first. It is easy to see that Axiom B3 is equivalent to $\beta < 1$.

Preferences described in Definition 8.2 do not in general satisfy dynamic consistency, which is a key requirement in any model of dynamic behavior. Maccheroni et al. (2006b) discuss the following definition of dynamic consistency (see also Epstein and Schneider, 2003b).

Definition 8.3: For each $(t, \omega) \in \mathcal{T} \times \Omega$ with $t < T$, and all $h, h' \in \mathcal{H}$, if $h_\tau = h'_\tau$ for all$\tau \leq t$ and $h \succsim_{t+1,\omega'} h'$ for all $\omega' \in \Omega$ then $h \succsim_{t,\omega} h'$.

As Theorem 1 of Maccheroni et al. (2006b) shows, Dynamic Consistency is equivalent to a certain restriction on the dynamic ambiguity index, notably the so-called "no-gain condition"

$$c_t(\omega, q) = \beta \sum_{\substack{G \in \mathcal{G}_{t+1} \\ q(G)>0}} q(G) c_{t+1}(G, q_G) + \min_{\{p \in \Delta(G_t(\omega)) | p_{|\mathcal{G}_{t+1}} = q_{|\mathcal{G}_{t+1}}\}} c_t(\omega, p). \tag{8.2}$$

Because of the "no-gain condition" it is possible to write the representation (8.1) recursively

$$V_t(\omega, h) = u(h_t) + \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\beta \int V_{t+1}(h) \,\mathrm{d}r + \gamma_t(\omega, r) \right), \tag{8.3}$$

where for all $r \in \Delta(\Omega, \mathcal{G}_{t+1})$

$$\gamma_t(\omega, r) = \min_{\{p \in \Delta(G_t(\omega)) | p_{|\mathcal{G}_{t+1}} = r\}} c_t(\omega, p). \tag{8.4}$$

8.3. Recursive Multiplier Preferences

8.3.1. Discounted Entropy

Dynamic multiplier preferences are a special case of dynamic variational preferences where the dynamic ambiguity index $\{c_t\}$ takes a form of relative entropy. In a dynamic setting the relative entropy can be defined on probability measures over the entire state space Ω or on measures over the one period signal. In the first case, given $p, q \in \Delta(\Omega)$ the relative entropy of p with respect to q is given by

$$R(p \| q) := \begin{cases} \sum_{\omega \in \Omega} \log \frac{p(\omega)}{q(\omega)} p(\omega) & \text{if } p \ll q, \\ \infty & \text{otherwise.} \end{cases} \tag{8.5}$$

In the second case, given $p, q \in \Delta(\Omega, \mathcal{G}_t)$ the relative entropy of p with respect to q on $\mathcal{G}_t$ is given by

$$R_{G_t}(p \| q) := \begin{cases} \sum_{G \in \mathcal{G}_t} \log \frac{p(G)}{q(G)} p(G) & \text{if } p \ll q, \\ \infty & \text{otherwise.} \end{cases} \tag{8.6}$$

It is well known that the relative entropy of p with respect to q can be decomposed into the sum of relative entropies between conditional one-step-ahead probabilities of one period signals

$$R(p\| q) = \sum_{\omega\in\Omega} \left[\sum_{\tau=0}^{T-1} R_{\mathcal{G}_{\tau+1}}\big(p^{+1}_{G_\tau(\omega)} \| \, q^{+1}_{G_\tau(\omega)}\big) \right] p(\omega). \tag{8.7}$$

The literature on applications of robust control to dynamic settings initiated by Hansen and Sargent (2001) uses a different—discounted—version of entropy

$$R^d(p\| q) := \sum_{\omega\in\Omega} \left[\sum_{\tau=0}^{T-1} \beta^{\tau+1} R_{\mathcal{G}_{\tau+1}}\big(p^{+1}_{G_\tau(\omega)} \| \, q^{+1}_{G_\tau(\omega)}\big) \right] p(\omega), \tag{8.8}$$

where $\beta \in (0,1)$ is the discount factor. Note that for any $t < T$

$$R^d\big(p_{G_t(\omega)} \| \, q_{G_t(\omega)}\big) = \sum_{\omega\in\Omega} \left[\sum_{\tau=t}^{T-1} \beta^{\tau+1} R_{\mathcal{G}_{\tau+1}}\big(p^{+1}_{G_\tau(\omega)} \| \, q^{+1}_{G_\tau(\omega)}\big) \right] p(\omega). \tag{8.9}$$

8.3.2. Multiplier Preferences

The direct discrete time analogue of multiplier preferences studied in Hansen and Sargent (2001) is defined using discounted relative entropy (see Definition 8.4 below). Maccheroni et al. (2006b) who also study multiplier preferences used undiscounted relative entropy (see Definition 8.5 below and Section 5.2 of Maccheroni et al., 2006b). Both of those definition yield dynamically consistent preferences, the difference lies in their stationarity properties: if the entropy is discounted then the decision maker's attitude towards ambiguity is constant over time, if the entropy is not discounted then the decision maker's attitude towards ambiguity is fading away as time passes.

Definition 8.4: The family $\succsim_t, \omega$ has a *discounted multiplier representation* if and only if it is represented by

$$(8.10) \qquad V_t(\omega, h) = \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau \geq T} \beta^{\tau - t} u(h_\tau) \, \mathrm{d}p_{G_t(\omega)} + \theta \beta^{-t} R^d \big(p_{G_t(\omega)} \, \| \, q_{G_t(\omega)} \big) \right),$$

where $u : \Delta(Z) \to \mathbb{R}$ is affine, $\beta \in (0,1)$, $\theta \in (0, \infty]$, and $q \in \Delta^{++}(\Omega)$.

It can be shown (by a reasoning analogous to the proof of Theorem 3 of Maccheroni et al., 2006b) that discounted multiplier preferences satisfy dynamic consistency and therefore have a recursive representation

$$(8.11) \qquad V_t(\omega, h) = u(h_t(\omega)) + \beta \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\int V_{t+1}(h) \, \mathrm{d}r + \theta R_{G_{t+1}} \big(r \, \| \, q^{+1}_{G_t(\omega)} \big) \right).$$

Definition 8.5: The family $\succsim_t, \omega$ has an *undiscounted multiplier representation* if and only if it is represented by

$$(8.12) \qquad V_t(\omega, h) = \inf_{p \in \Delta^{++}(\Omega)} \left(\int \sum_{\tau \geq T} \beta^{\tau - t} u(h_\tau) \, \mathrm{d}p_{G_t(\omega)} + \theta \beta^{-t} R \big(p_{G_t(\omega)} \, \| \, q_{G_t(\omega)} \big) \right),$$

where $u : \Delta(Z) \to \mathbb{R}$ is affine, $\beta \in (0,1)$, $\theta \in (0, \infty]$, and $q \in \Delta^{++}(\Omega)$.

By Theorem 3 of Maccheroni et al. (2006b) those preferences satisfy dynamic consistency and therefore have a recursive representation

$$(8.13) \quad V_t(\omega, h) = u(h_t(\omega)) + \beta \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\int V_{t+1}(h) \, \mathrm{d}r + \theta \beta^{-(t+1)} R_{G_{t+1}} \big(r \, \| \, q^{+1}_{G_t(\omega)} \big) \right).$$

As can be seen from equations (8.11) and (8.13), using the discounted relative entropy has an advantage in that it renders a *recursive* representation while using the undiscounted relative entropy necessitates an adjustment factor $\beta^{-(t+1)}$. This means that a decision maker with discounted multiplier preferences has a stationary attitude towards ambiguity: in any time period he attaches the same weight θ to the distortions of the one-step-ahead conditionals. On the other hand, a decision maker with discounted multiplier preferences has a fading attitude towards ambiguity: his multiplier increases with time.

These two preferences could be used for modeling the same decision maker is *different* informational environments. In the first scenario (discounted entropy) the uncertainty is inherently unlearnable: no matter how much information the decision maker receives he will still perceive ambiguity about the future. For example (see, e.g. Epstein and Schneider, 2007), imagine a decision maker facing a sequence of Ellsberg urns; the decision maker is given the same ex-ante information about each urn but doesn't know whether the composition of balls is actually identical in each urn. Each period a ball will be drawn from a different urn (only once). In this scenario it is impossible for the decision maker to make any inferences from past observations. Thus, the decision maker's ambiguity aversion is persistent.

On the other undiscounted entropy corresponds to a scenario where it is eventually possible to learn the probability distribution. For example (see, e.g. Epstein and Schneider, 2007) imagine that the decision maker draws over and over again from the *same* urns. At the beginning he will perceive ambiguity; however, as time passes he will learn his way out of ambiguity and in the limit will be certain of the composition of the urn. Thus, the decision maker's ambiguity aversion is fading as time passes.

The literature applying robust control to macroeconomics has focused on settings where ambiguity is unlearnable and the vast majority of papers use the discounted multiplier preferences. For this reason the axiomatization of this paper focuses on discounted entropy.

8.3.3. Axiomatization

The reference probability $q^{+1}_{G_t(\omega)}$ in representation (8.11) can be history dependent, which is natural in non-stationary environments or when learning takes place. However, the parameter θ is history-independent. Thus, a separation is achieved between the attitude towards model uncertainty, which is constant, and the uncertainty itself can depend on the history of shocks, reflecting possible persistence of shocks or learning about the environment.

In the static version of the model, Savage's axioms were used to characterize multiplier preferences. Because those axioms rely on infiniteness of the state space and in the present setting S is finite, a different approach will be used, that of Wakker's *tradeoff consistency* (see, e.g., Köbberling and Wakker, 2003). This axiom is expressed by means of the following relation $\asymp$. Roughly speaking this relation compares tradeoffs between pairs of lotteries: the pair $[\pi_1, \pi_2]$ is in relation $\asymp$ with pair $[\pi_3, \pi_4]$ if the "utility difference" between π_1 and π_2 is the same as the "utility difference" between π_3 and π_4. Because preferences are defined at every (t, ω) node, there will be a tradeoff relation defined at each node.

Definition 8.6: For any $\pi_1, \pi_2, \pi_3, \pi_4 \in \Delta(Z)$ define $[\pi_1, \pi_2] \asymp_{t,\omega} [\pi_3, \pi_4]$ if there exist acts $h'_{t+1}, h''_{t+1} \in \mathcal{H}_{t+1}$, and an t, ω-nonnull event $G \in \mathcal{G}_{t+1}$ such that

$$(\rho_0, \ldots, \rho_t, \pi_1 G h'_{t+1}, \rho_{t+2}, \ldots, \rho_T) \sim_{t,\omega} (\rho_0, \ldots, \rho_t, \pi_2 G h''_{t+1}, \rho_{t+2}, \ldots, \rho_T)$$

and

$$(\rho_0, \ldots, \rho_t, \pi_3 G h'_{t+1}, \rho_{t+2}, \ldots, \rho_T) \sim_{t,\omega} (\rho_0, \ldots, \rho_t, \pi_4 G h''_{t+1}, \rho_{t+2}, \ldots, \rho_T)$$

for some $\rho_0, \ldots, \rho_T \in \Delta(Z)$. Observe, that the choice of ρs does not matter.

Axiom B4—Tradeoff Consistency: For any $(t, \omega) \in \mathcal{T} \times \Omega$ and $\pi_1, \pi_2, \pi_3, \pi_4 \in \Delta(Z)$ if $[\pi_1, \pi_2] \asymp_{t,\omega} [\pi_3, \pi_4]$, then this relation does not hold if any of the lotteries $\pi_1, \ldots, \pi_4$ is exchanged for a more preferred one.

Imposing Axiom B4 on the recursive variational preferences yields a multiplier representation of preferences in each period but allows the concern for model misspecification to be time- and state-dependent.

Theorem 8.1: *Suppose that $\{\succsim_{t,\omega}\}$ is a recursive variational preference. Then Axioms B3 and B4 are necessary and sufficient for $\{\succsim_{t,\omega}\}$ to have a representation*

$$V_t(\omega, h) = u(h_t(\omega)) + \beta \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\int V_{t+1}(h) \, \mathrm{d}r + \theta_{t,\omega} R_{G_{t+1}}\big(r \,\|\, q^{+1}_{G_t(\omega)}\big) \right).$$

Moreover, $\big(\{\theta_{t,\omega}\}, u, q, \beta\big)$ and $\big(\{\theta'_{t,\omega}\}, u', q', \beta'\big)$ represent the same dynamic multiplier preference if and only if $\beta = \beta'$, $q = q'$, and there exists $a > 0$ and $b \in \mathbb{R}$ such that $u = au' + b$ and $\theta_{t,\omega} = a\theta'_{t,\omega}$.

The following axiom delivers constant θ, which, as discussed above, corresponds to the model of undiscounted entropy used by Hansen and Sargent (2001).

Axiom B5—Stationary Tradeoff Consistency: Relation $\asymp_{t,\omega}$ is independent of (t,ω).

Theorem 8.2: *Suppose that $\{\succsim_{t,\omega}\}$ is a recursive variational preference. Then Axioms B3–B5 are necessary and sufficient for $\{\succsim_{t,\omega}\}$ to have an undiscounted multiplier representation. Moreover, (θ, u, q, β) and $(\theta', u', q', \beta')$ represent the same dynamic multiplier preference if and only if $\beta = \beta'$, $q = q'$, and there exists $a > 0$ and $b \in \mathbb{R}$ such that $u = au' + b$ and $\theta = a\theta'$.*

8.4. Preference for earlier resolution of uncertainty and Variational Preferences

The main objective of this section is to determine which of the stationary variational preferences exhibit preference for earlier resolution of uncertainty. Fix a node (t,ω) and suppose that the only uncertainty that the decision maker faces will be paid off at time $t+2$, i.e., only f_{t+2} is a non-degenerate act. Consider two scenarios. In the first one, the uncertainty resolves early, that is the decision maker learns the realizations of f_{t+2} already at time $t+1$. Formally, f_{t+2} is $\mathcal{G}_{t+1}$-measurable.

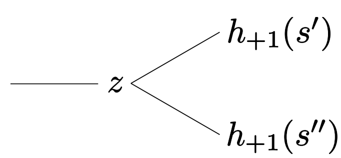

Figure 8.1. Uncertainty resolves tomorrow

In the second scenario, the uncertainty resolves late, that is the decision maker learns the realizations of f_{t+2} only at time $t+2$. Formally, f_{t+2} is not $\mathcal{G}_{t+1}$-measurable.

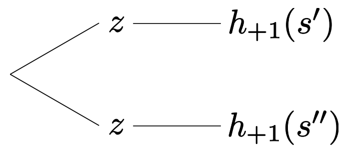

Figure 8.2. Uncertainty resolves today

A decision maker who prefers the first scenario over the second one is said to display a preference for earlier resolution of uncertainty.

In order to make these notions precise, some assumptions will be made. First, assume that the state space is a product space $\Omega = S^T$ with the naturally defined information structure $\mathcal{G}_t = \big\{\{(s_1, \ldots, s_t)\} \times S^{T-t} \mid (s_1, \ldots, s_t) \in S^t\big\}$. Given any $f : S \to \Delta(Z)$ define a $\mathcal{G}_t$-measurable act $f_t : \Omega \to \Delta(Z)$ by $f_t(s_1, \ldots, s_T) = f(s_t)$; that is, act f_t is a copy of act f that resolves at time t, i.e., that depends on the t-th component of the state space.

Given any $f : S \to \Delta(Z)$ the difference between f_t and f_{t+1} is twofold. First, those two acts differ in the timing of their resolution. Second, they differ to the extent to which the uncertainty about the t-th copy of S differs from the $t+1$-th copy of S. It is quite conceivable that the decision maker may prefer f_t to f_{t+1} not because of his preference for earlier resolution of uncertainty but rather because of the second reason, i.e., the difference in his beliefs. To isolate the pure effect of timing it will be assumed in the sequel that this second reason does not occur.

Recall that where for any $(t, \omega) \in \mathcal{T} \times \Omega$ the cost function $\gamma_t(\omega, \cdot) : \Delta(\Omega, \mathcal{G}_{t+1}) \to [0, \infty]$ takes value infinity outside $\Delta(G_t(\omega))$, i.e., is concentrated on one-step-ahead conditionals. In the present setup this means that it can be represented by a function $\bar{\gamma}_t(\omega, \cdot) : \Delta(S) \to [0, \infty]$. A decision maker displays IID ambiguity whenever his attitude towards those one-step-ahead conditionals is the same at each decision node (t, ω).

Definition 8.7: A family $\succsim_t, \omega$ of recursive variational preferences displays IID ambiguity if and only if $\bar{\gamma}_t(\omega, \cdot)$ does not depend on (t, ω).

This definition extends the notion of *IID Ambiguity* studied by Chen and Epstein (2002) and Epstein and Schneider (2003a) in the context of maxmin expected utility to the class of variational preferences. Intuitively, IID ambiguity means that every period the decision maker faces a new Ellsberg urn. His ex-ante beliefs about each urn are identical, but because he observes only one draw from each urn, he cannot make inferences across urns.[1]

Because the uncertainty that the decision maker faces in period t is identical to the uncertainty in period $t+1$, and the only property that distinguishes them is the timing of their resolution, attitudes towards such timing of resolution can be studied.

Definition 8.8: Relation $\succsim_{t,\omega}$ exhibits *preference for earlier resolution of uncertainty* if for any $f : S \to \Delta(Z)$

$$(\rho_0, \ldots, \rho_t, \rho_{t+1}, f_{t+1}, \rho_{t+3}, \ldots, \rho_T) \ \succsim_{t,\omega} \ (\rho_0, \ldots, \rho_t, \rho_{t+1}, f_{t+2}, \rho_{t+3}, \ldots, \rho_T)$$

for any choice of $\rho_0, \ldots, \rho_T \in \Delta(Z)$. Indifference to earlier resolution of uncertainty and preference for later resolution are defined similarly.

Given this definition, the preference for earlier resolution of uncertainty can be studied in the class of stationary variational preferences. One initial observation is that discounted multiplier preferences exhibit a strict preference for earlier resolution of uncertainty for all values of $\theta < \infty$.

[1] This failure of inference is known in econometrics as the problem of incidental parameters (see, e.g., Neyman and Scott, 1948).

Theorem 8.3: *Suppose the family* $\succsim_{t,\omega}$ *has a discounted multiplier representation with* $\theta < \infty$*. Then for any choice of* $\rho_0, \ldots, \rho_T \in \Delta(Z)$*, and any* $f : S \to \Delta(Z)$ *that does not yield a constant utility profile*

$$(\rho_0, \ldots, \rho_t, \rho_{t+1}, f_{t+1}, \rho_{t+3}, \ldots, \rho_T) \succ_{t,\omega} (\rho_0, \ldots, \rho_t, \rho_{t+1}, f_{t+2}, \rho_{t+3}, \ldots, \rho_T).$$

Theorem 8.4: *Suppose that* $\succsim_{t,\omega}$ *is a recursive variational preference displaying IID ambiguity. The relation* $\succsim_{t,\omega}$ *satisfies indifference to the timing of resolution of uncertainty if and only if it is a recursive maxmin expected utility preference, i.e., it is represented by*

$$V_t(\omega, h) = u(h_t(\omega)) + \beta \min_{r \in P} \left(\int V_{t+1}(h) \, \mathrm{d}r \right) \tag{8.14}$$

where $P \subseteq \Delta(\Omega, \mathcal{G}_{t+1})$*.*

Theorem 8.4 asserts that stationary variational preferences typically exhibit preference for earlier resolution of uncertainty. The only class that satisfies indifference is precisely the class of stationary maxmin expected utility preferences studied by Chen and Epstein (2002) and Epstein and Schneider (2003a).

8.5. Proofs

8.5.1. Proof of Theorem 8.1

For any $(t, \omega) \in \mathcal{T} \times \Omega$ define the functional $I_t(\omega, \cdot)$ on $B_0(\mathcal{G}_{t+1}, \mathcal{U})$ by

$$I_t(\omega, \xi_{t+1}) = \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\int u(\xi_{t+1}) \, \mathrm{d}r + \gamma_t(\omega, r) \right)$$

for any $\xi_{t+1} \in B_0(\Omega, \mathcal{G}_{t+1})$. Note that $I_t(\omega, \cdot)$ is a niveloid, hence it is monotone and continuous on $B_0(\Omega, \mathcal{G}_{t+1})$.

By Theorem 1 of Maccheroni et al. (2006b) for any $t, \omega \in \mathcal{T} \times \Omega$ the relation $\succsim_{t,\omega}$ is represented by

$$V_t(\omega, h) = u(h_t) + \beta \min_{r \in \Delta(\Omega, \mathcal{G}_{t+1})} \left(\int V_{t+1}(h) \, \mathrm{d}r + \gamma_t(\omega, r) \right).$$

Observe that for any $t \in \mathcal{T}$ the function $V_t(\cdot, h)$ is $\mathcal{G}_t$-measurable, hence the representation can be rewritten as

$$V_t(\omega, h) = u(h_t) + \beta I_t(\omega, V_{t+1}(h)).$$

Fix any any $\rho_0, \ldots, \rho_T \in \Delta(Z)$ and let $k := u(\rho_t) + 0 + \beta^2 u(\rho_{t+2}) + \beta^3 u(\rho_{t+3}) + \ldots + \beta^{T-t} u(\rho_T)$ and observe that for any $h_{t+1} \in \mathcal{H}_{t+1}$

$$V_t\big(\omega, (\rho_0, \ldots, \rho_t, h_{t+1}, \rho_{t+2}, \ldots, \rho_T)\big) = k + \beta I_t\big(\omega, u(h_{t+1})\big).$$

Define the relation $\succsim^*_{t,\omega}$ on $\mathcal{H}_{t+1}$ by $V_t(\omega, h_{t+1}) = I_t(\omega, u(h_{t+1}))$ and observe that it is monotone and continuous by monotonicity and continuity of $I_t(\omega, \cdot)$ and of $u(\cdot)$. Observe also that this relation satisfies tradeoff consistency. Thus, by Corollary 10 of Köbberling and Wakker (2003) there exists a unique probability $q^{+1}_{G_t(\omega)} \in \Delta(\Omega, \mathcal{G}_{t+1})$ and a continuous function $\psi_{\omega,t} : \Delta(Z) \to \mathbb{R}$ that represents $\succsim^*_{t+1}$. Moreover, function $\psi_{\omega,t}$ is unique up to positive affine transformations.

As in other proofs, u and $\psi_{\omega,t}$ are ordinally equivalent on lotteries; thus, there exits a strictly monotone function $\phi_{\omega,t}$ such that $\psi_{\omega,t} = \phi_{\omega,t} \circ u$. Uncertainty aversion implies that $\psi_{\omega,t}$ is concave and the translation invariance of $I_t(\omega, \cdot)$ leads to the Pexider equation for $\phi_{\omega,t}$ and the only solutions of the Pexider equation are $\phi_{\theta_{\omega,t}}$, where $\theta_{\omega,t} \in (0,\infty]$ is uniquely pinned down. □

8.5.2. Proof of Theorem 8.2

Because the relation $\precsim_{t,\omega}$ is constant across (t,ω) the scalar $\theta_{t,\omega}$ is constant across (t,ω). To see that, suppose that $\mathcal{U} = [0,\infty]$ (the cases $\mathcal{U} = [-\infty, 0]$ and $\mathcal{U} = \mathbb{R}$ are dealt with analogously). For any θ define $x(\theta) > 0$ which satisfies $\phi_\theta(1) - \phi_\theta(0) = \phi_\theta(x) - \phi_\theta(1)$. Thus, $x(\theta)$ is implicitly defined by $\Phi(\theta, x) = 0$ where $\Phi(\theta, x) = 2\phi_\theta(1) - \phi_\theta(0) - \phi_\theta(x)$. It can be verified that, by the implicit function theorem, $\frac{\mathrm{d}x}{\mathrm{d}\theta} \neq 0$. Thus, for any two different values of θ, the corresponding values of $x(\theta)$ are different.

Let $(t,\omega), (t',\omega')$, be distinct. For any $x \in \mathcal{U}$ let $\pi_x \in \Delta(Z)$ be such that $u(\pi_x) = x$. Observe that $[\pi_1, \pi_0] \asymp_{t,\omega} [\pi_{x(\theta_{t,\omega})}, \pi_1]$ and $[\pi_1, \pi_0] \asymp_{t',\omega'} [\pi_{x(\theta_{t',\omega'})}, \pi_1]$. If $\theta_{t,\omega} \neq \theta_{t',\omega'}$, then, wlog, $x(\theta_{t,\omega}) > x(\theta_{t',\omega'})$, so $\pi_{x(\theta_{t,\omega})}$ is an improvement over $\pi_{x(\theta_{t',\omega'})}$. This contradicts the equality $\precsim_{t,\omega} = \precsim_{t',\omega'}$ and tradeoff consistency of both $\precsim_{t,\omega}$ and $\precsim_{t',\omega'}$. □

For any $(t,\omega) \in \{0, 1, \ldots, T-1\} \times \Omega$ define the functional $\bar{I}_t(\omega, \cdot)$ by

$$\bar{I}_t(\omega, \xi) = \min_{r \in \Delta(S)} \left(\int u(\xi)\, \mathrm{d}r + \bar{\gamma}_t(\omega, r) \right)$$

for any $\xi : S \to \mathbb{R}$. Let $\omega = (s_1, \ldots, s_T)$ and notice that

$$I_t(\omega, \xi_{t+1}) = \bar{I}_t(\omega, \xi_{t+1}(s_1, \ldots, s_t, \cdot)) \tag{8.15}$$

for any $\xi_{t+1} \in B_0(\Omega, \mathcal{G}_{t+1})$, where $I_t(\omega, \cdot)$ is the the functional defined in the proof of Theorem 8.1.

Because of IID Ambiguity the functional $\bar{I}_t(\omega, \cdot)$ does not depend on (t, ω); thus, there exists some functional $\bar{I}$ such that for all $(t, \omega) \in \{0, 1, \ldots, T-1\} \times \Omega$ and for any $\xi : S \to \mathbb{R}$ $\bar{I}_t(\omega, \xi) = \bar{I}(\xi)$.

Lemma 8.1: *Fix some* $\rho_0, \ldots, \rho_T \in \Delta(Z)$. *For any* $f : S \to \Delta(Z)$

$$V_t(\omega, (\rho_0, \ldots, \rho_t, \rho_{t+1}, f_{t+1}, \rho_{t+3}, \ldots, \rho_T)) > V_t(\omega, (\rho_0, \ldots, \rho_t, \rho_{t+1}, f_{t+2}, \rho_{t+3}, \ldots, \rho_T))$$

if and only if

$$\bar{I}(\beta u(f)) > \beta \bar{I}(u(f)) \tag{8.16}$$

Proof. Let $k := \beta^2 u(\rho_{t+3}) + \ldots + \beta^{T-t-1} u(\rho_T)$ and observe that for any $f : S \to \Delta(Z)$

$$V_t(\omega, (\rho_0, \ldots, \rho_t, \rho_{t+1}, f_{t+1}, \rho_{t+3}, \ldots, \rho_T)) = u(\rho_t) + \beta I_t\big(\omega, u(\rho_{t+1}) + \beta I_{t+1}(\cdot, u(f_{t+1})) + k\big).$$

Because f_{t+1} is $\mathcal{G}_{t+1}$-measurable $I_{t+1}(\omega', u(f_{t+1})) = u(f_{t+1}(\omega'))$ for all $\omega' \in \Omega$. By (8.15), and by IID Ambiguity $I_t(\omega, \beta u(f_{t+1})) = \bar{I}_t(\omega, \beta u(f)) = \bar{I}(\beta u(f)))$. Thus, by translation invariance,

$$V_t(\omega, (\rho_0, \ldots, \rho_t, \rho_{t+1}, f_{t+1}, \rho_{t+3}, \ldots, \rho_T)) = u(\rho_t) + \beta u(\rho_{t+1}) + \beta k + \beta \bar{I}(\beta u(f)). \tag{8.17}$$

On the other hand,

$$V_t(\omega, (\rho_0, \ldots, \rho_t, \rho_{t+1}, f_{t+2}, \rho_{t+3}, \ldots, \rho_T)) = u(\rho_t) + \beta I_t\big(\omega, u(\rho_{t+1}) + \beta I_{t+1}(\cdot, u(f_{t+2})) + k\big).$$

Because f_{t+2} does not depend on s_{t+1} for all $\omega' \in \Omega$ by (8.15) and by IID Ambiguity $I_{t+1}(\omega', u(f_{t+2})) = \bar{I}_{t+1}(\omega', u(f))) = \bar{I}(u(f))$. Thus, by translation invariance

$$V_t(\omega, (\rho_0, \dots, \rho_t, \rho_{t+1}, f_{t+2}, \rho_{t+3}, \dots, \rho_T)) = u(\rho_t) + \beta u(\rho_{t+1}) + \beta k + \beta^2 \bar{I}(u(f)). \tag{8.18}$$

The conclusion follows from comparing expressions (8.17) and (8.19). $\square$

8.5.3. Proof of Theorem 8.3

By Lemma 8.1 the conclusion of the Theorem is equivalent to expression (8.16). Observe that $\bar{I}(\xi) = \phi_\theta^{-1}\big(\int_S \phi_\theta(\xi(s))\, \mathrm{d}q(s)\big)$ for some $q \in \Delta^{++}(S)$. Let $\xi := u(f)$ and observe that the conclusion is equivalent to

$$\frac{1}{\beta}\phi_\theta^{-1}\Big(\int \phi_\theta(\beta\xi)\, \mathrm{d}q\Big) > \phi_\theta^{-1}\Big(\int \phi_\theta(\xi)\, \mathrm{d}q\Big)$$

which is equivalent to

$$\phi_{\frac{\theta}{\beta}}^{-1}\Big(\int \phi_{\frac{\theta}{\beta}}(\xi)\, \mathrm{d}q\Big) > \phi_\theta^{-1}\Big(\int \phi_\theta(\xi)\, \mathrm{d}q\Big). \tag{8.19}$$

Recall that $\beta < 1$. It is routine to verify that the function ϕ_θ is a strictly concave transform of the function $\phi_{\frac{\theta}{\beta}}$. By assumption there exists $s', s'' \in S$ such that $u(s') \neq u(s'')$. Thus, inequality (8.19) follows from Jensen's inequality. $\square$

8.5.4. Proof of Theorem 8.4

By Lemma 8.1 the indifference to timing of uncertainty is equivalent to

$$\bar{I}(\beta u(f)) = \beta \bar{I}(u(f)) \text{ for any } f : S \to \Delta(Z).$$

which in turn is equivalent to

$$\bar{I}(\xi) = \beta \bar{I}(\xi) \text{ for any } \xi : S \to \mathcal{U}. \tag{8.20}$$

If $\succsim_{t,\omega}$ is a family of recursive maxmin expected utility preferences displaying IID ambiguity, then $\bar{I}$ is a positively homogeneous functional and equality (8.20) holds.

Conversely, suppose that equality (8.20) holds. The rest of the proof establishes that $\bar{I}(bu(f)) = b\bar{I}(u(f))$ for any $\xi : S \to \mathbb{R}$ and for any $b \in (0,1)$. Fix $\xi : S \to \mathcal{U}$ and suppose, toward contradiction, that there exists $b \in (0,1)$ such that $\bar{I}(b\xi) \neq b\bar{I}(\xi)$. Observe that, $\bar{I}(b\xi) = \bar{I}(b\xi + (1-b)0) \geq b\bar{I}(\xi)$, by concavity and because $\bar{I}(0) = 0$. Thus, $\bar{I}(b\xi) > b\bar{I}(\xi)$. First, suppose that $0 < b < \beta$. Observe that $\bar{I}(\beta^n \xi) = \bar{I}(\beta\beta^{n-1}\xi) = \beta\bar{I}(\beta^{n-1}\xi) = \cdots = \beta^n \bar{I}(\xi)$ for any $n \in \mathbb{N}$. Choose n such that $\beta^n < b$. For this n it follows that $\beta^n \bar{I}(\xi) = \bar{I}(\beta^n \xi) = \bar{I}\big(\frac{\beta^n}{b} b\xi + \frac{b-\beta^n}{b} 0\big) \geq \frac{\beta^n}{b}\bar{I}(b\xi) > \beta^n \bar{I}(\xi)$. Contradiction. Suppose now that $\beta < b < 1$. Observe, that $\bar{I}(b^n \xi) = \bar{I}(b^{n-1} b\xi) \geq b^{n-1}\bar{I}(b\xi) > b^n \bar{I}(\xi)$ for any $n \in \mathbb{N}$. Choose n such that $b^n < \beta$. Contradiction with the previous case.

As a consequence, $\bar{I}$ is a niveloid on $\mathcal{U}^S$ that is homogeneous of degree one. Extend $\bar{I}$ to $\mathbb{R}^S$ by homogeneity. Observe that the extension is a normalized niveloid, thus it satisfies the assumptions of Lemma 3.5 of Gilboa and Schmeidler (1989); therefore, there exists a closed and convex set $C \subseteq \Delta(S)$ such that $\bar{I}(\xi) = \min_{p \in C} \int \xi \, \mathrm{d}p$ for all $\xi : S \to \mathbb{R}$.

$\square$

Part 4

Certain Properties of Variational Preferences

This part studies certain properties of static variational preferences. Chapter 9 shows that probabilistic sophistication implies expected utility under an assumption that there exists a nontrivial unambiguous event. This means that although variational preferences are an excellent tool for studying behavior exemplified by the Ellsberg paradox, their ability to account for the Allais paradox is limited. Chapter 10 studies a certain subclass of variational preferences termed second order variational preferences, which is a generalization of multiplier preferences. Such preferences have two variational representations in the Savage setting while having a unique variational representation in the Anscombe–Aumann setting. This dichotomy shows that the uniqueness of the variational representation relies substantially on the structure of the Anscombe–Aumann framework. This feature distinguishes variational preferences from maxmin expected utility preference which have a unique representation in the Savage framework.

CHAPTER 9

Probabilistic Sophistication and Variational Preferences

9.1. Introduction

This paper studies two well known classes of preferences: the variational preferences of Maccheroni et al. (2006a) and the probabilistically sophisticated preferences of Machina and Schmeidler (1992).

Variational preferences are a very broad class of preferences that allow for modelling choices consistent with the Ellsberg (1961) praradox. This class of preferences includes the maxmin expected utility preferences of Gilboa and Schmeidler (1989), where the decision maker has a nonunique probability, as well as many other classes of preferences that violate separability across states.

The notion of probabilistic sophistication means that the decision maker bases his choices on probabilistic beliefs. This class includes expected utility, as well as many nonexpected utility criteria that allow for modelling the Allais (1953) paradox and related violations of linearity in probabilities.

These two types of preferences can coexist. In many situations involving ambiguity and ambiguity aversion, such as in the Ellsberg paradox, there exist events to which the decision maker can attach unambiguous probabilities. In principle, a decision maker could be probabilistically sophisticated but nonexpected utility over such events. The question is whether it is possible to model his attitude toward the remaining ambiguous evens using the model of variational preferences; thus, whether it is possible to jointly study the Ellsberg and Allais paradoxes using this class of preferences.

This paper studies to what extent such coexistence of these two models is possible. Marinacci (2002) studied this question for the subclass of maxmin expected utility preferences and showed that, under a mild assumption that all the probabilities of the decision maker agree on some event, probabilistic sophistication is equivalent to expected utility. This paper shows that this result holds generally for the whole class of variational preferences. This suggests that, although variational preferences are an excellent tool for studying behavior exemplified by the Ellsberg paradox, their ability to account for the Allais paradox is limited because probabilistically sophisticated preferences collapse to expected utility preferences which are inconsistent with the Allais paradox.

9.2. Preliminaries

9.2.1. Setting

Let S be the set of states of the world with a sigma algebra Σ of subsets of S. Let X be the set of consequences, assumed to be a convex subset of a vector space. An act is a Σ-measurable and finite-valued mapping $f : S \to X$ that attaches a consequence to each possible state. The preferences $\succsim$ are defined over such acts.

9.2.2. Probabilistic sophistication

The notion of *probabilistic sophistication*, introduced by Machina and Schmeidler (1992), means that the decision maker treats subjective uncertainty in the same manner as objective risk. In order to do so, the decision maker formulates a subjective probability measure q on the state space S. To evaluate an act $f : S \to X$, he first computes the lottery that the act induces on prizes, $q \circ f^{-1}$; second, he uses some criterion, M, of evaluating objective lotteries over prizes (see Figure **??**). The criterion used may be expected utility but it can also be one of the many nonexpected utility criteria, which allow for modeling choices consistent with the Allais paradox.

9.2.3. Variational Preferences

The *variational preferences*, introduced and axiomatized by Maccheroni et al. (2006a), are represented by

$$V(f) = \min_{p \in \Delta S} \int_S u(f)\, \mathrm{d}p + c(p), \tag{9.1}$$

where $c : \Delta S \to [0, \infty]$ is a nonnegative, convex, and weak* lower semincontinuous function taking value zero for at least one measure; and $u : \Delta Z \to \mathbb{R}$ is a nonconstant and affine utility function. An important subclass of variational preferences are those where the minimization is over the set of *countably* additive probabilities. Such preferences are called *continuous* variational preferences.

A classic example of variational preferences are maxmin expected utility preferences (MEU) of Gilboa and Schmeidler (1989) with representation

$$V(f) = \min_{p \in P} \int_S u(f)\, \mathrm{d}p, \tag{9.2}$$

where P is a nonempty, convex, and weak* compact set of probabilites in ΔS. Formula (9.2) is a special case of (9.1) for

$$c_{\mathrm{MEU}}(p) = \begin{cases} 0 & \text{for } p \in P \\ \infty & \text{for } p \notin P. \end{cases}$$

A special case of both of those classes are Anscombe–Aumann expected utility preferences represented by

$$V(f) = \int_S u(f)\,\mathrm{d}p;$$

in this case the set P is a singleton composed of p.

9.3. Main Result

The main question of this paper is whether variational preferences are flexible enough to allow for modelling the Allais paradox. Marinacci (2002) showed that for the subclass of Maxmin Expected Utility preferences the answer is negative under a weak assumption of agreement of probabilities.

Assumption 9.1: There exists an event $A_0 \in \Sigma$ such that if $c(p) = c(p') = 0$, then $0 < p(A_0) = p'(A_0) < 1$.

This assumption means that there exists an event A_0, such that any two measures with zero cost (i.e., any two measures belonging to the set of priors P) agree on A_0.

Theorem 9.1 below extends Marinacci's (2002) result to the whole class of variational preferences under an appropriately extended notion of agreement of probabilities. In principle, there are two possible extensions of this assumption to cost functions taking values other than zero and infinity.

Assumption 9.2: For any $r \in [0, \infty)$ there exists an event $A_r \in \Sigma$ such that if $c(p) = c(p') = r$, then $0 < p(A_r) = p'(A_r) < 1$.

This assumption requires that all measures with the same cost agree on some event. This assumption is equivalent to Marinacci's (2002) assumption for the subclass of Maxmin Expected Utility preferences.

Assumption 9.3: There exists an event $A \in \Sigma$ such that if $c(p), c(p') < \infty$, then $0 < p(A) = p'(A) < 1$.

This assumption means that all measures with finite cost attach the same probability to some event. This is a stronger requirement than Assumption 9.2 and it may be harder to verify for a given cost function.

The main result of this paper, Theorem 9.1 shows that the weaker Assumption 9.2 is sufficient.

Theorem 9.1: *Suppose that $\succsim$ is a continuous variational preference. If Assumption 9.2 holds, then the following two statements are equivalent*

(i) $\succsim$ is probabilistically sophisticated

(ii) $\succsim$ is an Anscombe–Aumann expected utility preference.

Remarks: (i) Strictly speaking, Theorem 9.1 is not a generalization of Marinacci's (2002) result because his theorem holds also for α-MEU preferences which do not belong to the class of variational preferences. Moreover, he uses a weaker notion of probabilistic sophistication, that of probabilistic beliefs, and his results for maxmin expected utility preferences do not rely on countable additivity. (ii) Marinacci's (2002) techniques rely on axiom P4 of Savage, which is generally violated by variational preferences. The proof of Theorem 9.1 uses different techniques; it builds on the elegant characterization of probabilistically sophisticated variational preferences obtained by Maccheroni et al. (2006a). (iii) There is a sense in which Assumption 9.2 cannot be weakened. As Proposition 2 in Marinacci (2002) shows, there exist MEU preferences that violate his Assumption 9.1 and are probabilistically sophisticated while not being expected utility. Such examples are inherited by Theorem 9.1; the class of such examples is even larger, as it includes some variational but non-MEU preferences, notably the multiplier preferences of Hansen and Sargent (2001). (iv) The statement of Theorem 9.1 needs to be changed when applied to settings where the set of outcomes X is the set of objective lotteries over some more primitive set of prizes Z, such as the Anscombe–Aumann framework. The definition of probabilistic sophistication in the Anscombe–Aumann setting formulated by Machina and Schmeidler (1995) (see also a recent analysis of Grant and Polak, 2006) requires that the same criterion of evaluating risk (denoted M in Figure **??**) be applied to lotteries over Z and to lotteries induced by acts. In the class of variational preferences such uniform decision attitudes can be satisfied only for Anscombe–Aumann expected utility. An appropriate translation of the results in this paper involves the weaker notion of second-order probabilistic sophistication, introduced by Ergin and Gul (2004). (v) It is necessary that the results in this paper be formulated for the case when X is a convex set because of the nonuniqueness of the cost function in the general setup of Savage, as exemplified by the case of multiplier preferences, see, e.g., **?**.

9.4. Proofs

Let S be a set and let Σ be a sigma algebra of its events. Let $\Delta^\sigma(S,\Sigma)$ denote the set of all countably additive probability measures on (S,Σ). Let $q \in \Delta^\sigma(S,\Sigma)$ and let $L^1(S,\Sigma,q)$ denote the set of all nonnegative measurable functions on (S,Σ) with $\int_S f \,\mathrm{d}q = 1$. For $f, g \in L^1(S,\Sigma,q)$ define $f \sim_{cx} g$ iff

$$q(s \in S \mid f(s) \leq t) = q(s \in S \mid g(s) \leq t)$$

for any $t \geq 0$. Similarly, for any measures $p, p' \in \Delta^\sigma(S,\Sigma)$ define $p \sim_{cx} p'$ iff $\frac{\mathrm{d}p}{\mathrm{d}q} \sim_{cx} \frac{\mathrm{d}p'}{\mathrm{d}q}$. For $p \in \Delta^\sigma(S,\Sigma)$, the set $O(p) = \{p' \in \Delta^\sigma(S,\Sigma) \mid p' \sim_{cx} p\}$ is called the *orbit* of p. A set of measures $\Gamma \subseteq \Delta^\sigma(q)$ is called *orbit-closed* iff $p \in \Gamma \Rightarrow O(p) \subseteq \Gamma$.

Lemma 9.1: *Let $f \in L^1(S,\Sigma,q)$ and let $F, G \in \Sigma$ be disjoint events, with $q(F) = q(G)$. Then, there exists $g \in L^1(S,\Sigma,q)$ such that $f = g$ on $(F \cup G)^c$, $\int_F f \,\mathrm{d}q = \int_G g \,\mathrm{d}q$, and $f \sim_{cx} g$.*

Proof. For each $n \in \mathbb{N}$ and for $1 \leq k \leq n2^n$ define sets

$$ {}_nF_0 = \{s \in F \mid f(s) \geq n\},\ {}_nF_k = \left\{ s \in F \,\middle|\, \frac{k-1}{2^n} \leq f(s) \leq \frac{k}{2^n} \right\}, $$
$$ {}_nG_0 = \{s \in G \mid f(s) \geq n\},\ {}_nG_k = \left\{ s \in G \,\middle|\, \frac{k-1}{2^n} \leq f(s) \leq \frac{k}{2^n} \right\}. $$

Because q is nonatomic, it is also convex-ranged (see, e.g., Villegas, 1964). Thus, for each n, partitions $\{{}_nF'_k\}_{k=0}^{n2^n}$ of F and $\{{}_nG'_k\}_{k=0}^{n2^n}$ of G can be constructed such that

$$q(F'_{n,k}) = q(G_{n,k}) \text{ and } q(G'_{n,k}) = q(F_{n,k})$$

for all $0 \leq k \leq n2^n$ and

$${}_{(n+1)}G'_{(2k)} \subseteq {}_{(n+1)}G'_{(k)} \text{ and } {}_{(n+1)}G'_{(2k+1)} \subseteq {}_{(n+1)}G'_{(k)}$$

for all $0 \leq k \leq n2^n$ and $n \in \mathbb{N}$.

Define functions

$$f_n = \sum_{k=1}^{n2^n} \left(\frac{k-1}{2^n} \mathbf{1}_{{}_nF_k} \right) + n\mathbf{1}_{{}_nF_k} + f_{|(E \cup G)^c} + \sum_{k=1}^{n2^n} \left(\frac{k-1}{2^n} \mathbf{1}_{{}_nG_k} \right) + n\mathbf{1}_{{}_nG_k},$$
$$g_n = \sum_{k=1}^{n2^n} \left(\frac{k-1}{2^n} \mathbf{1}_{{}_nF'_k} \right) + n\mathbf{1}_{{}_nF'_k} + f_{|(E \cup G)^c} + \sum_{k=1}^{n2^n} \left(\frac{k-1}{2^n} \mathbf{1}_{{}_nG'_k} \right) + n\mathbf{1}_{{}_nG'_k}.$$

Observe, that functions f_n satisfy $0 \leq f_n \leq f_{n+1}$, and converge pointwise to f. Similarly, functions g_n satisfy $0 \leq g_n \leq g_{n+1}$. Define $g = \lim_{n \to \infty} g_n$. Observe that $f = g$ on $(E \cup G)^c$. Moreover, $\int_S f_n \, \mathrm{d}q = \int_S g_n \, \mathrm{d}q$, so by the Monotone Convergence Theorem $\int_S f \, \mathrm{d}q = \int_S g \, \mathrm{d}q$.

To see that $f \sim_{cx} g$, let $t \geq 0$ and define sets

$$A_n = \{s \in S \mid f_n(s) \leq t\}, \; A = \{s \in S \mid f(s) \leq t\},$$

$$B_n = \{s \in S \mid g_n(s) \leq t\}, \; B = \{s \in S \mid g(s) \leq t\},$$

Verify, that by construction of f_n and g_n $A_n \downarrow A$, $B_n \downarrow B$, and $q(A_n) = q(B_n)$ for all n. By countable additivity of q, $\lim_{n\to\infty} q(A_n) = q(A)$ and $\lim_{n\to\infty} q(B_n) = q(B)$. □

Lemma 9.2: *Suppose that $\Gamma \subseteq \Delta^\sigma(q)$ is an orbit-closed set of measures. Suppose also that there exists $A \in \Sigma$ such that $0 < p(A) = p'(A) < 1$ for all $p, p' \in \Gamma$. Then $\Gamma = \{q\}$.*

Proof. Let $\alpha = q(A)$. Observe, that wlog $\alpha \leq \frac{1}{2}$, because if all measures in Γ agree on A, then they also agree on A^c. Also, if $\alpha = 0$, then for any $p \in \Gamma$ $q(A) = 0 \Rightarrow p(A) = 0$, contradicting the assumption. Thus, $\alpha \in (0, \frac{1}{2}]$.

Step 1: *$p(E) = p(A)$ for all $p \in \Gamma$ and for all events $E \in \Sigma$ with $q(E) = \alpha$.*

Let $E \in \Sigma$ be such that $q(E) = \alpha$ and observe that $q(A - E) = q(E - A)$. Let $p \in \Gamma$ and define $f = \frac{\mathrm{d}p}{\mathrm{d}q}$. By Lemma 9.1 applied to $(E - A)$ and $(A - E)$, there exists $g \in L^1(S, \Sigma, q)$ such that $f = g$ on $(A \cup E)^c \cup (A \cap E)$, $\int_{(E-A)} f \, \mathrm{d}q = \int_{(A-E)} g \, \mathrm{d}q$, and $f \sim_{cx} g$. Define measure $p' \in \Delta^\sigma(S, \Sigma)$ by $p'(F) = \int_F g \, \mathrm{d}q$ and observe that $p' \sim_{cx} p$. Moreover, $p(E - A) = p'(A - E)$ and $p(A \cap E) = p'(A \cap E)$. Thus, $p(E) = p(E - A) + p(A \cap E) = p'(A - E) + p'(A \cap E) = p'(A) = p(A)$, where the last equality holds by orbit-closedness of Γ.

Step 2: *$p(F) = p(F')$ for all $p \in \Gamma$ and for all disjoint events $F, F' \in \Sigma$ with $q(F) = q(F') = \beta < \alpha$.*

Observe that $\beta < \frac{1}{2}$, so $\alpha - \beta < 1 - 2\beta$. Thus, by range-convexity of q, there exists $H \subseteq (F \cup F')^c$ with $q(H) = \alpha - \beta$. By Step 1 applied to sets $F \cup H$ and $F' \cup H$, it follows that $p(F) + p(H) = p(F \cup H) = p(A) = p(F' \cup H) = p(F') + p(H)$; hence, $p(F) = p(F')$.

Step 3: *$p(G) = q(G)$ for all $p \in \Gamma$ and for $G \in \Sigma$..*

Let $\gamma = q(G)$ and for each $n \in \mathbb{N}$ define $k_n = \sup\{k \mid \frac{k}{n} \leq \gamma\}$. Observe, that $\lim_{n\to\infty} \frac{k_n}{n} = \gamma$. For each $n \in \mathbb{N}$, by range-convexity of q, there exists a partition $\{F_1, \ldots, F_n\}$ of F such that $q(F_k) = \frac{1}{n}$ for $k = 1, \ldots, n$, sets $F_1, \ldots, F_{k_n} \subseteq G$, and sets $F_{k_n+2}, \ldots, F_n \subseteq G^c$. By Step 2, $p(F_k) = \frac{1}{n}$ for $k = 1, \ldots n$, so $\frac{k_n}{n} \leq p(G) \leq \frac{k_n+1}{n}$. By letting n to infinity, $p(G) = \gamma$. □

Proof of Theorem 9.1. The direction $(ii) \Rightarrow (i)$ is trivial. For $(i) \Rightarrow (ii)$, observe that for any $r \in \mathbb{R}_+$ let $C_r = \{p \in \Delta(S, \Sigma) \mid c(p) = r\}$ denote the level set of the cost function c. Observe that

$$V(f) = \min_{p\in\Delta(S,\Sigma)} \int_S (u \circ f)\,\mathrm{d}p + c(p) = \min_{r\in\mathbb{R}_+} \min_{p\in C_r} \int_S (u \circ f)\,\mathrm{d}p + r$$

By Theorem 13 of Maccheroni et al. (2006a) the preference satisfies their axiom A.8 of (weak) monotone continuity. From the proof of Corollary 4 in Sarin and Wakker (2000) it follows that this axiom implies that $\succsim$ is probabilistically sophisticated with respect to some $q \in \Delta^\sigma(S)$. By Theorem 14 of Maccheroni et al. (2006a), if $\succsim$ is probabilistically sophisticated with respect to $q \in \Delta^\sigma(S)$, then c is rearrangement invariant, i.e., $p \sim_{cx} p' \Rightarrow c(p) = c(p')$ for all $p, p' \in \Delta(S, \Sigma)$. Thus, each C_r is orbit-closed. Therefore, by Assumption 9.2 and Lemma 9.2, $C_r = \{q\}$ for all $r \in \mathbb{R}_+$. Thus,

$$V(f) = \min_{r\in\mathbb{R}_+} \int_S (u \circ f)\,\mathrm{d}q + r = \int_S (u \circ f)\,\mathrm{d}q. \qquad \square$$

CHAPTER 10

Second Order Variational Preferences

10.1. Second-Order Variational Preferences

Multiplier preferences are an example of variational preferences having two representations:

$$V_1(f) = \min_{p\in\Delta(S)} \int_S u(f)\,\mathrm{d}p + \theta R(p\,\|\,q) \tag{3.2}$$

and

$$V_2(f) = \int_S \phi_\theta\big(u(f)\big)\,\mathrm{d}q. \tag{3.4}$$

One interpretation of this dichotomy is that model uncertainty in (3.2) manifests itself as second order risk aversion in (3.4). This motivates the following definition.

Definition 10.1: Preference relation $\succsim$ is a *Second-Order Variational Preference* if $\succsim$ is a variational preference with representation

$$V_1(f) = \min_{p\in\Delta S} \int_S u(f)\,\mathrm{d}p + c_1(p)$$

and it also has representation

$$V_2(f) = \min_{p\in\Delta S} \int_S \phi_\theta\big(u(f)\big)\,\mathrm{d}p + c_2(p)$$

for $\theta \in (0,\infty)$ and some grounded, convex, and lower semicontinous cost function c_2.

The following theorem characterizes this class of variational preferences.

Theorem 10.1: *Suppose that S is a Polish space and that $\succsim$ satisfies A1-A8. Preference $\succsim$ is a second-order variational preference if and only if* $c_1(p) = \min_{q\in Q} \theta R(p\,\|\,q)$

for some closed and convex set of measures $Q \subseteq \Delta^\sigma(S)$. In this case c_2 can be chosen to satisfy $c_2 = \delta_Q$, i.e., $V_2(f) = \min_{p\in Q} \int_S \phi_\theta\big(u(f_s)\big)\,\mathrm{d}p$.[1]

The analysis of probabilistic sophistication of Chapter 9 can be extended to second-order variational preferences. In order to do so, Marinacci's weak agreement assumption Assumption 9.1 will be used. Recall, that the assumption stipulates that there exists an event A_0, such that any two measures with zero cost agree on A_0.

Theorem 10.2: *Suppose that $\succsim$ is a Second-Order Variational Preference. If Assumption 9.1 holds, then the following two statements are equivalent*

(i) $\succsim$ is Second-Order Probabilistically Sophisticated

(ii) $\succsim$ is a Second-Order Expected Utility preference.

As a corollary of Theorem 9.1 another characterization of multiplier preferences is obtained.

Corollary 10.1: *Suppose that $\succsim$ satisfies Axioms A1-A8 and Assumption 9.1 holds. Then $\succsim$ is a multiplier preference if and only if $\succsim$ is a Second-Order Variational Preference and it is Second-Order Probabilistically Sophisticated.*

10.2. Proof of Theorem 10.1

Lemma 10.1 establishes that $c_1(p) = \min_{q\in Q} \theta R(p \,\|\, q)$ is a legitimate cost function. Lemma 10.2 is the main step in proving necessity. The rest of the proof deals with sufficiency.

[1]The function c_2 in representation V_2 may not be unique. Uniqueness is guaranteed if the function u is unbounded from below.

Lemma 10.1: *Suppose S is a Polish space. For any convex closed set $Q \subseteq \Delta^\sigma(S)$ the function $c_1(p) = \min_{q \in Q} \theta R(p \| q)$ is nonnegative, convex, lower semicontinuous, and $\{p \in \Delta(S) \mid c_1(p) \leq r\} \subseteq \Delta^\sigma(S)$ for each $r \geq 0$. Moreover, the function c_1 is grounded and $\{p \in \Delta(S) \mid c_1(p) = 0\} = Q$.*

Proof. Nonnegativity follows from $R(p \| q)$ being nonnegative for any $p, q \in \Delta(S)$.

By Lemma 1.4.3 (b) in Dupuis and Ellis (1997), $R(\cdot \| \cdot)$ is a convex, lower semicontinuous function on $\Delta^\sigma(S) \times \Delta^\sigma(S)$. Thus, $\arg\min_{q \in Q} \theta R(p \| q)$ is a nonempty compact and convex set for any $p \in \Delta^\sigma(S)$. Let $\lambda \in (0,1)$ and $p', p'' \in \Delta^\sigma(S)$. Let $q' \in \arg\min_{q \in Q} \theta R(p' \| q)$ and $q'' \in \arg\min_{q \in Q} \theta R(p'' \| q)$. Convexity follows from:

$$
\begin{aligned}
c_1(\lambda p' + (1-\lambda) p'') &= \min_{q \in Q} \theta R(\lambda p' + (1-\lambda) p'' \| q) \\
&\leq \theta R\big(\lambda p' + (1-\lambda) p'' \| \lambda q' + (1-\lambda) q''\big) \\
&\leq \lambda \theta R(p' \| q') + (1-\lambda) \theta R(p'' \| q'') \\
&= \lambda c_1(p') + (1-\lambda) c_1(p'').
\end{aligned}
$$

For lower semicontiuniuty define $\mathrm{Proj} : \Delta^\sigma(S) \times Q \times \mathbb{R} \to \Delta^\sigma(S) \times \mathbb{R}$ to be a projection $\mathrm{Proj}(p, q, r) = (p, r)$. Let $\mathrm{Epi}(R) = \{(p, q, r) \in \Delta^\sigma(S) \times Q \times \mathbb{R} \mid R(p \| q) \leq r\}$ be the epigraph of R and $\mathrm{Epi}(c_1) = \{(p, r) \in \Delta^\sigma(S) \times \mathbb{R} \mid c_1(p) \leq r\}$ be the epigraph of c_1. Observe that, by lower semicontinuity of R, the set $\mathrm{Epi}(R)$ is closed. Next, observe that $\mathrm{Epi}(c_1) = \mathrm{Proj}\big(\mathrm{Epi}(R)\big)$.

To verify that, let $(p, r) \in \mathrm{Epi}(c_1)$. Then $c_1(p) \leq r$; thus $\min_{q \in Q} R(p \| q) \leq r$. Let $q' \in \arg\min_{q \in Q} R(p \| q)$. It follows, that $R(p \| q') \leq r$; thus, $(p, q, r) \in \mathrm{Epi}(R)$. Conclude that $(p, r) \in \mathrm{Proj}\big(\mathrm{Epi}(R)\big)$. Conversely, let $(p, r) \in \mathrm{Proj}\big(\mathrm{Epi}(R)\big)$. Then there exists q'

such that $(p, q', r) \in \text{Epi}(R)$, so that $R(p\| q') \leq r$. Thus, $c_1(p) = \min_{q\in Q} R(p\| q) \leq R(p\| q') \leq r$. Conclude that $(p, r) \in \text{Epi}(c_1)$.

Finally, observe that $\text{Proj}(C)$ is closed for any closed set $C \in \Delta^\sigma(S) \times Q \times \mathbb{R}$. Let (p_n, r_n) be a sequence in $\text{Proj}(C)$ with limit (p, r). Because $(p_n, r_n) \in \text{Proj}(C)$, there exists a sequence q_n in Q such that $(p_n, q_n, r_n) \in C$. Because Q is a compact set subset of a metric space, $\lim_{n\to\infty} q_n = q \in Q$ by passing to a subsequence. By closedness of C, it follows that $\lim_{n\to\infty}(p_n, q_n, r_n) = (p, q, r) \in C$. Thus, $(p, r) \in C$.

To see that $\{p \in \Delta(S) \mid c_1(p) \leq r\} \subseteq \Delta^\sigma(S)$ for each $r \geq 0$, observe that $\{p \in \Delta(S) \mid R(p\| q) \leq r\} \subseteq \Delta^\sigma(S)$ and that by compactness of Q and lower-semicontinuity of $R(p\| \cdot)$

$$\{p \in \Delta(S) \mid c_1(p) \leq r\} = \bigcup_{q\in Q} \{p \in \Delta(S) \mid R(p\| q) \leq r\}.$$

For groundedness, recall that by Lemma 1.4.1 in Dupuis and Ellis (1997) $R(p\| q) = 0$ iff $p = q$. Thus, $c_1(q) \leq R(q \,\|\, q) = 0$ for any $q \in Q$. Conversely, if $c_1(p) = 0$, then $\min_{q\in Q} R(p \,\|\, q) = 0$. By lower semincontinuity of R, there exists $q \in Q$ such that $0 = c_1(p) = R(p\| q)$. Thus, by Lemma 1.4.1 in Dupuis and Ellis (1997), $p = q$; hence, $p \in Q$. $\square$

Lemma 10.2: *Suppose $\succsim$ is a variational preference and $Q \subseteq \Delta^\sigma(S)$ is a closed and convex set. Then V_1 with $c_1(p) = \min_{q\in Q} \theta R(p \,\|\, q)$ represents $\succsim$ if and only if V_2 with $c_2 = \delta_Q$ represents $\succsim$.*

Proof. Observe that

$$\begin{aligned} V_1(f) &= \min_{p\in\Delta S}\int_S u(f_s)\,\mathrm{d}p + \min_{q\in Q}\theta R(p\|\,q) \\ &= \min_{p\in\Delta S}\min_{q\in Q}\int_S u(f_s)\,\mathrm{d}p + \theta R(p\|\,q) \\ &= \min_{q\in Q}\min_{p\in\Delta S}\int_S u(f_s)\,\mathrm{d}p + \theta R(p\|\,q) \\ &= \min_{q\in Q}\phi_\theta^{-1}\Big(\int_S \phi_\theta\big(u(f_s)\big)\,\mathrm{d}q\Big) \\ &= \phi_\theta^{-1}\Big(\min_{q\in Q}\int_S \phi_\theta\big(u(f_s)\big)\,\mathrm{d}q\Big), \end{aligned}$$

where the fourth inequality follows from Proposition 1.4.2 in Dupuis and Ellis (1997) and the fifth from strict monotonicity of ϕ_θ^{-1}. Thus, V_1 is ordinally equivalent to $V_2(f) = \min_{q\in Q}\int_S \phi_\theta\big(u(f_s)\big)\,\mathrm{d}q = V_2(f) = \min_{p\in\Delta S}\int_S \phi_\theta\big(u(f_s)\big)\,\mathrm{d}p + c_2(p)$. □

Proof of Theorem 10.1. Suppose that V_1 with $c_1(p) = \min_{q\in Q}\theta R(p\,\|\,q)$ represents $\succsim$. By Lemma 10.1 an by Theorems 3 and 13 of Maccheroni et al. (2006a), $V_1(f) = \min_{p\in\Delta S}\int_S u(f_s)\,\mathrm{d}p + c_1(p)$ is a representation of a preference $\succsim$ that satisfies axioms A1-A8. By Lemma 10.2, V_2 with $c_2 = \delta_Q$ represents $\succsim$.

Conversely, suppose that $\succsim$ is a variational preference represented by

$$V_2(f) = \min_{p\in\Delta S}\int_S \phi_\theta\big(u(f_s)\big)\,\mathrm{d}p + c_2(p).$$

Define niveloid $I : B_0(\Sigma, \phi_\theta(\mathcal{U})) \to \mathbb{R}$ by $I(\xi) = \min_{p\in\Delta S} \int_S \xi \,\mathrm{d}p + c_2(p)$ and observe that $V_2(f) = I\big(\phi_\theta(u(f))\big)$. Therefore,

$$\begin{aligned} V_2(\alpha f + (1-\alpha)\pi) &= I\bigg(\phi_\theta\big(\alpha u(f) + (1-\alpha)u(\pi)\big)\bigg) \\ &= I\bigg(-\phi_\theta\big((1-\alpha)u(\pi)\big)\cdot\phi_\theta\big(\alpha u(f_s)\big)\bigg) \end{aligned} \tag{10.1}$$

for any $f \in \mathcal{F}(\Delta(Z))$, $\pi \in \Delta(Z)$, and $\alpha \in (0,1)$.

Niveloid I is homogeneous of degree one. To verify, suppose that $\mathcal{U} = u(\Delta(Z)) = \mathbb{R}_+$. (The case of $\mathcal{U} \in \{\mathbb{R}_-, \mathbb{R}\}$ is analogous.) Let $\xi \in \mathcal{B}_0(\Sigma, \phi_\theta(\mathbb{R}_+))$ and $b \in (0,1]$ (the case $b \geq 1$ follows from this). Let scalar $r = b^{-1}I(b\xi)$; observe that $I(br) = I(I(b\xi)) = I(b\xi)$. Let $f \in \mathcal{F}(\Delta(Z))$ be such that $\phi_\theta(\frac{1}{2}u(f)) = \xi$ and $\pi \in \Delta(Z)$ be such that $\phi_\theta(\frac{1}{2}u(\pi)) = r$. Their existence is guaranteed by unboundedness of $\mathcal{U}$. Furthermore, let $\rho, \rho' \in \Delta(Z)$ be such that $b = -\phi_\theta(\frac{1}{2}u(\rho))$ and $u(\rho') = 0$. (In the case of $\mathcal{U} = \mathbb{R}_-$, prove homogeneity for $b \geq 1$ and deduce for $b \in (0,1]$.) By (10.1), $I(b\xi) = I(br)$ this implies $V_2\big(\phi_\theta(\frac{1}{2}u(f) + \frac{1}{2}u(\rho))\big) = V_2\big(\phi_\theta(\frac{1}{2}u(\pi) + \frac{1}{2}u(\rho))\big)$. Because $\succsim$ satisfies Axiom A2, this implies $V_2\big(\phi_\theta(\frac{1}{2}u(f) + \frac{1}{2}u(\rho'))\big) = V_2\big(\phi_\theta(\frac{1}{2}u(\pi) + \frac{1}{2}u(\rho'))\big)$, which, by (10.1), implies $I(\xi) = I(r)$. Thus, $I(b\xi) = I(br) = bI(r) = bI(\xi)$.

If $\mathcal{U} = \mathbb{R}_+$ or $\mathcal{U} = \mathbb{R}_-$, then I is defined on $B_0(\Sigma, [-1, 0))$ or $B_0(\Sigma, (-\infty, -1])$, respectively. Extend I to $B_0(\Sigma, \mathbb{R}_-)$ by homogeneity. Note that I is monotone, homogeneous of degree one, and vertically invariant on $B_0(\Sigma, \mathbb{R}_-)$. If $\mathcal{U} = \mathbb{R}$, then I is already defined on $B_0(\Sigma, \mathbb{R}_-)$ and enjoys those properties.

By Lemma 23 of Maccheroni et al. (2004), I is niveloid on $B_0(\Sigma, \mathbb{R}_-)$. By Lemmas 21 and 22 of Maccheroni et al. (2004), the unique vertically invariant extension of I to

$\mathcal{B}_0(\Sigma)$, defined by $\tilde{I}(\xi + k) = I(\xi) + k$ for any $\xi + k \in B_0(\Sigma, \mathbb{R})$ such that $\xi \in B_0(\Sigma, \mathbb{R}_-)$ is monotonic. Note that $\tilde{I}$ is monotone homogeneous of degree one on $B_0(\Sigma, \mathbb{R})$.

Therefore, $\tilde{I}$ satisfies the assumptions of Lemma 3.5 of Gilboa and Schmeidler (1989). Thus, there exists a closed, convex set $Q \subseteq \Delta(S)$ such that $\tilde{I}(\xi) = \min_{p \in Q} \int \xi \, \mathrm{d}p$. Hence, $I(\xi) = \min_{p \in Q} \int \xi \, \mathrm{d}p$ for all $\xi \in B_0(\Sigma, \phi_\theta(\mathcal{U}))$.

Let E_n be a vanishing sequence of events and let $x < y$ be elements of $\phi_\theta(\mathcal{U})$. Observe that by Axiom A8, for any k there exists a N such that $I(xE_ny) > I(y - \frac{1}{k})$ for all $n \geq N$. Thus, $\min_{p \in Q} \int xE_ny \, \mathrm{d}p > y - \frac{1}{k}$. Therefore, $(x - y) \max_{p \in Q} p(E_n) > \frac{1}{k}$. Hence, $p(E_n) < (k(y - x))^{-1}$ for any $p \in Q$. Therefore $\lim_{n \to \infty} p(E_n) = 0$ for any $p \in Q$. Thus, $Q \subseteq \Delta^\sigma(S)$.

Finally, by Lemma 10.2, $c_1(p) = \min_{q \in Q} \theta R(p \| q)$. □

10.2.1. Proof of Theorem 10.2

The direction $(ii) \Rightarrow (i)$ is trivial. For $(i) \Rightarrow (ii)$, observe that by Theorem 10.1, $\succsim$ can be represented by

$$V_1(f) = \min_{p \in \Delta(S, \Sigma)} \int_S (u \circ f) \, \mathrm{d}p + c_1(p)$$

with $c_1(p) = \min_{q \in Q} R(p \| q)$ for some closed and convex set $Q \subseteq \Delta^\sigma(S)$. From the proof of Corollary 4 in Sarin and Wakker (2000) it follows that Axiom A8 implies that $\succsim$ is probabilistically sophisticated with respect to some $q \in \Delta^\sigma(S)$. By Theorem 14 of Maccheroni et al. (2006a), if $\succsim$ is probabilistically sophisticated with respect to $q \in \Delta^\sigma(S)$, then c_1 is rearrangement invariant, i.e., $p \sim_{cx} p' \Rightarrow c_1(p) = c_1(p')$ for all $p, p' \in \Delta(S, \Sigma)$. Thus, in particular, the set $\{p \in \Delta(S) \mid c_1(p) = 0\}$ is orbit-closed. Therefore, by Assumption 9.2 and Lemma 9.2, $\{p \in \Delta(S) \mid c_1(p) = 0\} = \{q\}$. But, by Theorem 10.1,

$\succsim$ can be represented by

$$V_2(f) = \min_{p \in Q} \int_S \phi_\theta(u \circ f) \, \mathrm{d}p.$$

Moreover, by Lemma 10.1, $Q = \{p \in \Delta(S) \mid c_1(p) = 0\}$. Conclude that $\succsim$ can be represented by

$$V_2(f) = \int_S \phi_\theta(u \circ f) \, \mathrm{d}q.$$

$\square$

Part 5

Subjective Beliefs and Ex Ante Trade

CHAPTER 11

Subjective Beliefs and Ex Ante Trade

11.1. Introduction

In a model with risk averse agents who maximize subjective expected utility, betting occurs if and only if agents' priors differ. This link between common priors and speculative trade in the absence of aggregate uncertainty is a fundamental implication of expected utility for risk-sharing in markets. A similar relationship holds when ambiguity is allowed and agents maximize the minimum expected utility over a set of priors, as in the model of Gilboa and Schmeidler (1989). In this case, purely speculative trade occurs when agents hold no priors in common; full insurance is Pareto optimal if and only if agents have at least one prior in common, as **?** show. This note develops a more general connection between subjective beliefs and speculative trade applicable to a broad class of convex preferences, which encompasses as special cases not only the previous results for expected utility and maxmin expected utility, but all the models central in studies of ambiguity in markets, including the convex Choquet model of Schmeidler (1989), the smooth second-order prior models of Klibanoff et al. (2005) and Nau (2006), the second-order expected utility model of Ergin and Gul (2004), the confidence preferences model of **?**, the multiplier model of Hansen and Sargent (2001), and the variational preferences model of Maccheroni et al. (2006a).

By casting our results in the general setting of convex preferences, we are able to focus on several simple underlying principles. We identify a notion of subjective beliefs based on market behavior, and show how it is related to various notions of belief that arise from different axiomatic treatments. We highlight the close connection between the fundamental welfare theorems of general equilibrium and results that link common beliefs and risk-sharing. Finally, by establishing these links for general convex preferences, we provide a framework for studying ambiguity in markets while allowing for heterogeneity in the way ambiguity is expressed through preferences. The generality of this approach identifies the forces underlying betting without being restricted to any one particular representation, and in so doing unifies our thinking about models of ambiguity aversion in economic settings.

The note is organized as follows. Section 11.2 studies subjective beliefs and behavioral characterizations, with illustrations for various familiar representations. Section 11.3 studies trade between agents with convex preferences. Appendix A develops an extension of these results to infinite state spaces, while Appendix B collects some proofs omitted in the text.

11.2. Beliefs and Convex Preferences

11.2.1. Convex Preferences

Let S be a finite set of states of the world. The set of consequences is $\mathbb{R}_+$, which we interpret as monetary payoffs. The set of acts is $\mathcal{F} = \mathbb{R}_+^S$ with the natural topology. Acts are denoted by f, g, h, while $f(s)$ denotes the monetary payoff from act f when state

s obtains. For any $x \in \mathbb{R}_+$ we abuse notation by writing $x \in \mathcal{F}$, which stands for the constant act with payoff x in each state of the world.

Let $\succsim$ be a binary relation on $\mathcal{F}$. We say that $\succsim$ is a *convex preference relation* if it satisfies the following axioms:

Axiom 11.1—Preference: $\succsim$ is complete and transitive.

Axiom 11.2—Continuity: For all $f \in \mathcal{F}$, the sets $\{g \in \mathcal{F} \mid g \succsim f\}$ and $\{g \in \mathcal{F} \mid f \succsim g\}$ are closed.

Axiom 11.3—Monotonicity: For all $f, g \in \mathcal{F}$, if $f(s) > g(s)$ for all $s \in S$, then $f \succ g$.

Axiom 11.4—Convexity: For all $f \in \mathcal{F}$, the set $\{g \in \mathcal{F} \mid g \succsim f\}$ is convex.

These axioms are standard, and well-known results imply that a convex preference relation $\succsim$ is represented by a continuous, increasing and quasi-concave function $V : \mathcal{F} \to \mathbb{R}$.[1] Convex preferences include as special cases many common models of risk aversion and ambiguity aversion. In many of these special cases, one element of the representation identifies a notion of beliefs. In what follows, we adopt the notion of subjective probability suggested in **?** to define *subjective beliefs* for general convex preferences. We then study characterizations of this concept in terms of market behavior, and illustrate particular special cases including maxmin expected utility, Choquet expected utility, and variational preferences.

[1] Axiom 11.4 captures convexity in monetary payoffs. For Choquet expected utility agents, who evaluate an act according to the Choquet integral of its utility with respect to a non-additive measure (capacity), the relation between payoff-convexity and uncertainty aversion has been studied by **?**. **?** studies the relation between payoff-convexity and risk aversion.

11.2.2. Supporting Hyperplanes and Beliefs

The decision-theoretic approach of de Finetti, Ramsey, and Savage identifies a decision maker's subjective probability with the odds at which he is willing to make small bets. In this spirit, ? identifies subjective probability with a hyperplane that supports the upper contour set.[2] If this set has kinks, for example because of non-differentiabilities often associated with ambiguity, there may be multiple supporting hyperplanes at some acts. To encompass such preferences, we consider the set of *all* (normalized) supporting hyperplanes.[3]

Definition 11.1: *The set of subjective beliefs at an act f is*

$$\boldsymbol{\pi}(f) := \{p \in \Delta S \mid p \cdot g \geq p \cdot f \text{ for all } g \succsim f\}$$

Given the interpretation of the elements of $\boldsymbol{\pi}(f)$ as beliefs, we will write $E_p g$ instead of $p \cdot g$. For any convex preference relation, $\boldsymbol{\pi}(f)$ is nonempty, compact and convex, and is equivalent to the set of (normalized) supports to the upper contour set of $\succsim$ at f. In the next section we explore behavioral implications of this definition, including willingness or unwillingness to trade, and their market consequences.

11.2.3. Market Behavior and Beliefs

We begin with a motivating example, set in the maxmin expected utility (MEU) model of Gilboa and Schmeidler (1989). The agent's preferences are represented using a compact,

[2] In the finance literature this is commonly called a risk-neutral probability, or risk-adjusted probability.
[3] Alternatively, ? define beliefs using superdifferentials of the *benefit function*. Their definition turns out to be equivalent to ours.

convex set of priors $P \subseteq \Delta S$ and a utility index $u : \mathbb{R}_+ \to \mathbb{R}$ that is concave and differentiable. The utility of an act f is given by the minimum expected utility over the set of priors P:

$$V(f) := \min_{p \in P} \sum_{s \in S} p_s u(f(s)) = \min_{p \in P} E_p u(f)$$

where we abuse notation by writing $u(f)$ for $\big(u(f(1)), \ldots, u(f(S))\big)$.

Imagine that the agent is initially endowed with a constant act x. First, consider an act g such that $E_p g = x$ for some $p \in P$, as depicted in the left panel of Figure 11.1 (the shaded area collects all such acts). One can see that the agent will have zero demand for g. Second, consider an act g such that $E_p g > x$ for all $p \in P$, as depicted in the right panel of Figure 11.1. One can see that there exists $\varepsilon > 0$ sufficiently small such that $\varepsilon g + (1 - \varepsilon)x \succ x$.

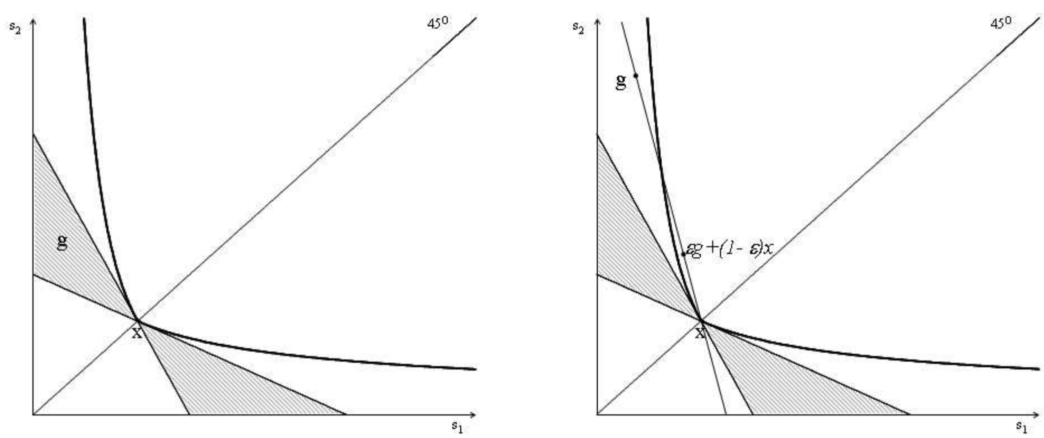

Figure 11.1. Behavioral properties of beliefs in the MEU model.

In the MEU model, the set P captures two important aspects of market behavior (both evident in Figure 11.1). First, agents are unwilling to trade from a constant bundle to a random one if the two have the same expected value for *some* prior in the set P. In particular, the set P is the largest set of beliefs revealed by this unwillingness to trade based on zero expected net returns. Second, agents are willing to trade from a constant bundle to (a possibly small fraction of) a random one whenever the random act has greater expected value according to *every* prior in the set P. In particular, the set P is the smallest set of beliefs revealing this willingness to trade based on positive expected net returns.

We introduce two notions of beliefs revealed by market behavior that attempt to capture these properties for general convex preferences. The first notion collects all beliefs that reveal an unwillingness to trade from a given act f.

Definition 11.2: *The set of beliefs revealed by unwillingness to trade at f is*

$$\boldsymbol{\pi}^u(f) := \{p \in \Delta S \mid f \succsim g \text{ for all } g \text{ such that } E_p g = E_p f\}.$$

This set gathers all beliefs for which the agent is unwilling to trade assets with zero expected net returns. It can also be interpreted as the set of Arrow-Debreu prices for which the agent endowed with f will have zero net demand. For a convex preference, it is straightforward to see that this gives a set of beliefs equivalent to that defined by our subjective beliefs in Definition 11.1.

Our second notion collects beliefs revealed by a willingness to trade from a given act f. To formalize this, let $\mathcal{P}(f)$ denote the collection of all compact, convex sets $P \subseteq \Delta S$

such that if $E_p g > E_p f$ for all $p \in P$ then $\varepsilon g + (1-\varepsilon) f \succ f$ for sufficiently small ε.[4] We define the willingness-to-trade revealed beliefs as the smallest such set.[5]

Definition 11.3: *The set of beliefs revealed by willingness to trade at an act f is*

$$\boldsymbol{\pi}^w(f) := \bigcap \mathcal{P}(f)$$

The following proposition establishes the equivalence between the different notions of belief presented in this section, and therefore gives behavioral content to Definition 11.1. Subjective beliefs are related to observable market behavior in terms of willingness or unwillingness to make small bets or trade small amounts of assets.

Proposition 11.1: *If $\succsim$ is a convex preference relation, then $\boldsymbol{\pi}(f) = \boldsymbol{\pi}^u(f) = \boldsymbol{\pi}^w(f)$ for every strictly positive act f.*

11.2.4. Special cases

In this section we explore the relationships between our notion of subjective belief and those arising in several common models of ambiguity. For the benchmark case of classical subjective expected utility, as observed by **?**, our subjective beliefs coincide with the local trade-offs or risk-neutral probabilities that play a central role in many applications of risk. If we restrict attention to constant acts, then subjective beliefs will coincide with the unique prior of the subjective expected utility representation. This property generalizes beyond SEU. The subjective beliefs we calculate at a constant act, at which risk

[4]Notice that $\mathcal{P}(f)$ is always nonempty, because $\Delta S \in \mathcal{P}(f)$ by Axiom 11.3.

[5]The proof of Proposition 11.1 shows that $\mathcal{P}(f)$ is closed under intersection.

and ambiguity are absent, coincide with the beliefs identified axiomatically in particular representations.

Maxmin Expected Utility Preferences

We begin with MEU preferences, represented by a particular set of priors P and utility index u.[6] These preferences also include the convex case of Choquet expected utility, for which P has additional structure as the core of a convex capacity.

To derive a simple characterization of the set $\boldsymbol{\pi}(f)$ for MEU preferences, let $U : \mathbb{R}^S_+ \to \mathbb{R}^S$ be the function $U(f) := (u(f(1)), \ldots, u(f(S)))$ giving ex-post utilities in each state. For any $f \in \mathbb{R}^S_{++}$, $DU(f)$ is the $S \times S$ diagonal matrix with diagonal given by the vector of ex-post marginal utilities $(u'(f(1)), \ldots, u'(f(S)))$. For each $f \in \mathbb{R}^S_+$, let

$$M(f) := \arg\min_{p \in P} E_p u(f)$$

be the set of minimizing priors realizing the utility of f. Note that $V(f) = E_p u(f)$ for each $p \in M(f)$. Using a standard envelope theorem, we can express the set $\boldsymbol{\pi}(f)$ as follows.

Proposition 11.2: *Let $\succsim$ be a MEU preference represented by a set of priors P and a concave, strictly increasing and differentiable utility index u. Then $\succsim$ is a convex preference, and*

$$\boldsymbol{\pi}(f) = \left\{ \frac{q}{\|q\|} \mid q = pDU(f) \text{ for some } p \in M(f) \right\}.$$

[6]The MEU model is a special case of the model of invariant biseparable preferences in **?**. **?** introduce a definition of beliefs for such preferences and propose a differential characterization. For invariant biseparable preferences that are also convex, their differential characterization is equivalent to ours when calculated at constant bundle. The only invariant biseparable preferences that are convex are actually MEU preferences, however, so these are already included in our present discussion.

In particular, $\boldsymbol{\pi}(x) = P$ *for all constant acts* x*.*

Variational Preferences

Introduced and axiomatized by Maccheroni et al. (2006a), variational preferences have the following representation:

$$V(f) = \min_{p \in \Delta S} [E_p u(f) + c^\star(p)]$$

where $c^\star \colon \Delta S \to [0, \infty]$, is a convex, lower semicontinuous function such that $c^\star(p) = 0$ for at least one $p \in \Delta S$. The function $c^\star$ is interpreted as the cost of choosing a prior. As special cases, this model includes MEU preferences, when $c^\star$ is 0 on the set P and ∞ otherwise, the multiplier preferences of Hansen and Sargent (2001), when $c^\star(p) = R(p \,\|\, q)$ is the relative entropy between p and some fixed reference distribution q, and the mean-variance preference of Markovitz and Tobin, when $c^\star(p) = G(p \,\|\, q)$ is the relative Gini concentration index between p and some fixed reference distribution q.

For each $f \in \mathbb{R}^S_+$, let

$$M(f) := \arg\min_{p \in \Delta S} \{E_p[u(f)] + c^\star(p)\}$$

be the set of minimizing priors realizing the utility of f. Note that $V(f) = E_p u(f) + c^\star(p)$ for each $p \in M(f)$. The set $\boldsymbol{\pi}(f)$ can be characterized as follows.

Proposition 11.3: *Let* $\succsim$ *be a variational preference for which* u *is concave, increasing, and differentiable. Then* $\succsim$ *is a convex preference and*

$$\boldsymbol{\pi}(f) = \left\{ \frac{q}{\|q\|} \mid q = pDU(x) \text{ for some } p \in M(f) \right\}.$$

In particular, $\boldsymbol{\pi}(x) = \{p \in \Delta S \mid c^\star(p) = 0\}$ *for all constant acts* x*.*

The set of subjective beliefs at a constant act x, $\boldsymbol{\pi}(x)$, is equal to the set of probabilities for which $c^\star$, the cost of choosing a prior, is zero. An interesting implication of this result is that at a constant act, the subjective beliefs of an agent with Hansen and Sargent (2001) multiplier preferences are equal to the singleton $\{q\}$ consisting of the reference probability, since $R(p \parallel q) = 0$ if and only if $p = q$.[7] A similar result holds for mean-variance preferences.

Confidence Preferences

? introduced and axiomatized a class of preferences in which ambiguity is measured by a confidence function $\varphi : \Delta S \to [0, 1]$. The value of $\varphi(p)$ describes the decision maker's confidence in the probabilistic model p; in particular $\varphi(p) = 1$ means that the decision maker has full confidence in p. By assumption, the set of such full confidence measures is nonempty; moreover, the function φ is assumed to be upper semi continuous and quasiconcave. Preferences in this model are represented by:

$$V(f) = \min_{p \in L_\alpha} \frac{1}{\varphi(p)} E_p u(f),$$

where $L_\alpha = \{q \in \Delta S \,|\, \varphi(q) \geq \alpha\}$ is a set of measures with confidence above α.

As before, for each $f \in \mathbb{R}^S_+$, let

$$M(f) := \arg\min_{p \in L_\alpha} \left\{ \frac{1}{\varphi(p)} E_p u(f) \right\}$$

[7]This result also follows from an alternate representation $V(f) = -E_q \exp\big(-\theta^{-1} \cdot u(f)\big)$ of those preferences. **?** obtains an axiomatization of multiplier preferences along these lines.

be the set of minimizing priors realizing the utility of f. Note that $V(f) = \frac{1}{\varphi(p)} E_p u(f)$ for each $p \in M(f)$. By standard envelope theorems, $\boldsymbol{\pi}(f)$ can be characterized in this case as follows.

Proposition 11.4: *Let $\succsim$ be a confidence preference for which u is concave, increasing, and continuously differentiable. Then $\succsim$ is a convex preference and*

$$\boldsymbol{\pi}(f) = \left\{ \frac{q}{\|q\|} \mid q = pDU(x) \text{ for some } p \in M(f) \right\}.$$

In particular, $\boldsymbol{\pi}(x) = \{p \in \Delta S \mid \varphi(p) = 1\}$ for all constant acts x.

Smooth Model

The *smooth model* of ambiguity developed in Klibanoff et al. (2005) allows preferences to display non-neutral attitudes towards ambiguity, but avoids kinks in the indifference curves.[8] This model has a representation of the form

$$V(f) = E_\mu \phi(E_p u(f))$$

where μ is interpreted as a probability distribution on the set of possible probability measures, $\phi : \mathbb{R} \to \mathbb{R}$ and $u : \mathbb{R}_+ \to \mathbb{R}$. When the indexes ϕ and u are concave, increasing, and differentiable, this utility represents a convex preference relation, and the set of subjective beliefs is a singleton consisting of a weighted mixture of all probabilities in the support of the measure μ.

[8]For similar models, see Segal (1990), Nau (2006) and Ergin and Gul (2004).

Proposition 11.5: *Let $\succsim$ be a smooth model preference for which u and ϕ are concave, increasing, and differentiable. Then $\succsim$ is a convex preference and*

$$\boldsymbol{\pi}(f) = \frac{1}{\|E_\mu[\phi'(E_p u(f))pDU(f)]\|} E_\mu[\phi'(E_p u(f))pDU(f)].$$

In particular, $\boldsymbol{\pi}(x) = \{E_\mu p\}$ for all constant x.

Ergin-Gul Model

Ergin and Gul (2004) introduce a model in which the state space takes the product form $S = S_a \times S_b$. This model permits different decision attitudes toward events in S_a and S_b, thereby inducing Ellsberg-type behavior. Consider a product measure $p = p_a \otimes p_b$ on S; for any $f \in \mathbb{R}^S$ let $E_a f$ be the vector of conditional expectations of f computed for all elements of S_b (thus $E_a f \in \mathbb{R}^{S_b}$) and for any $g \in \mathbb{R}^{S_b}$ let $E_b g$ denote the expectation of g according to p_b. The preferences are represented by

$$V(f) = E_b \phi(E_a u(f)).$$

In order to express subjective beliefs, let $U(f)$ and $DU(f)$ be defined as before, with the convention that the states in S are ordered lexicographically first by a, then by b. Analogously, for each f define the vector $\Phi(E_a u(f)) \in \mathbb{R}^{S_b}$ and the diagonal matrix $D\Phi(E_a u(f))$.

Proposition 11.6: *Let $\succsim$ be an Ergin-Gul preference for which u and ϕ are concave, increasing, and differentiable. Then $\succsim$ is a convex preference and*

$$\boldsymbol{\pi}(f) = \frac{1}{\|pDU(f)[I_a \otimes D\Phi(E_a u(f))]\|} pDU(f)[I_a \otimes D\Phi(E_a u(f))],$$

where I_a is the identity matrix of order S_a and $\otimes$ is the tensor product. In particular, $\boldsymbol{\pi}(x) = \{p\}$ for all constant x.

Remark 1: Our notion of beliefs may not agree with the beliefs identified by some representations, in part because we have focused on beliefs revealed by market behavior rather than those identified axiomatically. An illustrative case in point is rank-dependent expected utility (RDEU) of **?** and **?** in which probability distributions are distorted by a transformation function. When the probability transformation function is concave, this model reduces to Choquet expected utility with a convex capacity, a special case of MEU. By using the MEU representation, beliefs would be identified with a set of priors P, in general not a singleton. As we showed above, this set P coincides with the set $\boldsymbol{\pi}(x)$, the subjective beliefs given by any constant act x. However, RDEU preferences are also probabilistically sophisticated in the sense of Machina and Schmeidler (1992), with respect to some measure p^*.[9] Using the alternative representation arising from probabilistic sophistication, beliefs would instead be identified with this unique measure p^* rather than with the set P. Although $p^* \in P$, these different representations nonetheless lead to different ways of identifying subjective beliefs, each justified by differing behavioral axioms.[10] This indeterminacy could lead to different ways of attributing market behavior to beliefs. For example, **?** attribute unwillingness to trade to probabilistic first-order risk aversion, while **?** instead attribute unwillingness to trade to non-probabilistic ambiguity aversion.

[9]For more on probabilistic sophistication, RDEU and MEU, see **?**.

[10]A similar issue arises in the differing definitions of ambiguity found in the ambiguity aversion literature. One definition of ambiguity, due to Ghirardato and Marinacci (2002), takes the SEU model as a benchmark and attributes all deviations from SEU to non-probabilistic uncertainty aversion. Another definition, due to Epstein (1999), uses the probabilistic sophistication model as a benchmark and hence attributes some deviations from SEU to probabilistic first-order risk aversion rather than non-probabilistic uncertainty aversion.

11.3. Ex-Ante Trade

In this section, we use subjective beliefs to characterize efficient allocations. As our main result, we show that in the absence of aggregate uncertainty, efficiency is equivalent to full insurance under a "common priors" condition. While we maintain the assumption of a finite state space for simplicity, all of these results extend directly to the case of an infinite state space with appropriate modifications; for details see Appendix A.

We study a standard two-period exchange economy with one consumption good in which uncertainty at date 1 is described by the set S. There are m agents in the economy, indexed by i. Each agent's consumption set is the set of acts $\mathcal{F}$. The aggregate endowment is $e \in \mathbb{R}^S_{++}$. An allocation $f = (f_1, \ldots, f_m) \in \mathcal{F}^m$ is *feasible* if $\sum_{i=1}^m f_i = e$. An allocation f is *interior* if $f_i(s) > 0$ for all s and for all i. An allocation f is a *full insurance* allocation if f_i is constant across states for all i; any other allocation will be interpreted as betting. An allocation f is *Pareto optimal* if there is no feasible allocation g such that $g_i \succsim_i f_i$ for all i and $g_j \succ_j f_j$ for some j.

Proposition 11.7: *Suppose $\succsim_i$ is a convex preference relation for each i. An interior allocation $(f_1, \ldots, f_m)$ is Pareto optimal if and only if $\bigcap_i \boldsymbol{\pi}_i(f_i) \neq \emptyset$.*

Proof. First, suppose $(f_1, \ldots, f_m)$ is an interior Pareto optimal allocation. By the second welfare theorem, there exists $p \in \mathbb{R}^S$, $p \neq 0$, supporting this allocation, that is, such that $p \cdot g \geq p \cdot f_i$ for all $g \succsim_i f_i$ and each i. By monotonicity, $p > 0$, thus after normalizing we may take $p \in \Delta S$. By definition, $p \in \boldsymbol{\pi}_i(f_i)$ for each i, hence $\bigcap_i \boldsymbol{\pi}_i(f_i) \neq \emptyset$. For the other implication, take $p \in \bigcap_i \boldsymbol{\pi}_i(f_i)$. By standard arguments, $(f_1, \ldots, f_m; p)$ is a Walrasian equilibrium in the exchange economy with endowments $(f_1, \ldots, f_m)$. By the first welfare theorem, $(f_1, \ldots, f_m)$ is Pareto optimal. $\square$

This result provides a helpful tool to study mutual insurance and contracting between agents, regardless of the presence of aggregate uncertainty. The following example illustrates. Consider an exchange economy with two agents. The first agent has MEU preferences with set of priors P_1 and linear utility index, while the second agent has SEU preferences with prior p_2, also with a linear utility index. Assume p_2 belongs to the relative interior of P_1 (and hence that P_1 has a nonempty relative interior).[11] Thus this is an economy in which one agent is risk and ambiguity neutral, while the other is risk neutral but strictly ambiguity averse; moreover, the second agent is more ambiguity averse than the first, using the definition of **?**. In this case, an interior allocation is Pareto optimal if and only if it fully insures the ambiguity averse agent. This is because Proposition 11.7 implies an interior allocation f can be Pareto optimal if and only if $p_2 \in \boldsymbol{\pi}_1(f_1)$. If f_1 does not involve full insurance for agent 1, then $\boldsymbol{\pi}_1(f_1)$ will be the convex hull of a strict subset of the extreme points of P_1, and in particular, will not contain p_2. Alternatively, at any constant bundle x_1, $\boldsymbol{\pi}_1(x_1) = P_1 \ni p_2 = \boldsymbol{\pi}_2(e - x_1)$, so any such allocation is Pareto optimal. This result can be easily extended to the case in which agent 1 is also ambiguity averse, with MEU preferences given by the same utility index and a set P_2, provided P_2 is contained in the relative interior of P_1. Similarly, risk aversion can be introduced, although for given beliefs the result will fail for sufficiently high risk aversion.

Our main results seek to characterize desire for insurance and willingness to bet as a function of shared beliefs alone. To isolate the effects of beliefs, we first rule out aggregate

[11]By relative interior, here we mean relative to the affine hull of P_1.

uncertainty by taking the aggregate endowment e to be constant across states. In addition, we must rule out pure indifference to betting, as might occur in an SEU setting with risk neutral agents. The following two axioms guarantee that such indifference to betting is absent.

Axiom 11.5—Strong Monotonicity: For all $f \neq g$, if $f \geq g$, then $f \succ g$.

Axiom 11.6—Strict Convexity: For all $f \neq g$ and $\alpha \in (0,1)$, if $f \succsim g$, then $\alpha f + (1-\alpha)g \succ g$.

Finally, we focus on preferences for which local trade-offs in the absence of uncertainty are independent of the (constant) level of consumption. These preferences are characterized by the fact that the directions of local improvement, starting from a constant bundle at which uncertainty is absent, are independent of the particular constant.

Axiom 11.7—Translation Invariance at Certainty: For all $g \in \mathbb{R}^S$ and all constant bundles $x, x' > 0$, if $x + \lambda g \succsim x$ for some $\lambda > 0$, then there exists $\lambda' > 0$ such that $x' + \lambda' g \succsim x'$.

This axiom will be satisfied by all of the main classes of preferences we have considered. A simple example violating this axiom is the SEU model with state-dependent utility; in this case, the slopes of indifference curves can change along the 45^o line. In fact, in the class of SEU preferences Axiom 11.7 is equivalent to a state-independent and differentiable utility function. We show below that for a convex preference relation, translation invariance at certainty suffices to ensure that subjective beliefs are instead constant across constant bundles.

Proposition 11.8: *Let $\succsim$ be a convex preference relation satisfying Axiom 11.7. Then $\boldsymbol{\pi}(x) = \boldsymbol{\pi}(x')$ for all constant acts $x, x' > 0$.*

By this result, we can write $\boldsymbol{\pi}$ in place of $\boldsymbol{\pi}(x)$ when translation invariance at certainty is satisfied; we maintain this notational simplification below.

Our main result follows. For any collection of convex preferences satisfying translation invariance at certainty, the sets $\boldsymbol{\pi}_i$ of subjective beliefs contain all of the information needed to predict the presence or absence of purely speculative trade. Regardless of other features of the representation of preferences, the existence of a common subjective belief, understood to mean $\bigcap_i \boldsymbol{\pi}_i \neq \emptyset$, characterizes the efficiency of full insurance. Moreover, these results can be understood as straightforward consequences of the basic welfare theorems.

Proposition 11.9: *If the aggregate endowment is constant across states and $\succsim_i$ satisfies Axioms 11.1–11.7 for each i, then the following statements are equivalent:*

(i) There exists an interior full insurance Pareto optimal allocation.

(ii) Any Pareto optimal allocation is a full insurance allocation.

(iii) Every full insurance allocation is Pareto optimal.

(iv) $\bigcap_i \boldsymbol{\pi}_i \neq \emptyset$.

Proof. We show the sequence of inclusions:

(i) ⇒ (iv): Suppose that $x = (x_1, \ldots, x_m)$ is an interior full insurance allocation that is Pareto optimal. By the second welfare theorem, there exists $p \neq 0$ such that p supports the allocation x, that is, such that for each i, $p \cdot f \geq p \cdot x_i$ for all $f \succsim_i x_i$. By monotonicity,

$p > 0$, so after normalizing we can take $p \in \Delta S$. By definition $p \in \boldsymbol{\pi}_i$ for all i, hence $\bigcap_i \boldsymbol{\pi}_i \neq \emptyset$.

(iv) $\Rightarrow$ *(ii)*: Let $p \in \bigcap_i \boldsymbol{\pi}_i$ and suppose f is a Pareto optimal allocation such that f_j is not constant for some j. Define $x_i := E_p f_i$ for each i. By strict monotonicity, $p \gg 0$. Thus $x_i \geq 0$ for all i, and $x_i = 0 \iff f_i = 0$. Since $p \in \bigcap_{\{i:x_i>0\}} \boldsymbol{\pi}_i(x_i) = \bigcap_{\{i:x_i>0\}} \boldsymbol{\pi}_i^u(x_i)$, $x_i \succsim f_i$ for all i, and by strict convexity, $x_j \succ_j f_j$. Then the allocation $x = (x_1, \ldots, x_m)$ is feasible, and Pareto dominates f, which is a contradiction.

(ii) $\Rightarrow$ *(iii)*: Suppose that x is a full insurance allocation that is not Pareto optimal. Then there is a Pareto optimal allocation f that Pareto dominates x. By *(ii)*, f must be a full insurance allocation, which is a contradiction.

(iii) $\Rightarrow$ *(i)*: The allocation $(\frac{1}{m}e, \ldots, \frac{1}{m}e)$ is an interior full insurance allocation. By *(iii)* it is Pareto optimal. $\square$

Figure 11.2 illustrates Proposition 11.9 using an Edgeworth box: x is a full-insurance allocation and the two individuals' preferences and subjective beliefs are drawn in black and gray. One can easily verify that x is Pareto optimal and that the intersection of the subjective beliefs is not empty in this case.

Remark 2: **?** derive a version of this result for the particular case of maxmin preferences using an ingenious separation argument.[12] In this case, the common prior condition *(iv)* becomes the intuitive condition $\cap_i P_i \neq \emptyset$.[13] **?** also consider the case of an infinite state space. In the appendix, we show that our result can be similarly extended to an infinite state space, although the argument is somewhat more delicate.

[12]In **?** there is an imprecision in the proof that *(ii)* $\Rightarrow$ *(iii)*, which implicitly uses condition *(iv)*.
[13]See **?** for related results regarding purely speculative trade and no-trade theorems.

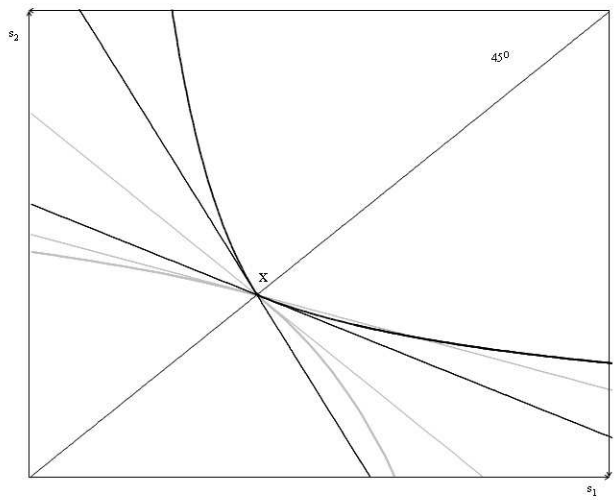

Figure 11.2. Full insurance and common subjective beliefs.

We view a main contribution of our result (and its extension to the infinite state space case) not as establishing the link between efficiency and notions of common priors per se, but in illustrating that these results are a simple consequence of the welfare theorems linking Pareto optimality to the existence of linear functionals providing a common support to agents' preferred sets, coupled with the particular form these supports take for various classes of preferences.

Proposition 11.9 can be articulated in the language of specific functional forms discussed in Section 11.2.4. For SEU preferences, condition *(iv)* becomes the standard common prior assumption, whereas for MEU preferences we recover the result of **?**. For the smooth model of Klibanoff et al. (2005) condition *(iv)* means that the *expected* measures have to coincide, while for variational preferences of Maccheroni et al. (2006a) the sets of measures with zero cost have to intersect. Interestingly, it follows that for Hansen and Sargent (2001) multiplier preferences condition *(iv)* means that the reference measures coincide.

Finally, we note that extending Propositions 11.7 and 11.9 to allow for incomplete preferences is fairly straightforward, after appropriately modifying axioms 11.1 and 11.2. [14]

11.4. Infinite State Space

Now we imagine that the state space S may be infinite, and let Σ be a σ-algebra of measurable subsets of S. Let $B(S, \Sigma)$ be the space of all real-valued, bounded, and measurable functions on S, endowed with the sup norm topology. Let $ba(S, \Sigma)$ be the space of bounded, finitely additive measures on (S, Σ), endowed with the weak* topology, and let ΔS be the subset of finitely additive probabilities. As in the finite case, we let $\mathcal{F}$ denote the set of acts, which is now $B(S, \Sigma)_+$. We continue to use $x \in \mathbb{R}_+$ interchangeably for the constant act delivering x in each state s. For an act f, a constant $x \in \mathbb{R}_+$ and an event $E \subset S$, let xEf denote the act such that

$$(xEf)(s) = \begin{cases} x & \text{if } s \in E \\ f(s) & \text{if } s \notin E \end{cases}$$

The goal of this section is to establish an analogue of our main result regarding the connection between the efficiency of full insurance and the existence of shared beliefs, Proposition 11.9, for infinite state spaces. Our work in section 3 renders this analogue fairly straightforward by highlighting the close link between these results and the fundamental welfare theorems, appropriate versions of which hold in infinite-dimensional settings as well.

[14] A similar observation is made by **?**, while a recent paper by **?** studies Pareto optima for general incomplete preferences.

Because topological issues are often subtle in infinite-dimensional spaces due to the multiplicity of non-equivalent topologies, we begin by emphasizing the meaning of our basic continuity axiom in this setting.

Axiom 11.8—Continuity: For all $f \in \mathcal{F}$, the sets $\{g \in \mathcal{F} \mid g \succsim f\}$ and $\{g \in \mathcal{F} \mid f \succsim g\}$ are closed in the sup-norm topology.

To accommodate an infinite state space, we will need several additional axioms that serve to restrict agents' beliefs, first by ensuring that beliefs are countably additive, and that beliefs are all mutually absolutely continuous both for a given agent and between different agents. To that end, consider the following:

Axiom 11.9—Countable Additivity: For each f, each $p \in \boldsymbol{\pi}(f)$ is countably additive.

Axiom 11.10—Mutual Absolute Continuity: If $xEf \sim f$ for some event E and some acts x, f with $x > \sup f$, then $yEg \sim g$ for every y and every act g.

Proposition 11.10: *Let $\succsim$ be monotone, continuous, convex, and satisfy mutual absolute continuity. If f, g are acts such that $\inf f, \inf g > 0$, then $\boldsymbol{\pi}(f)$ and $\boldsymbol{\pi}(g)$ contain only measures that are mutually absolutely continuous.*

Proof. Suppose, by way of contradiction, that acts f, g with $\inf f, \inf g > 0$, an event E, and measures $p \in \boldsymbol{\pi}(f), \bar{p} \in \boldsymbol{\pi}(g)$ such that $p(E) = 0$ while $\bar{p}(E) > 0$. Choose $x > \sup f$. By monotonicity, $x \succ f$ and $xEf \succsim f$. Since $p(E) = 0$,

$$p \cdot (xEf) = p \cdot f$$

Together with $p \in \boldsymbol{\pi}(f)$ this implies $xEf \sim f$. Choose y such that $y < \inf g$. By mutual continuity, $yEg \sim g$. Since $\bar{p}(E) > 0$,

$$\bar{p} \cdot (yEg) < \bar{p} \cdot g$$

But $\bar{p} \in \boldsymbol{\pi}(g)$, which yields a contradiction. □

The same argument will show that if mutual continuity holds across agents, then all beliefs of all agents are mutually absolutely continuous. We say that a collection $\{\succsim_i : i = 1, \ldots, m\}$ of preference orders on $\mathcal{F}$ *satisfies mutual absolute continuity* if whenever $xEf \sim_i f$ for some agent i, some event E, and some $x > \sup f$, then $yEg \sim_j g$ for every agent j, every y, and every act g.

Proposition 11.11: *Let $\succsim_i$ be monotone, continuous, and convex for each i, and let $\{\succsim_i : i = 1, \ldots, m\}$ satisfy mutual absolute continuity. Then for every i, j and any acts f, g such that $\inf f, \inf g > 0$, $\boldsymbol{\pi}_i(f)$ and $\boldsymbol{\pi}_j(g)$ contain only measures that are mutually absolutely continuous.*

Mutual absolute continuity is a strong assumption, and is close to the desired conclusion of mutual absolute continuity of agents' beliefs. Without more structure on preferences, it does not seem possible to weaken, however. Without the additional structure available in various representations, nothing needs to tie together beliefs at different acts. This gives us very little to work with for general convex preferences. In contrast, in particular special cases, much weaker conditions would suffice to deliver the same conclusion. For example, **?** show that a version of the modularity condition of **?** is equivalent to mutual absolute continuity of priors in the MEU model.

For a complete analogue of our main result regarding the connection between common priors and the absence of betting, we must ensure that individually rational Pareto optimal allocations exist given any initial endowment allocation. This is needed to show that *(ii)* $\Rightarrow$ *(iii)* in Proposition 11.9 without the additional assumption of a common prior, that is, to show that if every Pareto optimal allocation must involve full insurance, then all full insurance allocations are in fact Pareto optimal. Since no two full insurance allocations can be Pareto ranked, this conclusion will follow immediately from the existence of individually rational Pareto optimal allocations. Instead **?** use the existence of a common prior, condition *(iv)*, to argue that any Pareto improvement must itself be Pareto dominated by the full insurance allocation with consumption equal to the expected values, computed with respect to some common prior. In the finite state space case, it is straightforward to give an alternative argument that does not make use of the common prior condition. If a full insurance allocation is not Pareto optimal, then there must exist a Pareto optimal allocation that dominates it, as a consequence of the existence of individually rational Pareto optimal allocations. When all Pareto optimal allocations involve full insurance, this leads to a contradiction that establishes the desired implication.

With an infinite state space, the existence of individually rational Pareto optimal allocations is more delicate. Typically, this existence is derived from continuity of preferences in some topology in which order intervals, and hence sets of feasible allocations, are compact. In our setting, such topological assumptions are problematic, as order intervals in $B(S, \Sigma)$ fail to be compact in topologies sufficiently strong to make continuity a reasonable and not overly restrictive assumption. Instead we give a more subtle argument that makes use of countable additivity and mutual continuity to give an equivalent formulation of the problem recast in $L_\infty(S, \Sigma, \mu)$ for an appropriately chosen measure μ.

More precisely, suppose that $\{\succsim_i : i = 1, \ldots, m\}$ satisfy mutual absolute continuity. Choose a measure $\mu \in \boldsymbol{\pi}_1(x)$ for some constant x. We can extend each $\succsim_i$ to $L_\infty(S, \Sigma, \mu)_+$ in the natural way, first by embedding $B(S, \Sigma)_+$ in $L_\infty(S, \Sigma, \mu)_+$ via the identification of an act f with its equivalence class $[f] \in L_\infty(S, \Sigma, \mu)_+$, and then by noticing that a preference order satisfying our basic axioms will be indifferent over any acts $f, f' \in B(S, \Sigma)_+$ such that $f' \in [f]$. This allows us to extend each preference order $\succsim_i$ to $L_\infty(S, \Sigma, \mu)_+$ in the natural way, by defining $[f] \succsim_i [g] \iff f \succsim_i g$ for any $f, g \in B(S, \Sigma)_+$. Similarly, given a utility representation V_i of $\succsim_i$ on $B(S, \Sigma)_+$, define $V_i : L_\infty(S, \Sigma, \mu)_+ \to \mathbb{R}$ by $V_i([f]) = V_i(f)$ for each $f \in B(S, \Sigma)_+$.

With this recasting of the problem, the existence of individually rational Pareto optimal allocations follows from an additional type of continuity.

Axiom 11.11—Countable Continuity: There exists $\bar{x}$ and $\mu \in \boldsymbol{\pi}(\bar{x})$ such that for all $g, f, x \in \mathcal{F}$, if $\{f^\alpha\}$ is a net in $\mathcal{F}$ with $f^\alpha \succsim x$ and $f^\alpha \leq g$ for all α, and $q \cdot f^\alpha \to q \cdot f$ for all $q \in ca(S, \Sigma)$ such that $q \ll \mu$, then $f \succsim x$.

Proposition 11.12: *Let $\succsim_i$ be monotone, continuous, countably continuous, countably additive, and convex for each i, and let $\{\succsim_i : i = 1, \ldots, m\}$ satisfy mutual absolute continuity. For any initial endowment allocation $(e_1, \ldots, e_m)$, individually rational Pareto optimal allocations exist.*

Proof. Fix a constant act $x > 0$ and choose a measure $\mu \in \boldsymbol{\pi}_1(x)$. If f and g are μ-equivalent, so $\mu(\{s : f(s) \neq g(s)\}) = 0$, then $f \sim_i g$ for each i. To see this, fix μ-equivalent acts f and g, and an agent i. Without loss of generality suppose $g \succsim_i f$. First suppose that $\inf f, \inf g > 0$. In this case, every $p \in \boldsymbol{\pi}_i(f)$ is absolutely continuous with respect to μ, so

$$p \cdot g = p \cdot f \quad \forall p \in \boldsymbol{\pi}_i(f)$$

Thus $f \succsim_i g$, and we conclude $g \sim_i f$ as desired. For the general case, consider the sequence of constant acts $\{x^n\}$ with $x^n = \frac{1}{n}$ for each n: $\inf x^n > 0$ for each n while $x^n \to 0$ in the sup-norm topology. For each n, the acts $f + x^n$ and $g + x^n$ are μ-equivalent, and $\inf(f + x^n), \inf(g + x^n) > 0$. By the previous argument, $f + x^n \sim_i g + x^n$ for each n, and by continuity $f \sim_i g$ as desired.

For each i, extend V_i to $L_\infty(S, \Sigma, \mu)_+$ using this observation, by defining $V_i([f]) := V_i(f)$ for each $f \in B(S, \Sigma)_+$.

Fix an initial endowment allocation $(e_1, \ldots, e_m)$, and set $e := \sum_i e_i$. By the Banach-Alaoglu Theorem, the order interval $[0, e]$ is weak*-compact in $L_\infty(S, \Sigma, \mu)_+$, and by mutual absolute continuity and countable continuity, V_i is weak*-upper semi-continuous on $[0, e]$.

From this it follows by standard arguments that for every initial endowment allocation $(e_1, \ldots, e_m)$, an individually rational Pareto optimal allocation exists; for completeness we reproduce an argument from **?**; see also Theorem 1.5.3 in **?**.

Define a preorder on the compact set of feasible allocations

$$\mathcal{A} := \{f \in [L_\infty(S, \Sigma, \mu)_+]^m : \sum_i f_i = e\}$$

as follows. Given feasible allocations $(f_1, \dots, f_m)$ and $(g_1, \dots, g_m)$, define $f \succsim g$ if $f_i \succsim_i g_i$ for each i. Set

$$\mathcal{B}(g) := \{f \in \mathcal{A} : f \succsim g\}$$

and

$$\mathcal{S} := \mathcal{B}((e_1, \dots, e_m)) = \{f \in \mathcal{A} : f \succsim (e_1, \dots, e_m)\}$$

Let $\mathcal{R}$ be a chain in $\mathcal{S}$. For any finite subset $\bar{\mathcal{R}}$ of $\mathcal{R}$, $\cap_{g \in \bar{\mathcal{R}}} \mathcal{B}(g) = \mathcal{B}(\max\ \bar{\mathcal{R}})$ is nonempty, by transitivity. Thus $\{\mathcal{B}(g) : g \in \mathcal{R}\}$ has the finite intersection property. Each $\mathcal{B}(g)$ is weak*-closed, hence, by compactness of $\mathcal{A}$, $\cap_{g \in \mathcal{R}} \mathcal{B}(g) \neq \emptyset$, and any element of $\cap_{g \in \mathcal{R}} \mathcal{B}(g)$ provides an upper bound for $\mathcal{R}$. By Zorn's lemma for preordered sets (see, e.g., **?**, p. 6), $\mathcal{S}$ has a maximal element, which is then an individually rational Pareto optimal allocation.

$\square$

With this in place, we turn to the infinite version of Proposition 11.9. The proof is analogous, making use of an infinite-dimensional version of the second welfare theorem and our previous result establishing the existence of individually rational Pareto optimal allocations in our model. As in the finite case, the aggregate endowment e is constant, with $e > 0$, hence $\inf e > 0$. We say that $f = (f_1, \dots, f_m) \in \mathcal{F}^m$ is a norm-interior allocation if $\inf f_i > 0$ for $i = 1, 2, \dots m$.

Proposition 11.13: *Let $\{\succsim_i : i = 1, \dots, m\}$ satisfy Axioms 11.1–11.10. Then the following statements are equivalent:*

(i) There exists a norm-interior full insurance Pareto optimal allocation.

(ii) Any Pareto optimal allocation is a full insurance allocation.

(iii) Every full insurance allocation is Pareto optimal.

(iv) $\bigcap_i \boldsymbol{\pi}_i \neq \emptyset$.

Proof. As in the proof of Proposition 11.9, we show the sequence of inclusions:

(i) $\Rightarrow$ *(iv)*: Suppose that $x = (x_1, \ldots, x_m)$ is a norm-interior full insurance allocation that is Pareto optimal. Each x_i is contained in the norm interior of $B(S, \Sigma)_+$, hence by the second welfare theorem, there exists $p \in ba(S, \Sigma)$ with $p \neq 0$ such that p supports the allocation x, that is, such that for each i, $p \cdot f \geq p \cdot x_i$ for all $f \succsim_i x_i$. By monotonicity, $p > 0$, so after normalizing we can take $p \in \Delta S$. By definition $p \in \boldsymbol{\pi}_i$ for all i, hence $\bigcap_i \boldsymbol{\pi}_i \neq \emptyset$.

(iv) $\Rightarrow$ *(ii)*: Let $p \in \bigcap_i \boldsymbol{\pi}_i$ and suppose f is a Pareto optimal allocation such that f_j is not constant for some j. Define $x_i := E_p f_i$ for each i. By strict monotonicity, p is strictly positive, that is, $p \cdot g > 0$ for any act $g > 0$. Together with countable additivity, this yields $x_i \geq 0$ for all i, and $x_i = 0 \iff f_i = 0$. Since $p \in \bigcap_{\{i : x_i > 0\}} \boldsymbol{\pi}_i(x_i) = \bigcap_{\{i : x_i > 0\}} \boldsymbol{\pi}_i^u(x_i)$, $x_i \succsim f_i$ for all i, and by strict convexity, $x_j \succ_j f_j$. Then the allocation $x = (x_1, \ldots, x_m)$ is feasible, and Pareto dominates f, which is a contradiction.

(ii) $\Rightarrow$ *(iii)*: Suppose that x is a full insurance allocation that is not Pareto optimal. Using Proposition 11.12, there must be a Pareto optimal allocation f that Pareto dominates x. By *(ii)*, f must be a full insurance allocation, which is a contradiction.

(iii) $\Rightarrow$ *(i)*: The allocation $(\frac{1}{m}e, \ldots, \frac{1}{m}e)$ is a norm-interior full insurance allocation. By *(iii)* it is Pareto optimal. $\square$

We close with an example illustrating how the additional axioms arising in the infinite state space case might naturally be satisfied. We consider the version of the MEU model studied by **?**. They consider an MEU model in which each agent i has a weak*-closed, convex set of priors $P_i \subset ba(S, \Sigma)$ consisting only of countably additive measures, and a utility index $u_i : \mathbb{R}_+ \to \mathbb{R}$ that is strictly increasing, strictly concave, and differentiable. In addition, they assume that all measures in P_i and P_j are mutually absolutely continuous for all i and j. It straightforward to verify that $P_i = \boldsymbol{\pi}_i$ for each i, as in the finite state case, and that the model satisfies countable additivity. To verify mutual absolute continuity, suppose that $x > \sup f$ but $xEf \sim_i f$ for some event E and some agent i. Using Theorems 3 and 5 of **?**, there must exist $p \in P_i$ such that $p(E) = 0$. Because all measures in P_i and P_j for any other j are assumed to be mutually absolutely continuous, it must be the case that $p(E) = 0$ for any $p \in P_j$ for any agent j, which guarantees that $yEg \sim_j g$ for all j and any other acts y, g.

To see that continuity and countable continuity are also satisfied, first take $\{f^n\}, f$ in $\mathcal{F}$ with $\|f^n - f\| \to 0$. Then

$$\begin{aligned} |V_i(f^n) - V_i(f)| &= |\min_{p \in \boldsymbol{\pi}_i} E_p(u_i(f^n)) - \min_{p \in \boldsymbol{\pi}_i} E_p(u_i(f))| \\ &\leq \max\{|E_{p^{n*}}(u_i(f^n) - u_i(f))|, |E_p(u_i(f^n) - u_i(f))|\} \end{aligned}$$

where $p^{n*} \in M(f^n)$ and $p^* \in M(f)$.[15] Since $\|u_i(f^n) - u_i(f)\| \to 0$, $|V_i(f^n) - V_i(f)| \to 0$, and the desired conclusion follows.

Next, to see that countable continuity is also satisfied, fix $\mu \in \boldsymbol{\pi}_1$ and an agent i. Take $g, f, x \in \mathcal{F}$ and a net $\{f^\alpha\}$ in $\mathcal{F}$ with $f^\alpha \succsim_i x$ and $f^\alpha \leq g$ for all α. Notice that it suffices to show that the set $\{f \in L_\infty(S, \Sigma, \mu)_+ : f \succsim_i x, f \in [0, g]\}$ is $\sigma(L_\infty(S, \Sigma, \mu), L_1(S, \Sigma, \mu))$-closed, with $\succsim_i$ and acts recast in $L_\infty(S, \Sigma, \mu)$ as in Proposition 12. Using convexity, this is equivalent to showing that this set is closed in the Mackey topology $\tau := \tau(L_\infty(S, \Sigma, \mu), L_1(S, \Sigma, \mu))$. Thus suppose $f^\alpha \xrightarrow{\tau} f$. By way of

[15]As in the finite state space case, $M(f) := \arg\min_{p \in \boldsymbol{\pi}_i} E_p(u_i(f))$.

contradiction, suppose that $x \succ_i f$, thus $V_i(x) = E_{p^*}(u_i(x)) > E_{p^*}(u_i(f))$, where as above $p^* \in M(f)$. Then for every α,

$$E_{p^*}(u_i(f^\alpha)) \geq V_i(f^\alpha) \geq E_{p^*}(u_i(x)) > E_{p^*}(u_i(f))$$

while

$$\begin{aligned} 0 < E_{p^*}(u_i(x)) - E_{p^*}(u_i(f)) \leq E_{p^*}(u_i(f^\alpha)) - E_{p^*}(u_i(f)) &= E_{p^*}(u_i(f^\alpha) - u_i(f)) \\ &= |E_{p^*}(u_i(f^\alpha) - u_i(f))| \\ &\leq E_{p^*}(|u_i(f^\alpha) - u_i(f)|) \\ &\leq E_{p^*}(K|f^\alpha - f|) \end{aligned}$$

for some $K > 0$, where the last inequality follows from the assumption that u_i is strictly concave, strictly increasing, and differentiable, hence Lipschitz continuous. Since τ is locally solid, $|f^\alpha - f| \xrightarrow{\tau} 0$, from which it follows that $|f^\alpha - f| \xrightarrow{w^*} 0$ as well. Since $p^* \ll \mu$ and p^* is countably additive, by appealing to the Radon-Nikodym Theorem, $E_{p^*}(K|f^\alpha - f|) \to 0$. As this yields a contradiction, $f \succsim_i x$ as desired.

11.5. Proofs

We will use the fact that $\{g|g \succ f\} = \text{int}\,\{g|g \succsim f\}$ and $\{g|g \succsim f\} = \text{cl}\,\{g|g \succ f\}$. Let $\langle f, g\rangle$ denote the inner product of f and g and ∂I be the superdifferential of a concave function I.

Proof of Proposition 11.1. Using continuity, monotonicity, and convexity, standard arguments yield the equivalence of $\boldsymbol{\pi}(f)$ and $\boldsymbol{\pi}^u(f)$ for any strictly positive act f.

To show that $\boldsymbol{\pi}(f) = \boldsymbol{\pi}^w(f)$ as well, we first observe that by definition, the set $\boldsymbol{\pi}(f)$ is the set of normals to the convex upper contour set $B(f) := \{g \in \mathbb{R}^S : g \succsim f\}$ at f, normalized to lie in ΔS. Let $T_{B(f)}(f)$ denote the tangent cone to $B(f)$ at f, which is given by:

$$T_{B(f)}(f) = \{g \in \mathbb{R}^S : f + \lambda g \succsim f \text{ for some } \lambda > 0\}$$

From standard convex analysis results, $\boldsymbol{\pi}(f)$ is also the set of normals to $T_{B(f)}(f)$, again normalized to lie in ΔS. Thus

$$\boldsymbol{\pi}(f) = \{p \in \Delta S : p \cdot g \geq 0 \text{ for all } g \in T_{B(f)}(f)\}$$

and $g \in T_{B(f)}(f) \iff p \cdot g \geq 0$ for all $p \in \boldsymbol{\pi}(f)$. Then

$$\begin{aligned} & g' \in T_{B(f)}(f) + \{f\} = \{h \in \mathbb{R}^S : (1-\varepsilon)f + \varepsilon h \succsim f \text{ for some } \varepsilon > 0\} \\ \iff \quad & p \cdot g' \geq p \cdot f \text{ for all } p \in \boldsymbol{\pi}(f) \end{aligned}$$

Thus $\boldsymbol{\pi}(f) = \boldsymbol{\pi}^w(f)$. ■

For many of the results in the section on special cases, we make use of the following lemma.

Lemma 11.1: *Assume that $\succsim$ satisfies Axioms 11.1-11.4 and the representation V of $\succsim$ is concave. Then $\boldsymbol{\pi}(f) = \boldsymbol{\pi}^\partial(f) := \{\frac{q}{\|q\|} | q \in \partial V(f)\}$.*

Proof. First, we show that $\boldsymbol{\pi}^\partial(f) \subseteq \boldsymbol{\pi}(f)$. Let $p = \frac{q}{\|q\|}$ for some $q \in \partial V(f)$. Let $V(g) \geq V(f)$. We have $0 \leq V(g) - V(f) \leq \langle q, g - f\rangle$, hence $\langle q, f\rangle \leq \langle q, g\rangle$, so $E_p g \geq E_p f$. Second, we show that $\boldsymbol{\pi}^\partial(f) \in \mathcal{P}(f)$, thus $\boldsymbol{\pi}^w(f) \subseteq \boldsymbol{\pi}^\partial(f)$. Let g be such that $E_p g > E_p f$

for all $p \in \boldsymbol{\pi}^{\partial}(f)$. We need to find $\varepsilon > 0$ with $V(\varepsilon g + (1-\varepsilon)f) > V(f)$. The one-sided directional derivatives $V'(f;h)$ exist for all $h \in \mathbb{R}^S$, and $V'(f;h) = \min\{\langle l, h\rangle | l \in \partial V(f)\}$.[16] Hence, for some $q \in \partial V(f)$:

$$\begin{aligned} V(\varepsilon g + (1-\varepsilon)f) &= V(f + \varepsilon(g-f)) \\ &= V(f) + \varepsilon V'(f; g-f) + o(\varepsilon) \\ &= V(f) + \varepsilon \min\{\langle l, g-f\rangle |\, l \in \partial V(f)\} + o(\varepsilon) \\ &= V(f) + \varepsilon \langle q, g-f\rangle + o(\varepsilon) \\ &= V(f) + \varepsilon[\langle q, g-f\rangle + o(1)]. \end{aligned}$$

Because $q = \|q\| p$ for some $p \in \boldsymbol{\pi}^{\partial}(f)$, $\langle q, g-f\rangle = \|q\| E_p(g-f) > 0$. Therefore, there exists a $\delta > 0$ such that for all $\varepsilon \in (0,\delta)$, $\varepsilon[E_p(g-f)+o(1)] > 0$, hence $V(\varepsilon g + (1-\varepsilon)f) > V(f)$. □

Proof of Proposition 11.3. It follows from the proof of Theorem 3 in Maccheroni et al. (2006a) that $I(\xi) = \min_{p\in\Delta S}(E_p\xi + c^{\star}(p))$ is concave. This, together with concavity of u, yields the concavity of V. Continuity and monotonicity follow from the fact that I is monotonic and sup-norm Lipschitz continuous. By Theorem 18 of Maccheroni et al. (2006a),

$$\partial V(f) = \{q \in \mathbb{R}^S : q = pDU(f) \text{ for some } p \in M(f)\}.$$

The result follows from Lemma 11.1. ■

[16] Theorem 23.4 of **?** implies that $V'(f;h) = \inf\{\langle l, h\rangle | l \in \partial V(f)\}$ for all h. Because V is a proper concave function, $\partial V(f)$ is a compact set, hence the infimum is achieved.

Proof of Proposition 11.2. This follows from Proposition 11.3 by noting that MEU is the special case of variational preferences for which

$$c^\star(p) = \begin{cases} 0 & \text{if } p \in P \\ \infty & \text{if } p \notin P. \end{cases}$$

■

Proof of Proposition 11.4. It follows from Lemma 8 in **?** that $I(\xi) = \min_{p \in L_\alpha} \frac{1}{\varphi(p)} E_p \xi$ is concave. This, together with concavity of u, yields the concavity of V. Continuity and monotonicity follow from the fact that I is monotonic and sup-norm Lipschitz continuous (**?**)see Lemma 6 in. By **?** (2.8, Cor. 2),

$$\partial V(f) = \{q \in \mathbb{R}^S : q = pDU(f) \text{ for some } p \in M(f)\}.$$

The result follows from Lemma 11.1. ■

Proof of Proposition 11.5. Continuity, monotonicity and convexity are routine. When u and ϕ are concave and differentiable, it is straightforward to see that V is also concave and differentiable, and that $\partial V(f) = \{DV(f)\} = \{E_\mu[D\phi(E_p u(f))pDU(f)]\}$. ■

Proof of Proposition 11.6. Continuity, monotonicity and convexity are routine. When u and ϕ are concave and differentiable, it is straightforward to see that V is also concave and differentiable. A direct calculation of directional derivatives reveals that $\partial V(f) = \{DV(f)\} = \{pDU(f)[I_a \otimes D\Phi(E_a u(f))]\}$. ■

Proof of Proposition 11.8. Fix constant acts $x, x' > 0$, and let $B(x) := \{f \in \mathbb{R}^S_+ : f \succsim x\}$ denote the upper contour set of $\succsim$ at x. As in the proof of Proposition 11.1, let

$T_{B(x)}(x)$ denote the tangent cone to $B(x)$ at x:

$$T_{B(x)}(x) = \{g \in \mathbb{R}^S : x + \lambda g \succsim x \text{ for some } \lambda > 0\}$$

Again as in the proof of Proposition 11.1, $\boldsymbol{\pi}(x)$ is the normal cone to $T_{B(x)}(x)$, analogously for $\boldsymbol{\pi}(x')$. By translation invariance at certainty, $T_{B(x)}(x) = T_{B(x')}(x')$, from which we conclude that $\boldsymbol{\pi}(x) = \boldsymbol{\pi}(x')$. ■

References

ABDELLAOUI, M., A. BAILLON, AND P. P. WAKKER (2007): "Combining Bayesian Beliefs and Willingness to Bet to Analyze Attitudes towards Uncertainty," *mimeo.*

ACZÉL, J. (1966): *Lectures on Functional Equations and Their Applications.*, NewYork: Academic Press.

ALLAIS, M. (1953): "Le Comportement de l'Homme Rationnel devant le Risque: Critique des Postulats et Axiomes de l'Ecole Americaine," *Econometrica*, 21, 503–546.

ANDERSON, E. W., L. P. HANSEN, AND T. J. SARGENT (2000): "Robustness, Detection, and the Price of Risk," *mimeo.*

ANSCOMBE, F. AND R. AUMANN (1963): "A Definition of Subjective Probability," *The Annals of Mathematical Statistics*, 34, 199–205.

ARROW, K. (1970): *Essays in the Theory of Risk-Bearing*, Amsterdam: North-Holland.

BARILLAS, F., L. P. HANSEN, AND T. J. SARGENT (2007): "Doubts or Variability?" *mimeo.*

CHEN, Z. AND L. EPSTEIN (2002): "Ambiguity, Risk, and Asset Returns in Continuous Time," *Econometrica*, 70, 1403–1443.

CHEW, S. H. AND J. S. SAGI (2007): "Small Worlds: Modeling Attitudes towards Sources of Uncertainty," *Journal of Economic Theory*, forthcoming.

DUPUIS, P. AND R. S. ELLIS (1997): *A Weak Convergence Approach to the Theory of Large Deviations*, Wiley, New York.

Ellsberg, D. (1961): "Risk, Ambiguity, and the Savage Axioms," *The Quarterly Journal of Economics*, 75, 643–669.

Epstein, L. (1983): "Stationary cardinal utility and optimal growth under uncertainty," *Journal of Economic Theory*, 31, 133–152.

Epstein, L. and S. Zin (1989): "Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–969.

Epstein, L. G. (1999): "A Definition of Uncertainty Aversion," *Review of Economic Studies*, 66, 579 – 608.

Epstein, L. G. and M. Schneider (2003a): "IID: independently and indistinguishably distributed," *Journal of Economic Theory*, 113, 32–50.

——— (2003b): "Recursive multiple-priors," *Journal of Economic Theory*, 113, 1–31.

——— (2007): "Learning Under Ambiguity," *Review of Economic Studies*, 74, 1275–1303.

Ergin, H. and F. Gul (2004): "A Subjective Theory of Compound Lotteries," *mimeo*.

Fishburn, P. C. (1970): *Utility Theory for Decision Making*, Wiley, New York.

Ghirardato, P., F. Maccheroni, M. Marinacci, and M. Siniscalchi (2003): "A Subjective Spin on Roulette Wheels," *Econometrica*, 71, 1897–1908.

Ghirardato, P. and M. Marinacci (2002): "Ambiguity Made Precise: A Comparative Foundation," *Journal of Economic Theory*, 102, 251–289.

Gilboa, I. and D. Schmeidler (1989): "Maxmin expected utility with non-unique prior," *Journal of Mathematical Economics*, 18, 141–153.

Grandmont, J.-M. (1972): "Continuity Properties of a von Neumann-Morgenstern Utility," *Journal of Economic Theory*, 4, 45–57.

GRANT, S. AND B. POLAK (2006): "Bayesian beliefs with stochastic monotonicity: An extension of Machina and Schmeidler," *Journal of Economic Theory*, 130, 264–282.

——— (2007): "Absolute Ambiguity Aversion and Mean-Dispersion Preferences," *mimeo.*

HANSEN, L. P. AND T. J. SARGENT (2001): "Robust Control and Model Uncertainty," *The American Economic Review*, 91, 60–66.

——— (2007): *Robustness*, Princeton University Press.

HANSEN, L. P., T. J. SARGENT, G. TURMUHAMBETOVA, AND N. WILLIAMS (2006): "Robust control and model misspecification," *Journal of Economic Theory*, 128, 45–90.

HAYASHI, T. (2005): "Intertemporal substitution, risk aversion and ambiguity aversion," *Economic Theory*, 25, 933–956.

HERSTEIN, I. N. AND J. MILNOR (1953): "An Axiomatic Approach to Measurable Utility," *Econometrica*, 21, 291–297.

JACOBSON, D. J. (1973): "Optimal Linear Systems with Exponential Performance Criteria and their Relation to Differential Games," *IEEE Transactions on Automatic Control*, 18, 124–131.

KARANTOUNIAS, A. G., L. P. HANSEN, AND T. J. SARGENT (2007): "Ramsey taxation and fear of misspeciÂŕcation," *mimeo.*

KEYNES, J. (1921): *A Treatise on Probability*, MacMillan.

KLESHCHELSKI, I. AND N. VINCENT (2007): "Robust Equilibrium Yield Curves," *mimeo.*

KLIBANOFF, P., M. MARINACCI, AND S. MUKERJI (2005): "A smooth model of decision making under ambiguity," *Econometrica*, 73, 1849–1892.

KNIGHT, F. (1921): *Risk, Uncertainty and Profit*, Houghton Mifflin Company.

Köbberling, V. and P. P. Wakker (2003): "Preference Foundations for Nonexpected Utility: A Generalized and Simplified Technique," *Mathematics of Operations Research*, 28, 395–423.

Kreps, D. and E. Porteus (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46, 185–200.

Maccheroni, F., M. Marinacci, and A. Rustichini (2004): "Ambiguity Aversion, Malevolent Nature, and the Variational Representation of Preferences," *mimeo*.

——— (2006a): "Ambiguity Aversion, Robustness, and the Variational Representation of Preferences," *Econometrica*, 74, 1447 – 1498.

——— (2006b): "Dynamic variational preferences," *Journal of Economic Theory*, 128, 4–44.

Machina, M. and D. Schmeidler (1995): "Bayes Without Bernoulli: Simple Conditions for Probabilistically Sophisticated Choice," *Journal of Economic Theory*, 108–128.

Machina, M. J. and D. Schmeidler (1992): "A More Robust Definition of Subjective Probability," *Econometrica*, 60, 745–780.

Marinacci, M. (2002): "Probabilistic Sophistication and Multiple Priors," *Econometrica*, 70, 755–764.

Markowitz, H. M. (1952): "Portfolio Selection," *Journal of Finance*, 7, 77–91.

Nau, R. (2001): "Uncertainty Aversion with Second–Order Utilities and Probabilities," in *Proceedings of the Second International Symposium on Imprecise Probabilities and Their Applications*, http://www.sipta.org/isipta01/proceedings/063.html.

——— (2006): "Uncertainty Aversion with Second–Order Utilities and Probabilities," *Management Science*, 52, 136–145.

NEILSON, W. S. (1993): "Ambiguity Aversion: An Axiomatic Approach Using Second Order Probabilities," *mimeo.*

NEYMAN, J. AND E. SCOTT (1948): "Consistent Estimates Based on Partially Consistent Observations," *Econometrica*, 16, 1–32.

SARIN, R. AND P. P. WAKKER (2000): "Cumulative dominance and probabilistic sophistication," *Mathematical Social Sciences*, 40, 191–196.

SAVAGE, L. (1972): *The Foundations of Statistics*, Dover.

SCHMEIDLER, D. (1989): "Subjective Probability and Expected Utility without Additivity," *Econometrica*, 57, 571–587.

SEGAL, U. (1990): "Two-Stage Lotteries without the Reduction Axiom," *Econometrica*, 58, 349–377.

SKIADAS, C. (2003): "Robust control and recursive utility," *Finance and Stochastics*, 7, 475 – 489.

TOBIN, J. (1958): "Liquidity Preference as Behavior Toward Risk," *Review of Economic Studies*, 25, 65–86.

VILLEGAS, C. (1964): "On Qualitative Probability σ-Algebras," *The Annals of Mathematical Statistics*, 35, 1787–1796.

WHITTLE, P. (1981): "Risk-Sensitive Linear/Quadratic/Gaussian Control," *Advances in Applied Probability*, 13, 764–777.

WOODFORD, M. (2006): "An Example of Robustly Optimal Monetary Policy with Near-Rational Expectations," *Journal of the European Economic Association*, 4, 386–395.