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Infinitesimal p-adic Manin-Mumford and Applications to Hida Theory

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ABSTRACT

Infinitesimal p-adic Manin-Mumford and Applications to Hida Theory

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We study analytic functions on the open unit poly-disk in $\mathbb{C}_p^n$ centered at $(1, \ldots, 1)$ and prove that such functions only vanish at finitely many n-tuples $(\zeta_1 - 1, \ldots, \zeta_n - 1)$, where ζ_i are roots of unity, unless they vanish along a translate of formal $\mathbb{G}_m$. (Note that a root of unity ζ lies on the disk only if it has p-power order). For polynomial functions, this follows from the multiplicative Manin–Mumford Conjecture. Our results however allow for a much wider class of analytic functions; in particular we establish a rigidity result for formal tori. Moreover, we extend these results to Lubin–Tate formal groups beyond formal $\mathbb{G}_m$.

We then apply our methods to Hida theory where the rings of analytic functions in question parametrize families of automorphic forms by weight. In particular, this allows us to describe which families of cohomological automorphic forms for GL_2 over an imaginary quadratic field contain forms in infinitely many classical weights and addresses a question arising from the work of Calegari and Mazur [**CM09**, Section 8.5].

Finally, we discuss analogous results for functions vanishing near the torsion of the formal group of an abelian variety. In this case, Coleman's theory of p-adic abelian integrals [**Col85, Col87**] allows us to obtain a p-adic infinitesimal refinement of the Manin–Mumford Conjecture.

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Dedication

To my family. S.D.G.

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CHAPTER 1

Introduction and Background

In this Chapter we give an outline of the contents together with some relevant background, including some related work throughout the literature.

1.1. The Manin-Mumford Conjecture

The classical Manin-Mumford Conjecture, proven by M. Raynaud [**Ray83**], states that for an algebraic curve C of genus greater than one defined over a number field K together with an embedding defined over K of C into its Jacobian, there are only finitely many torsion points of the Jacobian on the curve, i.e. $C(\overline{K}) \cap \mathrm{Jac}(C)(\overline{K})_{\mathrm{tor}}$ is a finite set. One may ask a similar question replacing the Jacobian by an abelian variety, or more generally a commutative group variety G. Then in general an embedded algebraic curve contains a Zariski dense set of torsion points if and only if it is the translate of a subgroup by a torsion point. See also the formulation in 2.1.1 and the survey [**Tze00**] for more details.

1.1.1. The multiplicative case and p-adic infinitesimal strengthenings

When the ambient group G is $\mathbb{G}_m^n$, one has the so-called multiplicative Manin-Mumford Conjecture, which was already proven by S. Lang in [**Lan60, Lan65**]. It states that if an irreducible curve C embedded in $\mathbb{G}_m^n$ contains infinitely many torsion points, it must be a translate of $\mathbb{G}_m$ by a torsion point. Considering the case of $n = 2$ for ease of exposition, this amounts to the following explicit statement on polynomials:

Proposition 1.1.1 ([**Lan60**], p.28)**.** *Let C be an absolutely irreducible plane curve given by the zero set of a polynomial $f(X,Y) = 0$. Assume C passes through the multiplicative origin and*

$$f(\zeta, \xi) = 0$$

for infinitely many pairs of roots of unity (ζ, ξ). Then $f(X,Y) = X^m - Y^l$ or $f(X,Y) = X^m Y^l - 1$ for a pair of nonnegative integers $(m, l) \neq (0,0)$.

The proof relies on the algebraic properties of the polynomial f and one might not expect this statement to still hold when f is only analytic. In fact, for p-adic analytic functions, if one does not require f to have integral coefficients one can cook up power series vanishing at arbitrary sequences of points. There is also no guarantee the series converges at arbitrary roots of unity. However we obtain the following statement for power series as a special case of our results:

Proposition 1.1.2. *Let $\mathcal{O}_F$ denote the ring of integers of a finite extension $F/\mathbb{Q}_p$ and $\phi \in \mathcal{O}_F[[X,Y]]$ an irreducible power series passing through the origin. If*

$$\phi(\zeta - 1, \xi - 1) = 0$$

for infinitely many pairs of p-power roots of unity (ζ, ξ), then after possibly switching X and Y there is $m \in \mathbb{Z}_p$ so that $\phi = (X+1)^m - (Y+1)$, where $(X+1)^m = 1 + \sum_{i=1}^{\infty} \frac{m \cdots (m-i+1)}{i!} X^i$.

Denoting $\mathcal{S} = \{\zeta - 1 | \zeta \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)\}$, we consider more generally the set of n-tuples $\mathcal{S}^n$ and an ideal $I \subset \mathcal{O}_F[[X_1, \ldots, X_n]]$. In Chapter 2, we prove the following and deduce Proposition 1.1.2 when $n = 2$ and $I = (\phi)$:

Theorem 1.1.3. *Put $A = \mathcal{O}_F[[X_1, \ldots, X_n]]/I$, where $\mathcal{O}_F$ denotes the ring of integers of a finite extension of $\mathbb{Q}_p$. Then exactly one of the following occurs:*

(1) The formal scheme $\mathrm{Spf}(A)$ contains a translate of $\widehat{\mathbb{G}}_m$ by a torsion point of $\widehat{\mathbb{G}}_m^n$.

(2) There exist an explicit constant $C_I > 0$ depending only on I and the choice of p-adic absolute value $|-|_p$, as well as a finite set $\mathcal{F} \subset \mathcal{S}^n$ such that for any $\phi \in I$

$$|\phi(\zeta_1 - 1, \ldots, \zeta_n - 1)|_p > C_I$$

if $(\zeta_1 - 1, \ldots, \zeta_n - 1) \in \mathcal{S}^n \setminus \mathcal{F}$.

A power series $\phi \in \mathcal{O}_F[[X_1, \ldots, X_n]]$ gives rise to an analytic function on the open p-adic unit polydisk $\mathbb{B}^n(e, \mathbb{C}_p)$ centered at $e = (1, \ldots, 1)$ by evaluating points $Q \in \mathbb{B}^n(e, \mathbb{C}_p)$ at $Q - e$. The only roots of unity on the open disk have p-power order, which is why our results can be thought of as p-adic infinitesimal strengthenings of the multiplicative Manin-Mumford Conjecture. The proof of Theorem 1.1.3, given in Section 2.1, relies on the fact that any infinite sequence in $\mathcal{S}^n$ must approach the boundary of the polydisk since the normalized valuation is $v_p(\zeta_{p^k} - 1) = 1/(p^k - p^{k-1})$ when ζ_{p^k} has exact order p^k. We also utilize the action of automorphisms of the formal group $\widehat{\mathbb{G}}_m^n$ on the torsion points in $\mathcal{S}^n$ together with the algebraic properties of formal power series rings. In fact, our methods can be used to prove that Theorem 1.1.3 holds in greater generality replacing

$\widehat{\mathbb{G}}_m$ by a Lubin-Tate formal group $\mathcal{F}_{LT}$ and replacing $\mathcal{S}^n$ by n-tuples of torsion points of $\mathcal{F}_{LT}$. This is shown in Section 2.2.

Many generalizations of the Manin-Mumford Conjecture are known by work of S. Zhang [**Zha95, Zha98**], E. Ullmo [**Ull98**] and others, however to our knowledge these infinitesimal p-adic strengthenings have not previously been considered. Rather, they are related to some rigidity results appearing in the work of C-L. Chai and H. Hida which we discuss in Section 1.2 below. We also note that P. Monsky studied the p-adic power series rings in question with applications to Iwasawa theory in mind and [**Mon81**, Section 2] can be used to establish some of the results in Section 2.1.

1.1.2. A p-adic phenomenon

Observe that Theorem 1.1.3 makes the additional claim that there cannot be infinitely many torsion points arbitrarily close to $\mathrm{Spf}(A)(\mathbb{C}_p) \subset \mathbb{B}^n(0, \mathbb{C}_p)$ unless for a specific geometric reason. This is a purely p-adic phenomenon, for instance torsion points on abelian varieties are dense even in the complex analytic topology. It was observed by J. Tate and F. Voloch [**TV96**] that given a linear form f with zeroes $Z(f)$ and a choice of p-adic absolute value $|-|_p$, there is a uniform bound ϵ_f such that for any n-tuple of roots of unity,

$$f(\zeta_1, \ldots, \zeta_n) \neq 0 \Rightarrow |(\zeta_1, \ldots, \zeta_n) - P|_p > \epsilon_f \ \forall P \in Z(f).$$

They formulated a general conjecture for algebraic varieties which was proven by T. Scanlon [**Sca98**]. Our results exhibit the same phenomenon for formal power series and p-power roots of unity, whereas in her thesis A. Neira [**Nei02**] proves this result replacing linear forms with analytic functions on the closed disk. In this case one need not restrict to p-power roots of unity.

1.1.3. Infinitesimal results beyond the multiplicative case

In Chapter 4, we briefly address the natural question about p-adic infinitesimal Manin-Mumford statements when one is no longer dealing with the multiplicative group but with abelian varieties. We show how the p-adic integration techniques of R. Coleman [**Col85, Col87**] yield some p-adic infinitesimal refinements of the Manin-Mumford Conjecture in the case of abelian varieties.

1.2. Formal schemes and rigidity results

We now review some relevant rigidity results found in the literature. Over a field k of characteristic p, C-L. Chai [**Cha08**] proves a rigidity result for p-divisible formal groups, which is used in his work together with F. Oort on the Hecke Orbit Conjecture for Siegel modular varieties (see e.g. [**Cha05**]). Considering the torus

$$\widehat{\mathbb{G}}_{m/k}^n = \mathrm{Spf}(k[[X_1, \ldots, X_n]]),$$

let $X_*(\widehat{\mathbb{G}}_m^n) = \mathrm{Hom}_k(\widehat{\mathbb{G}}_m, \widehat{\mathbb{G}}_m^n) \cong \mathbb{Z}_p^n$ denote the group of cocharacters, so that $\mathrm{GL}(X_*) \cong \mathrm{GL}_n(\mathbb{Z}_p^n)$ naturally acts on the torus. One has from [**Cha08**, Theorem 4.3]:

Theorem 1.2.1 (Chai)**.** *Let $k = \overline{\mathbb{F}}_p$ and $Z \subset \widehat{\mathbb{G}}_{m/k}^n$ a closed formal subscheme, equidimensional of dimension r. If Z is stable under the diagonal action for all u in an open subgroup U of $(\mathbb{Z}_p^\times)^n \subset \mathrm{GL}(X_*)$, then there are finitely many $\mathbb{Z}_p$-direct summands $T_1, \ldots, T_s$ of rank r of $X_*(\widehat{\mathbb{G}}_m^n)$ so that*

$$Z = \bigcup_{i=1}^{s} \widehat{\mathbb{G}}_{m/k} \otimes T_i.$$

Hida uses Chai's rigidity results [**Hid10**, Section 3.4] and establishes characteristic zero versions thereof in [**Hid11**, Lemma 1.2] and [**Hid14**, Section 4]. He proves:

Lemma 1.2.2 ([**Hid14**], Lemma 4.1)**.** *Let $Z = \mathrm{Spf}(\mathcal{O}_F[[X_1, \ldots, X_n]]/I)$ be a closed formal subscheme of $\widehat{\mathbb{G}}_m^n$ that is flat and geometrically irreducible. Suppose there is an open subgroup $U \subseteq \mathbb{Z}_p^\times$ such that Z is stable under the action $(1 + X_i) \mapsto (1 + X_i)^u$ for all $u \in U$. If there exists a subset $\Omega \subseteq Z(\mathbb{C}_p) \cap \mu_{p^\infty}^n(\mathbb{C}_p)$ Zariski dense in Z, then Z is the translate of a formal subtorus by a torsion point in Ω.*

In particular, he obtains a rigidity result for formal power series by applying Lemma 1.2.2 to their graph in [**Hid14**, Corollary 4.2]:

Corollary 1.2.3 (Hida)**.** *Let $\phi \in \mathcal{O}_F[[X_1, \ldots, X_n]]$ be a power series such that there is a Zariski-dense subset $\Omega \subset \mu_{p^\infty}^n(\mathbb{C}_p)$ in $\widehat{\mathbb{G}}_m^n(\mathbb{C}_p)$ with $\phi(\zeta - 1) \subseteq \mu_{p^\infty}(\mathbb{C}_p)$ for all $\zeta \in \Omega$. Then there exist $\zeta_0 \in \mu_{p^\infty}(\mathcal{O}_F)$ and $N = (N_1, \ldots, N_n) \in \mathbb{Z}_p^n$ such that $\phi(X_1, \ldots, X_n) = \zeta_0 \prod_{i=1}^n (1 + X_i)^{N_i}$.*

Writing $A = \mathcal{O}_F[[X_1, \ldots, X_n]]/I$ and assuming $\mathrm{Spf}(A)$ is geometrically irreducible, we therefore consider the following statements:

(I) The formal subscheme $\mathrm{Spf}(A) \subset \widehat{\mathbb{G}}_m^n$ is the translate of a formal subtorus by a torsion point.

(II) There is a Zariski-dense set of torsion points of $\widehat{\mathbb{G}}_m^n$ on $\mathrm{Spf}(A)(\overline{\mathbb{Q}}_p)$.

(III) The formal subscheme $\mathrm{Spf}(A) \subset \widehat{\mathbb{G}}_m^n$ is stable under the action of an open subgroup U of the diagonal in the cocharacters $\mathrm{GL}(X_*) \cong \mathrm{GL}_n(\mathbb{Z}_p)$.

The first statement implies the two others. Chai's result is a version of (III) $\Rightarrow$ (I) in characteristic p. In characteristic zero, Lemma 1.2.2 shows that (II & III) $\Rightarrow$ (I), whereas Corollary 1.2.3 shows in some special cases that (II) $\Rightarrow$ (I). In Proposition 2.1.6 we prove a vanishing result for arbitrary formal power series and deduce in all generality that (II) $\Rightarrow$ (I), see in particular Corollary 2.1.13.

1.3. Applications

The power series rings in question are related to many interesting arithmetic objects. They occur naturally as completed group rings such as $\mathbb{Z}_p[[\mathbb{Z}_p^n]] \cong \mathbb{Z}_p[[X_1, \ldots, X_n]]$ and taking coinvariants of a $\mathbb{Z}_p[[\mathbb{Z}_p^n]]$-module by some finite index subgroup amounts to specializing $X_i \mapsto \zeta_i - 1$ for roots of unity $\zeta_i \in \mu_{p^\infty}$. Iwasawa theory studies class groups and Selmer groups as modules over these rings, together with their relationship to various p-adic L-functions. They also occur as completed local rings at smooth points of schemes over $\mathbb{Z}_p$ and as weight spaces parametrizing p-adically continuous families of automorphic forms. We restrict ourselves here to applications to the latter.

The idea of p-adic families of modular forms first appears in the study of congruences between Fourier coefficients of Eisenstein series by J-P. Serre [**Ser73**]. Hida vastly generalized this and constructed p-adic families of cuspforms, leading to the construction of

p-adic families of attached p-adic Galois representations (see e.g. [**Hid86a**]). These motivated the work of B. Mazur on deformation spaces of Galois representations [**Maz89**], which crucially appear in R. Taylor and A. Wiles' work [**Wil95, TW95**].

The theory has been generalized by Hida ([**Hid94**]) to automorphic forms for GL_2 over arbitrary number fields and there has also been work by A. Ash and G. Stevens (e.g., [**AS97**]) for GL_n. In Chapter 3 we consider automorphic forms for GL_2 over an imaginary quadratic field. We answer a question arising from the work of F. Calegari and B. Mazur [**CM09**, Section 8.5] about which p-adic families contain a Zariski-dense set of classical automorphic forms. For arbitrary number fields, this is not known (or expected) to hold in general unless F is totally real. Our main result, Theorem 3.2.6, shows the existence of a Zariski-dense set of classical automorphic forms on a p-adic family $\mathcal{H}$ forces that either:

(1) Every specialization of $\mathcal{H}$ in parallel weight and nebentypus is classical or

(2) Cofinitely many classical forms on $\mathcal{H}$ appear in a single weight $(k, k) \in \mathbb{Z}^2$ and infinitely many non-parallel nebentypi.

The first condition can occur via base change from $\mathbb{Q}$, however the latter seems improbable. In fact one can say a little more, so that in practice both can be excluded with a finite amount of computation and one can show that there are only finitely many classical automorphic forms on $\mathcal{H}$, when this is true. We include an example of such a computation in Section 3.3.

CHAPTER 2

Strengthenings of the multiplicative Manin-Mumford Conjecture

2.1. Infinitesimal p-adic multiplicative Manin-Mumford

In this Section, we prove an "unlikely intersection" result for affine schemes Spec(A), where $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$ and $\mathcal{O}_F$ is the valuation ring of a finite extension of $\mathbb{Q}_p$. Were we just dealing with algebraic varieties, this would follow from the multiplicative version of the Manin-Mumford Conjecture.

2.1.1. A formulation for power series

We use terminology inspired by the following formulation in [**Ull07**, Section 3.1.] of the classical Manin-Mumford Conjecture: let $X/\mathbb{C}$ be an algebraic variety. Define a set of *special subvarieties* S_X to be the following irreducible subvarieties of X:

- If X is an abelian variety, the special subvarieties are the translates by torsion points of abelian subvarieties of X.
- If X is a torus, the special subvarieties are given by the products of torsion points with subtori.

A *special point* is a zero-dimensional special subvariety. The conjecture may then simply be stated as:

Conjecture 2.1.1 (Manin-Mumford)**.** *An irreducible component of the Zariski closure of a set of special points is a special subvariety.*

We are interested in the $\overline{\mathbb{Q}}_p$-points on $\mathrm{Spec}(\mathcal{O}_F[[X_1, \ldots, X_n]])$ with coordinates:

$$\mathcal{S}^n = \{(\zeta_1 - 1, \ldots, \zeta_n - 1) | \zeta_i \in \mu_{p^\infty}(\overline{\mathbb{Q}}_p)\}$$

which are precisely the torsion points of the formal Lie group $\widehat{\mathbb{G}}_m^n$. For an $\mathcal{O}_F$-algebra $A = \mathcal{O}_F[[X_1, \ldots, X_n]]/I$ we define the set of *special points* $\mathcal{S}_A \subset \mathrm{Spec}(A)$ to be the points of $\mathcal{S}^n$ lying on $\mathrm{Spec}(A)$, i.e. prime ideals $\mathfrak{p} = (X_1 + 1 - \zeta_1, \ldots, X_n + 1 - \zeta_n)$ containing I. We want $\mathrm{Spec}(A)$ to be a *special subscheme* exactly when $\mathrm{Spf}(A) \subset \widehat{\mathbb{G}}_m^n$ is the product of a formal subtorus by a torsion point of $\widehat{\mathbb{G}}_m^n$. Note that endomorphisms of the formal group law act on a choice of coordinates via

$$(X_1, \ldots, X_n) \mapsto \left(\prod_{j=1}^{n} (X_j + 1)^{a_{1,j}} - 1, \ldots, \prod_{j=1}^{n} (X_j + 1)^{a_{n,j}} - 1 \right)$$

for matrices $(a_{i,j}) \in M_n(\mathbb{Z}_p) \cong \mathrm{End}(\widehat{\mathbb{G}}_m^n)$.

For any choice of coordinates, the set of special subschemes should account for twists by automorphisms in $\mathrm{GL}_n(\mathbb{Z}_p)$. We therefore make the following definitions:

Definition.

(1) A *multiplicative change of variables* on $\mathcal{O}_F[[X_1, \ldots, X_n]]$ is given by possibly swapping the roles of X_i and X_j and a series of transformations of the form:

$$X_i \mapsto (1 + X_i) \prod_{1 \leq j < i} (1 + X_j)^{B_j} - 1$$

for $1 \leq i \leq n$, where $B_j \in \mathbb{Z}_p$.

(2) A *special (multiplicative) subscheme* of $\mathrm{Spec}(\mathcal{O}_F[[X_1, \ldots, X_n]])$ is a closed affine subscheme that after a multiplicative change of variables is a finite union of intersections of hyperplanes of the form $X_i = \zeta_i - 1$ for $\zeta_i \in \mu_{p^\infty}$.

Remarks. (1) The multiplicative change of variables corresponds to, after possibly switching variables, acting on $\mathcal{S}^n$ via lower triangular matrices with coefficients in $\mathbb{Z}_p$ as follows :

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ B_{21} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \ldots & 1 \end{pmatrix} : \begin{pmatrix} \zeta_1 - 1 \\ \zeta_2 - 1 \\ \vdots \\ \zeta_n - 1 \end{pmatrix} \mapsto \begin{pmatrix} \zeta_1 - 1 \\ \zeta_2 \zeta_1^{B_{21}} - 1 \\ \vdots \\ \zeta_n \prod_{1 \leq i < n} \zeta_i^{B_{ni}} - 1 \end{pmatrix}$$

We note here that this operation together with swapping coordinates generates the subgroup of $GL_n(\mathbb{Z}_p)$ of determinant ± 1. For our purposes, we may ignore scalar matrices $\lambda \cdot I_n \in \mathrm{GL}_n(\mathbb{Z}_p)$, where $\lambda \in \mathbb{Z}_p^\times$.

(2) We must allow for general hyperplanes $X_i = \zeta - 1$ as opposed to $X_i = 0$ to account for the non-invertible endomorphisms of $\widehat{\mathbb{G}}_m^n$. For example, if $n = 2$ a special multiplicative subscheme is $\mathrm{Spec}(\mathcal{O}_F[[X_1, X_2]]/(X_1+1)^p - (X_2+1))$ which after the change of variable $X_2 \mapsto (X_1+1)^p(X_2+1) - 1$ becomes a union of hyperplanes $X_1 = \zeta_p - 1$ for $\zeta_p \in \mu_p$ and $X_2 = 0$.

(3) We can define the dimension of a special subscheme as the largest dimension of a component, so that a multiplicative change of variable does not change the dimension of a special subscheme, although it changes the number of components as in the example of $\operatorname{Spec}(\mathcal{O}_F[[X_1, X_2]]/(X_1+1)^p - (X_2+1))$.

We can now state the p-adic multiplicative Manin-Mumford result as follows:

Theorem 2.1.2. *Let $A = \mathcal{O}_F[[X_1, \ldots, X_n]]/I$. An irreducible component of the Zariski closure of the special points $\mathcal{S}_A$ on* $\operatorname{Spec}(A)$ *is a special multiplicative subscheme.*

We also establish the stronger result in this p-adic setting:

Theorem 2.1.3. *Let $A = \mathcal{O}_F[[X_1, \ldots, X_n]]/I$. Then exactly one of the following occurs:*

(1) The affine scheme $\operatorname{Spec}(A)$ *contains a positive dimensional special multiplicative subscheme.*

(2) There exist an explicit constant $C_I > 0$, depending only on I and the choice of p-adic absolute value $|-|_p$ on $\mathbb{B}^n(0, \mathbb{C}_p)$, as well as a finite set $\mathcal{F} \subset \mathcal{S}^n$ such that for any $\phi \in I$,

$$|\phi(\zeta_1 - 1, \ldots, \zeta_n - 1)|_p > C_I$$

if $(\zeta_1 - 1, \ldots, \zeta_n - 1) \in \mathcal{S}^n \setminus \mathcal{F}$.

2.1.2. Almost vanishing loci

Let π denote a uniformizer for $\mathcal{O}_F$ and $\mathbb{F}_q = \mathcal{O}_F/\pi\mathcal{O}_F$ the residue field. For any $\epsilon > 0$ and any ideal $I \subset \mathcal{O}_F[[X_1, \ldots, X_n]]$, we consider the special points lying close to the affine scheme $\mathrm{Spec}(\mathcal{O}_F[[X_1,\ldots,X_n]]/I)$:

$$S_I(\epsilon) := \{(\zeta_1-1,\ldots,\zeta_n-1)\in\Omega^n \text{ such that } \forall\phi\in I,\ |\phi(\zeta_1-1,\ldots,\zeta_n-1)|_p<\epsilon\}.$$

We note that $\mathcal{O}_F[[X_1,\ldots,X_n]]$ is a Noetherian ring and that it suffices to check the conditions on the finitely many generators of I. If $I=(\phi)$, we simply write $S_\phi(\epsilon)$.

The endomorphisms of $\widehat{\mathbb{G}}_m^n$ transform $S_I(\epsilon)$. In particular, performing a multiplicative change of variables on $\mathcal{O}_F[[X_1,\ldots,X_n]]/I$ acts on $S_I(\epsilon)$ via an automorphism of $\widehat{\mathbb{G}}_m^n$ as follows:

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ B_{21} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \ldots & 1 \end{pmatrix} : \begin{pmatrix} \zeta_1-1 \\ \zeta_2-1 \\ \vdots \\ \zeta_n-1 \end{pmatrix} \mapsto \begin{pmatrix} \zeta_1-1 \\ \zeta_2\zeta_1^{B_{21}}-1 \\ \vdots \\ \zeta_n\prod_{1\le i<n}\zeta_i^{B_{ni}}-1 \end{pmatrix} \tag{2.1}$$

Since our goal is to understand when $S_I(\epsilon)$ can be infinite, we may after twisting by an automorphism arrange for an explicit subsequence of special points:

Lemma 2.1.4. *Assume that $S_I(\epsilon)$ is infinite for some $\epsilon>0$. After a multiplicative change of variables there is a sequence $\{\zeta_k\}_{k\in\mathbb{N}}\in\mu_{p^\infty}$ for which the following hold:*

- *The elements $(\zeta_k-1,\zeta_k^{a_{2k}}-1,\zeta_k^{a_{2k}a_{3k}}-1,\ldots,\zeta_k^{a_{2k}a_{3k}\ldots a_{nk}}-1)$ are in $S_I(\epsilon)$.*
- *The set $\{\zeta_k\in\mu_{p^\infty}|k\in\mathbb{N}\}$ is infinite.*
- *The set $\{\zeta_k^{a_{2k}a_{3k}\ldots a_{nk}}|k\in\mathbb{N}\}$ is finite or each sequence $\{a_{ik}\}_{k\in\mathbb{N}}\in\mathbb{Z}_p$ converges to zero p-adically.*

Proof. We may swap the roles of the indeterminates X_i so that infinitely often elements $(\zeta_1 - 1, \dots, \zeta_n - 1) \in S_I(\epsilon)$ have decreasing orders, or equivalently $v_p(\zeta_i - 1) \leq v_p(\zeta_{i+1} - 1)$. Therefore there is an infinite sequence $\{\zeta_k \in \mu_{p^\infty} | k \in \mathbb{N}\}$ with

$$(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \zeta_k^{a_{2k}a_{3k}} - 1, \dots, \zeta_k^{a_{2k}a_{3k}\dots a_{nk}} - 1) \in S_I(\epsilon)$$

for exponents $\{a_{ik}\}_{k\in\mathbb{N}} \in \mathbb{Z}_p$. By compactness of $\mathbb{Z}_p$, we may after passing to a subsequence assume that the exponents a_{ik} converge p-adically to $A_i \in \mathbb{Z}_p$. We may now find $B_{i,j} \in \mathbb{Z}_p$ with

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ B_{21} & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \dots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ A_2 \\ \vdots \\ A_2 \cdots A_n \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

and twist by the corresponding multiplicative change of variables. It follows from (2.1) that all the exponent sequences $a_{2k} \dots a_{jk}$ converge to zero, in particular $A_2 = 0$. Now apply the same procedure to the $(n-1)$ last coordinates

$$(\xi_k - 1, \xi_k^{a_{3k}} - 1, \dots, \xi_k^{a_{3k}\dots a_{nk}} - 1)$$

where $\xi_k = \zeta_k^{a_{2k}}$ to arrange for $A_3 = 0$. Iterating this process, we get that $A_i = 0$ for $2 \leq i \leq n$. □

As for the non-invertible multiplication-by-p endomorphism of $\widehat{\mathbb{G}}_m^n$ one has:

Lemma 2.1.5. *Assume* $(\zeta_1 - 1, \dots, \zeta_n - 1) \in S_I(\epsilon)$ *for some* $\epsilon \leq |\pi|_p$. *Then*

$$(\zeta_1^p - 1, \dots, \zeta_n^p - 1) \in S_I(\epsilon^p).$$

Proof. This follows from evaluating at roots of unity the congruence

$$\phi((X_1+1)^p-1,\ldots,(X_n+1)^p-1) \equiv \phi(X_1,\ldots,X_n)^p \mod \pi$$

for functions $\phi \in I$. □

We turn to the proof of the general rigidity result for power series:

Proposition 2.1.6. *Let $\phi \in \mathcal{O}_F[[X_1,\ldots,X_n]]$ be a power series. Then exactly one of the following occurs:*

(1) After a multiplicative change of variables, there are fixed $(\xi_2-1,\ldots,\xi_n-1) \in \mathcal{S}^{n-1}$ such that $\phi(s,\xi_2-1,\ldots,\xi_n-1)=0$ for all $s \in \mathbb{B}(0,\mathbb{C}_p)$.

(2) There exist a constant $C_\phi > 0$ and a finite set $\mathcal{F}_\phi$ such that for all $(\zeta_1-1,\ldots,\zeta_n-1) \in \mathcal{S}^n \setminus \mathcal{F}_\phi$ there is a lower bound

$$|\phi(\zeta_1-1,\ldots,\zeta_n-1)|_p \geq C_\phi.$$

The proof proceeds by induction and the following deals with the base case $n=1$:

Lemma 2.1.7. *Let $\phi \in \mathcal{O}_F[[X]]$ be a power series and $\{r_k\}_{k\in\mathbb{N}} \in \mathfrak{m}_{\mathbb{C}_p} = \mathbb{B}(0,\mathbb{C}_p)$ be a sequence so that*

$$|\phi(r_k)|_p < \epsilon$$

for some $0 \leq \epsilon < 1$. If $v_p(r_k) \to 0$ then $\phi \in \pi\mathcal{O}_F[[X]]$.

Proof. Assume $\phi \notin \pi\mathcal{O}_F[[X]]$. Writing $\phi = \sum_{n=0}^{\infty} a_n X^n$, let M be the smallest integer such that $v_p(a_M) = 0$. Then, provided k is large enough so that $v_p(r_k) < v_p(\pi)$, we have

$$v_p(\phi(r_k)) = v_p(a_M r_k^M) = M v_p(r_k)$$

However, the right hand side becomes arbitrarily small, which is a contradiction. □

The next Lemma is crucial and provides the induction step in our proof. It puts a strong restriction on the subsets of $\mathcal{S}^n$ that can be realized as $S_\phi(\epsilon)$ for some $\phi \in \mathcal{O}_F[[X_1, \ldots, X_n]]$.

Lemma 2.1.8. *For any power series $\phi \in \mathcal{O}_F[[X_1, \ldots, X_n]]$, if*

$$(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \ldots, \zeta_k^{a_{2k}a_{3k}\ldots a_{nk}} - 1) \in S_\phi(p^{-(1-c)v_p(\pi)}),$$

where $0 \leq c < 1$ is a constant and each $\{a_{ik}\}_{k\in\mathbb{N}} \in \mathbb{Z}_p$ converges to zero for an infinite sequence $\{\zeta_k\}_{k\in\mathbb{N}} \in \mu_{p^\infty}$, then either:

(1) the sequence $\{\zeta_k^{a_{2k}a_{3k}\ldots a_{nk}}\}_{k\in\mathbb{N}}$ belongs to a finite set of roots of unity or

(2) the power series $\phi \in \pi\mathcal{O}_F[[X_1, \ldots, X_n]]$.

Proof. We proceed by induction. If $n = 1$ the result follows from Lemma 2.1.7. For arbitrary n, let c_k denote the order of $\zeta_k^{a_{2k}a_{3k}\ldots a_{nk}}$. Assume the set $\{\zeta_k^{a_{2k}a_{3k}\ldots a_{nk}} | k \in \mathbb{N}\}$ is infinite, then the same is true of $\{c_k | k \in \mathbb{N}\}$. We want to show $\phi \in \pi\mathcal{O}_F[[X_1, \ldots, X_n]]$. Suppose not; there is then a largest integer M with X_n^M dividing the reduction $\bar{\phi} \in \mathbb{F}_q[[X_1, \ldots, X_n]]$ modulo π. We choose $\psi \in \mathcal{O}_F[[X_1, \ldots, X_n]]$ so that

$$\phi(X_1, \ldots, X_n) \equiv X_n^M \psi(X_1, \ldots, X_n) \mod \pi.$$

Since $v_p(\zeta_k^{a_{2k}a_{3k}\cdots a_{nk}} - 1)$ becomes arbitrarily small, after passing to a subsequence

$$(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \ldots, \zeta_k^{a_{2k}a_{3k}\ldots a_{nk}} - 1) \in S_\psi(p^{-(1-c/2)v_p(\pi)})$$

for infinitely many $\zeta_k \in \mu_{p^\infty}$. It follows from Lemma 2.1.5 that:

$$(\zeta_k, \zeta_k^{c_k a_{2k}}, \ldots, \zeta_k^{c_k a_{2k}\cdots a_{(n-1)k}}, 0) \in S_\psi(p^{-(1-c/2)v_p(\pi)}).$$

By definition of c_k the set $\{\zeta_k^{c_k a_{2k}a_{3k}\cdots a_{(n-1)k}} | k \in \mathbb{N}\}$ consists of primitive $p^{v_p(a_{nk})}$-roots of unity. Since $a_{nk} \neq 0$ infinitely often by assumption, the set is infinite. Moreover the exponent sequence $\{a'_{2k} := c_k \cdot a_{2k}\}_{k\in\mathbb{N}}$ still converges to zero. Therefore by induction

$$\psi(X_1, \ldots, X_{n-1}, 0) \in \pi\mathcal{O}_F[[X_1, \ldots, X_{n-1}]].$$

We may write $\psi(X_1, \ldots, X_n) = \psi(X_1, \ldots, X_{n-1}, 0) + X_n\theta(X_1, \ldots, X_n)$ for some power series $\theta \in \mathcal{O}_F[[X_1, \ldots, X_n]]$, so that $\psi \equiv X_n\theta \mod \pi$. This results in the congruence

$$\phi(X_1, \ldots, X_n) \equiv X_n^{M+1}\theta(X_1, \ldots, X_n) \mod \pi,$$

which contradicts the maximality of M. □

We now prove the main result of this section.

Proof of Proposition 2.1.6. We proceed by induction on n. Assume that for any integer $M \geq 1$ the set $S_\phi(p^{-v_p(\pi^M)})$ is infinite. If $n = 1$ it follows from the Weierstrass Preparation Theorem or Lemma 2.1.7 that ϕ vanishes identically. For $n > 1$, after applying Lemma 2.1.4 we conclude that there are infinitely many $\zeta_k \in \mu_{p^\infty}$ for which

$$(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \ldots, \zeta_k^{a_{2k}a_{3k}\ldots a_{nk}} - 1) \in S_\phi(p^{-Mv_p(\pi)})$$

for all M. By Lemma 2.1.8 the sequence $\{\zeta_k^{a_{2k}a_{3k}\ldots a_{nk}}\}_{k\in\mathbb{N}}$ must belong to a finite set of roots of unity. Thus there is a fixed $\xi \in \mu_{p^\infty}$ such that $|\phi(X_1, \ldots, X_{n-1}, \xi - 1)|_p$ is arbitrarily small for infinitely many roots of unity in $\mathcal{S}^{n-1}$. We conclude by applying our induction hypothesis to $\phi(X_1, \ldots, X_{n-1}, \xi - 1) \in \mathcal{O}_{F[\xi]}[[X_1, \ldots, X_{n-1}]]$. □

We obtain in the same way:

Proof of Theorem 2.1.3. The ideal $I \subset \mathcal{O}_F[[X_1, \ldots, X_n]]$ is finitely generated. Writing $I = (\phi_1, \ldots, \phi_d)$, we run the proof above for the d generators simultaneously. □

When $n = 2$, we get the explicit formulation:

Proposition 2.1.9. *Let $\phi \in \mathcal{O}_F[[X, Y]]$ be an irreducible power series passing through the origin. If for infinitely many pairs (ζ, ξ) of p-power roots of unity*

$$\phi(\zeta - 1, \xi - 1) = 0$$

then for some $m \in \mathbb{Z}_p$, after possibly switching the roles of X and Y, one has

$$\phi = (X + 1)^m - (Y + 1).$$

Proof. After possibly switching the roles of X and Y, it follows from Lemmas 2.1.7 and 2.1.4 that there are fixed $m \in \mathbb{Z}_p$ and $\xi \in \mu_{p^\infty}$ such that $\phi(X, (Y + 1)(X + 1)^m - 1)$

vanishes at $(\zeta_k - 1, \xi - 1)$ for an infinite sequence $\{\zeta_k\}_{k\in\mathbb{N}}$. We deduce that

$$\phi(\zeta_k - 1, \xi\zeta_k^m - 1) = 0$$

for an infinite sequence $\{\zeta_k\}_{k\in\mathbb{N}}$. Over $\mathcal{O}_{F[\xi]}$ we may then write

$$\phi(X,Y) = \phi(X, \xi(X+1)^m - 1) + (\xi(X+1)^m - (Y+1))G(X,Y)$$

for some power series $G(X,Y)$. Since the power series $H(X) := \phi(X, \xi(X+1)^m - 1)$ vanishes at infinitely many points $\{\zeta_k - 1\}_{k\in\mathbb{N}}$ it follows that $H = 0$ and thus

$$\phi(X,Y) = (\xi(X+1)^m - (Y+1))G(X,Y) \tag{2.2}$$

for some $G(X,Y) \in \mathcal{O}_{F[\xi]}[[X,Y]]$. Taking conjugates under the group $\mathrm{Gal}(F[\xi]/F)$ of order g in (2.2), it follows that

$$\phi(X,Y)^g = \left(\prod_{\sigma\in\mathrm{Gal}(F[\xi]/F)} ((X+1)^m - \sigma(\xi)(Y+1))\right)\left(\prod_{\sigma\in\mathrm{Gal}(F[\xi]/F)} G^\sigma(X,Y)\right) \tag{2.3}$$

where $G^\sigma(X,Y)$ is obtained from G by acting on the coefficients. The two factors on the right hand side of (2.3) have coefficients in $\mathcal{O}_F$. Since ϕ is irreducible, it must be an irreducible factor of $\phi = \prod_{\sigma\in\mathrm{Gal}(F[\xi]/F)}((X+1)^m - \sigma(\xi)(Y+1))$. But requiring $\phi(0,0) = 0$ forces $\xi = g = 1$ which shows that $\phi(X,Y) = (X+1)^m - (Y+1)$, as desired.

□

2.1.3. Geometric statements

We are interested in irreducible components Z of the Zariski closure of the set of special points $\mathcal{S}_A$ on $A = \mathcal{O}_F[[X_1, \ldots, X_n]]/I$ for I a non-trivial ideal. As irreducible special subschemes correspond to products of a formal subtorus with a torsion point of $\widehat{\mathbb{G}}_m^n$, their dimension should be the dimension of the subtorus. While one has to be careful with dimensions in our setting as the ring of coefficients is one-dimensional, it follows from the definitions of Z that $(\pi) \subset \mathcal{O}_F$ is never an associated prime. The codimension of Z can be read off on the set of special points:

Lemma 2.1.10. *Let Z be an irreducible component of the Zariski closure of $\mathcal{S}_A$ and let S_Z denote the set of special points lying on Z. Then Z has codimension the maximum number r of columns with finite projections in the $\mathrm{Aut}(\widehat{\mathbb{G}}_m^n) \cong \mathrm{GL}_n(\mathbb{Z}_p)$-orbit of S_Z.*

Proof. The inequality $\mathrm{codim}(Z) \geq r$ is straightforward: if r coordinates have finite projections we may cover S_Z by finitely many sets of the form $\mathcal{S}^r \times (\xi_1 - 1) \times \cdots \times (\xi_{n-r} - 1)$. Their Zariski closure are codimension r hyperplanes. We proceed to show $\mathrm{codim}(Z) \leq r$. We may assume $\mathrm{codim}(Z) \geq 1$ and $n > r$. By definition of r, after a multiplicative change of variables as in Lemma 2.1.4 we may assume there are sequences $(\zeta_k)_{k\in\mathbb{N}}$ and $(a_{ik})_{k\in\mathbb{N}}$ with

$$S_0 := \{(\zeta_k - 1, \zeta_k^{a_{2k}} - 1, \zeta_k^{a_{2k}a_{3k}} - 1, \ldots, \zeta_k^{a_{2k}a_{3k}\ldots a_{nk}} - 1) | k \in \mathbb{N}\} \subseteq S_Z$$

and the projection $\{\zeta_k^{a_{2k}\cdots a_{(n-r)k}} - 1 | k \in \mathbb{N}\}$ to coordinate $(n-r)$ of S_0 is an infinite set. Let $r' \leq r$ be such that coordinate $(n - r')$ of S_0 is the last one with infinite projection. In particular, there is a height r' prime ideal $\mathfrak{p}_0 = (X_{n-r'+1} - \xi_{n-r'+1}, \ldots, X_n - \xi_n)$ of Z for some fixed roots of unity $(\xi_{n-r'+1}, \ldots, \xi_n) \in \mu_{p^\infty}^{r'}$. Suppose $\mathrm{codim}(Z) > r$. Thus we may find a height $r+1$ prime ideal $\mathfrak{p}$ strictly containing $\mathfrak{p}_0$ and pick $f \in \mathfrak{p}$ that is not in $(\pi, \mathfrak{p}_0)$. Then

$$f(X_1, \dots, X_{n-r'}, \xi_{n-r'+1} - 1, \dots, \xi_n - 1) \in \mathcal{O}_{F[\xi_{n-r'+1}, \dots, \xi_n]}[[X_1, \dots, X_{n-r'}]] \neq 0$$

and we may apply Lemma 2.1.8 to $f(X_1, \dots, X_{n-r'}, \xi_{n-r'+1} - 1, \dots, \xi_n - 1)$, whence the projection to coordinate $(n - r')$ of S_0 is a finite set, a contradiction.

□

We now prove the geometric formulation of infinitesimal p-adic Manin-Mumford:

Theorem 2.1.11. *Let $A = \mathcal{O}_F[[X_1, \dots, X_n]]/I$. An irreducible component of the Zariski closure of the special points $\mathcal{S}_A$ on* Spec(A) *is a special multiplicative subscheme.*

Proof. We denote by Z an irreducible component of the Zariski closure of $\mathcal{S}_A$ and by S_Z the special points on Z. We apply a multiplicative change of variables realizing $r = \text{codim}(Z)$ as the number of columns of the set S_Z with finite projection by Lemma 2.1.10. There are then finitely many sets $H_{r,i} = \mathcal{S}^{n-r} \times (\xi_{i,n-r+1} - 1) \times \cdots \times (\xi_{i,n} - 1)$ with

$$S_Z \subset \bigcup_{i=1}^{f} H_{r,i}.$$

Therefore by definition $Z \subset \bigcup_{i=1}^{f} Z_{r,i}$, where $Z_{r,i}$ denotes the Zariski closure of $H_{r,i}$. Since the $Z_{r,i}$ are irreducible special multiplicative subschemes of the same codimension as Z, it must be that $Z = Z_{r,i}$ for some i, as desired. □

In particular, we note that:

Corollary 2.1.12. *With notations as above, if the Zariski closure of* $\mathcal{S}_A$ *is* d*-dimensional, then* $\mathrm{Spec}(A)$ *contains a* d*-dimensional special multiplicative subscheme.*

One also deduces the rigidity result for formal schemes:

Corollary 2.1.13. *If the closed formal subscheme* $\mathrm{Spf}(A) \subset \widehat{\mathbb{G}}_m^n$ *contains infinitely many* $\overline{\mathbb{Q}}_p$*-torsion points of* $\widehat{\mathbb{G}}_m^n$*, it contains a translate of a formal subtorus by a torsion point. Moreover, if* $\mathrm{Spf}(A)$ *is geometrically irreducible and the set of special points* $\mathcal{S}_A$ *are Zariski dense,* $\mathrm{Spf}(A)$ *is precisely the translate of a formal subtorus by a torsion point.*

Proof. The formal subschemes associated to special subschemes are translates of formal subtori. Therefore the statement follows from Theorem 2.1.11. □

2.2. Generalization to Lubin-Tate formal groups

At this point, the reader may wonder whether similar results hold for a larger class of formal groups than the multiplicative group. We proceed to show that essentially all the results of Section 2.1 hold replacing the role of $\widehat{\mathbb{G}}_m$ by a one-dimensional Lubin-Tate formal group $\mathcal{F}_{LT}$. We come back to this question and discuss further generalizations in Chapter 4.

Let $E/\mathbb{Q}_p$ be a subfield of F with ring of integers $\mathcal{O}_E$, uniformizer π_E and residue field $\mathbb{F}_{q'}$. Given a power series $f \in \mathcal{O}_E[[X]]$ with

$$f(X) \equiv \pi X \mod X^2 \text{ and } f(X) \equiv X^{q'} \mod \pi_E,$$

Lubin and Tate show [**LT65**, Lemma 1] that for all $a \in \mathcal{O}_E$ there is a unique power series $[a](X) \in \mathcal{O}_E[[X]]$ with $[a](X) \equiv aX \mod X^2$ and $f([a](X)) = [a](f(X))$. They construct [**LT65**, Theorem 1] a commutative one-dimensional formal group law $L(X,Y) \in \mathcal{O}_E[[X,Y]]$ such that for all $a, b \in \mathcal{O}_E$ the following hold:

(1) $L([a](X),[a](Y)) = [a](L(X,Y))$

(2) $L([a](X),[b](X)) = [a+b](X)$

(3) $[a]([b](X)) = [ab](X)$

(4) $[\pi_E](X) = f(X)$ and $[1](X) = X$.

Up to isomorphism, the group law is independent of the choice of $f \in \mathcal{O}_E[[X]]$ with the desired properties. We denote by $\mathcal{F}_{LT}$ the formal group over $\mathcal{O}_E$ resulting from this construction. In particular, the properties above show there is an injective ring homomorphism

$$\mathcal{O}_E \hookrightarrow \mathrm{End}(\mathcal{F}_{LT}),$$

and the $\overline{\mathbb{Q}}_p$-torsion points of $\mathcal{F}_{LT}$ form a divisible $\mathcal{O}_E$-module. We will abuse notations and write $\mathcal{F}_{LT}[\pi_E^\infty]$ for the $\overline{\mathbb{Q}}_p$-points. It follows from the Newton polygon of $[\pi_E](X)$ and the congruence $[\pi_E](X) \equiv X^{q'} \mod \pi_E$ that a torsion point $\zeta \in \mathcal{F}_{LT}[\pi_E^\infty]$ of exact order π^k has normalized valuation $v_p(\zeta) = 1/(q^k - q^{k-1})$. Adjoining torsion points gives a totally ramified abelian extension $E(\mathcal{F}_{LT}[\pi_E^\infty])$ of E. As these properties suggest, we may take as set of *special points* the $\overline{\mathbb{Q}}_p$-points in $\mathcal{F}_{LT}^n[\pi_E^\infty]$ and define as before:

Definition.

(1) An $\mathcal{F}_{LT}$-*multiplicative* change of variables on $\mathcal{O}_F[[X_1,\dots,X_n]]$ is given by a series of transformations using the formal group law of $\mathcal{F}_{LT}$:

$$X_i \mapsto L(\cdots L(L(X_i,[B_{i-1}](X_{i-1})),[B_{i-2}](X_{i-2})),\dots,[B_1](X_1))$$

for $1 \le i \le n$ and $B_j \in \mathcal{O}_E$, composed with possibly swapping variables $X_i \leftrightarrow X_j$.

(2) An $\mathcal{F}_{LT}$-*special subscheme* of $\mathrm{Spec}(\mathcal{O}_F[[X_1,\dots,X_n]])$ is a closed subscheme that after an $\mathcal{F}_{LT}$-multiplicative change of variables becomes a finite union of intersections of hyperplanes $X_i = \zeta_i$ where $\zeta_i \in \mathcal{F}_{LT}[\pi^\infty]$.

Remark. The $\mathcal{F}_{LT}$-*multiplicative* changes of variables correspond to automorphisms of the Lubin-Tate formal group law. An irreducible scheme $\mathrm{Spec}(\mathcal{O}_F[[X_1,\dots,X_n]]/I)$ is $\mathcal{F}_{LT}$-special if and only if $\mathrm{Spf}(\mathcal{O}_F[[X_1,\dots,X_n]]/I)$ is the translate of a formal subtorus $\mathcal{F}_{LT}^d$ by a torsion point of $\mathcal{F}_{LT}^n$. When $\mathcal{F}_{LT} = \widehat{\mathbb{G}}_m$, we simply have that

$$L(X,Y) = (X+1)(Y+1) - 1$$

and we recover all the previous definitions as a special case.

We revisit the proofs of the key Lemmas 2.1.4 and 2.1.8 in this more general setting. Fix an ideal $I \subset \mathcal{O}_F[[X_1,\dots,X_n]]$ and let $A = \mathcal{O}_E[[X_1,\dots,X_n]]/I$. For any $\epsilon > 0$ we consider as before the special points almost on $\mathrm{Spec}(A)$:

$$S_I(\epsilon) = \{(\zeta_1,\dots,\zeta_n) \in \mathcal{F}_{LT}^n[\pi^\infty] \text{ such that } \forall \phi \in I \ |\phi(\zeta_1,\dots,\zeta_n)|_p < \epsilon\}.$$

Using the properties of the endomorphism ring of $\mathcal{F}_{LT}^n$ we again obtain:

Lemma 2.2.1. *Assume that $S_I(\epsilon)$ is infinite for some $\epsilon > 0$. Then after an $\mathcal{F}_{LT}$-multiplicative change of variables there is a sequence $\{\zeta_k\}_{k\in\mathbb{N}} \in \mathcal{F}_{LT}[\pi^\infty]$ for which*

$$(\zeta_k, [a_{2k}](\zeta_k), \ldots, [a_{2k} \cdots a_{nk}](\zeta_k)) \in S_I(\epsilon)$$

and:

- *The set $\{\zeta_k \in \mathcal{F}_{LT}[\pi^\infty] | k \in \mathbb{N}\}$ is infinite.*
- *The set $\{[a_{2k} \cdots a_{nk}](\zeta_k) | k \in \mathbb{N}\}$ is finite or each sequence $\{a_{ik}\}_{k\in\mathbb{N}} \in \mathcal{O}_E$ converges to zero p-adically.*

Proof. After swapping indeterminates X_i so that infinitely often elements $(\zeta_1, \ldots, \zeta_n) \in S_I(\epsilon)$ have decreasing orders, we again get an explicit sequence

$$(\zeta_k, [a_{2k}](\zeta_k), \ldots, [a_{2k} \cdots a_{nk}](\zeta_k)) \in S_I(\epsilon),$$

for exponents $a_{ik} \in \mathcal{O}_E$ and infinitely many $\zeta_k \in \mathcal{F}_{LT}[\pi^\infty]$. By compactness of $\mathcal{O}_E$, we may after passing to a subsequence assume that the exponents a_{ik} converge p-adically to $A_i \in \mathcal{O}_E$. As before, choose $B_{i,j} \in \mathcal{O}_E$ such that

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ B_{21} & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \ldots & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ A_1 \\ \vdots \\ A_1 \cdots A_{n-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Using crucially that the map $\mathcal{O}_F \to \mathrm{End}(\mathcal{F}_{LT})$ is a ring homomorphism, we see that the $\mathcal{F}_{LT}$-multiplicative change of variables corresponding to $B_{i,j}$ acts on our sequence via:

$$\begin{pmatrix} \zeta \\ [a_2](\zeta) \\ \vdots \\ [a_2 \cdots a_n](\zeta) \end{pmatrix} \mapsto \begin{pmatrix} \zeta \\ L([B_{21}](\zeta), [a_2](\zeta)) \\ \vdots \\ L([B_{n1}](\zeta), L(\cdots L([B_{nn-1}][a_2 \cdots a_{n-1}](\zeta), [a_2 \cdots a_n](\zeta)) \cdots) \end{pmatrix}$$

$$= \begin{pmatrix} \zeta \\ [B_{21} + a_2](\zeta) \\ \vdots \\ [B_{n1} + \cdots + B_{nn-1} a_2 \cdots a_{n-1} + a_2 \cdots a_n](\zeta) \end{pmatrix}$$

so that after changing variables we may assume the products $a_{2k} \cdots a_{jk} \in \mathcal{O}_E$ converge to zero for any $2 \leq j \leq n$. Now repeat this process as in the proof of Lemma 2.1.4 to conclude. □

Lemma 2.2.2. *Assume* $(\zeta_1, \ldots, \zeta_n) \in S_I(\epsilon)$ *for some* $\epsilon \leq |\pi|_p$. *Then*

$$([\pi_E](\zeta_1), \ldots, [\pi_E](\zeta_n)) \in S_I(\epsilon^{q'}).$$

Proof. The congruence $[\pi_E](X) \equiv X^{q'} \mod \pi_E$ yields

$$\phi([\pi_E](X_1), \ldots, [\pi_E](X_n)) \equiv \phi(X_1^{q'}, \ldots, X_n^{q'}) \equiv \phi(X_1, \ldots, X_n)^{q'} \mod \pi$$

for functions $\phi \in I$ and the result follows. □

It follows that the key Lemma 2.1.8 generalizes:

Lemma 2.2.3. *For any power series* $\phi \in \mathcal{O}_F[[X_1, \ldots, X_n]]$*, if*

$$(\zeta_k, [a_{2k}](\zeta_k), \ldots, [a_{2k} \cdots a_{nk}](\zeta_k)) \in S_\phi(p^{-(1-c)v_p(\pi)}),$$

where $0 \leq c < 1$ *is a constant and each sequence* $\{a_{ik}\}_{k \in \mathbb{N}} \in \mathcal{O}_E$ *converges to zero for an infinite set* $\{\zeta_k | k \in \mathbb{N}\} \subset \mathcal{F}_{LT}[\pi_E^\infty]$*, then either:*

(1) the sequence $\{[a_{2k}a_{3k} \ldots a_{nk}](\zeta_k)\}_{k \in \mathbb{N}}$ *belongs to a finite set of torsion points or*

(2) the power series $\phi \in \pi\mathcal{O}_F[[X_1, \ldots, X_n]]$*.*

Proof. We proceed by induction. If $n = 1$ the result again follows from Lemma 2.1.7. For arbitrary n, let c_k be the smallest power π_E^k such that $[\pi_E^k] \cdot ([a_{2k} \cdots a_{nk}](\zeta_k)) = 0$. Assume the set $\{[a_{2k} \cdots a_{nk}](\zeta_k) | k \in \mathbb{N}\}$ is infinite, then the same is true of $\{c_k | k \in \mathbb{N}\}$. We want to show $\phi \in \pi\mathcal{O}_F[[X_1, \ldots, X_n]]$. Suppose not; there is then a largest integer M with X_n^M dividing the reduction $\bar{\phi} \in \mathbb{F}_q[[X_1, \ldots, X_n]]$ modulo π.
We choose $\psi \in \mathcal{O}_F[[X_1, \ldots, X_n]]$ so that

$$\phi(X_1, \ldots, X_n) \equiv X_n^M \psi(X_1, \ldots, X_n) \mod \pi.$$

Since the valuation $v_p([a_{2k} \cdots a_{nk}](\zeta_k))$ becomes arbitrarily small, after passing to a subsequence we get that $(\zeta_k, [a_{2k}](\zeta_k), \ldots, [a_{2k} \cdots a_{nk}](\zeta_k)) \in S_\psi(p^{-(1-c/2)v_p(\pi)})$. We now see from Lemma 2.2.2 that:

$$(\zeta_k, [c_k a_{2k}](\zeta_k), \ldots, [c_k a_{2k} \cdots a_{(n-1)k}](\zeta_k), 0) \in S_\psi(p^{-(1-c/2)v_p(\pi)}).$$

By definition of c_k the torsion point $[c_k a_{2k} \cdots a_{(n-1)k}](\zeta_k)$ has exact order the largest power of π dividing $[a_{nk}]$. Since $a_{nk} \neq 0$ infinitely often by assumption, the set of torsion points $\{[c_k a_{2k} \cdots a_{(n-1)k}](\zeta_k) | k \in \mathbb{N}\}$ is infinite. It follows by induction that $\psi(X_1, \ldots, X_{n-1}, 0) \in \pi\mathcal{O}_F[[X_1, \ldots, X_{n-1}]]$. We may now write for some power series $\theta \in \mathcal{O}_F[[X_1, \ldots, X_n]]$ $\psi(X_1, \ldots, X_n) = \psi(X_1, \ldots, X_{n-1}, 0) + X_n\theta(X_1, \ldots, X_n)$, so that $\psi \equiv X_n\theta \mod \pi$. This results in the congruence

$$\phi(X_1, \ldots, X_n) \equiv X_n^{M+1}\theta(X_1, \ldots, X_n) \mod \pi,$$

which contradicts the maximality of M. □

The main results are now deduced exactly as in Section 2.1 for $\mathcal{F}_{LT} = \widehat{\mathbb{G}}_m$. Below are the adapted geometric formulations. The proof is left to the reader.

Theorem 2.2.4. *Let $A = \mathcal{O}_F[[X_1, \ldots, X_n]]/I$. An irreducible component of the Zariski closure of the $\mathcal{F}_{LT}$-special points on* Spec(A) *is a special $\mathcal{F}_{LT}$-multiplicative subscheme.*

Theorem 2.2.5. *Let $A = \mathcal{O}_F[[X_1, \ldots, X_n]]/I$. Exactly one of the following occurs:*

(1) The formal scheme Spf(A) *contains a translate of the Lubin-Tate formal group $\mathcal{F}_{LT}$ by a torsion point of $\mathcal{F}_{LT}^n$.*

(2) There exist an explicit constant $C_I > 0$, depending only on I and the choice of p-adic absolute value $|-|_p$, as well as a finite set $\mathcal{F} \subset \mathcal{F}_{LT}[\pi_E^\infty]^n$ such that for any $\phi \in I$,

$$|\phi(\zeta_1, \ldots, \zeta_n)|_p > C_I$$

if $(\zeta_1, \ldots, \zeta_n) \in \mathcal{F}_{LT}[\pi_E^\infty]^n \setminus \mathcal{F}$.

CHAPTER 3

Applications to Hida theory

In this Chapter, we apply our results to study p-adic families of automorphic forms parametrized by weight. Under appropriate assumptions, the weights of a family are points on $\mathrm{Spec}(\mathbb{Z}_p[[X_1, \ldots, X_n]])$. We will use the results in Chapter 2 to study special points on the space of p-adic weights of classical automorphic forms for GL_2 over an imaginary quadratic field. As a result, we are able to describe exactly when a family passes through infinitely many such special points in Section 3.2.4.

3.1. Modular forms in p-adic families

We first highlight some aspects of the theory of Hida families for modular forms over $\mathbb{Q}$. For a more extensive introduction to the subject, we refer the reader to the survey articles [**Hid87, Eme11**] and Hida's paper [**Hid86a**].

Modular forms are holomorphic functions f on the complex upper half-plane $\mathbb{H}_2$ invariant under the weight k action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ for some congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ holomorphic at the cusp at $i\infty$. The action is given by:

$$(f|_k\gamma)(z) := (cz + d)^{-k}(f(\gamma \cdot z)),$$

where the action on $\mathbb{H}_2$ is by fractional linear transformations. We shall consider the standard congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$:

$$\Gamma_0(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) | c \equiv 0 \mod N \}$$

$$\Gamma_1(N) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) | a, d \equiv 1 \mod N \}$$

and denote by $M_k(\Gamma_1(N))$ the space of modular forms of level N and weight k as well as by $S_k(\Gamma_1(N))$ the space of cuspforms, i.e. vanishing at the cusps of $\Gamma_1(N)\backslash\mathbb{H}_2$. The weight k action of $\Gamma_0(N)$ preserves $M_k(\Gamma_1(N))$ and factors through $\Gamma_0(N)/\Gamma_1(N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$. If $(l, N) = 1$ we let $\langle l \rangle$ denote the operator on $M_k(\Gamma_1(N))$ corresponding to the class $l \in (\mathbb{Z}/N\mathbb{Z})^\times$. The Hecke operators T_l also act on these spaces and can be characterized by their effect on the Fourier expansion at the cusp at $i\infty$, which uniquely determines modular forms:

$$T_l f(\tau) = \sum_{n=0}^{\infty} a_{nl}(f) q^n + \sum_{n=0}^{\infty} l^{k-1} a_n(\langle l \rangle f) q^{nl},$$

where $a_n(f)$ is the n-th Fourier coefficient of f. If $l|N$ we also consider operators U_l given by:

$$U_l f(\tau) = \sum_{n=0}^{\infty} a_{nl}(f) q^n.$$

The Hecke algebra $\mathbb{T}_k(N)$ is the $\mathbb{Z}$-subalgebra of endomorphisms of $M_k(\Gamma_1(N))$ generated by T_l and $S_l = \langle l \rangle l^{k-2}$ for primes l not dividing N and U_l for $l|N$. It is well-known that $\mathbb{T}_k(N)$ is commutative, reduced and finite rank over $\mathbb{Z}$.

Now fix an embedding $i : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ via which we may view the Fourier coefficients as p-adic numbers. If R is a ring and $S_k(\Gamma_1(N), R)$ denotes the space of cusp forms with Fourier coefficients in R, we have a non-degenerate bilinear pairing

$$\mathbb{T}_k(N) \otimes_{\mathbb{Z}} R \times S_k(\Gamma_1(N), R) \to R$$

$$(T, f) \mapsto a_1(Tf)$$

If $R = \overline{\mathbb{Q}}_p$ or $\overline{\mathbb{Q}}$, we obtain in this way isomorphisms:

$$\mathrm{Hom}_R(\mathbb{T}_k(N) \otimes_{\mathbb{Z}} R, R) \cong S_k(\Gamma_1(N)),$$

since the Fourier coefficients are algebraic integers. The Hecke operators are simultaneously diagonalizable and there is a basis of eigenvectors for $S_k(\Gamma_1(N))$. Since $a_l(f) = a_1(T_l f)$, the R-algebra homomorphisms

$$\lambda : \mathbb{T}_k(N) \otimes_{\mathbb{Z}} R \to R$$

correspond bijectively to eigenforms, normalized so that $a_1 = 1$. One than simply has $\lambda(T_l) = a_l(f)$, i.e. λ is a system of Hecke eigenvalues. Hida constructs p-adic families of cusp forms parametrized by weight by exhibiting a p-adic Hecke algebra that interpolates all the weight k algebras for $k \geq 2$.

3.1.1. The universal ordinary p-adic Hecke algebra

Fix a level N and an odd prime p such that $(p, N) = 1$ and let $\mathcal{O}$ denote the ring of integers of a finite extension of $\mathbb{Q}_p$. The p-adic Hecke algebras $\mathbb{T}_k(Np^r, \mathcal{O}) = \mathbb{T}_k(Np^r) \otimes_{\mathbb{Z}} \mathcal{O}$ now act on spaces of p-adic modular forms with coefficients in $\mathcal{O}$. The technical issue one runs into is that the operator U_p acting on these spaces now has large kernel, so one needs to project to the part where U_p acts invertibly:

The Hecke algebra $\mathbb{T}_k(Np^r, \mathcal{O})$ is a product of local rings and U_p projects to each of the factors. Define an idempotent e_{Np^r} as the sum of the idempotents for local rings where U_p is invertible. The *ordinary* part $\mathbb{T}_k^{\mathrm{ord}}(Np^r, \mathcal{O}) = e_{Np^r}\mathbb{T}_k(Np^r, \mathcal{O})$ is then the maximal direct summand on which the image of U_p is invertible. We may increase the level for fixed weight k and denote

$$\mathbb{T}_k(Np^\infty, \mathcal{O}) := \varprojlim_r \mathbb{T}_k(Np^r, \mathcal{O}),$$

where the limit is taken with respect to the surjective $\mathbb{Z}_p$-algebra homomorphisms

$$\mathbb{T}_k(Np^r, \mathcal{O}) \to \mathbb{T}_k(Np^s, \mathcal{O}), \text{ for } r \geq s.$$

These map U_p to U_p and one may also consider the ordinary parts

$$\mathbb{T}_k^{\mathrm{ord}}(Np^\infty) = \varprojlim_r \mathbb{T}_k^{\mathrm{ord}}(Np^r, \mathcal{O}) = (\varprojlim_r e_{Np^r})\mathbb{T}_k(Np^\infty, \mathcal{O}).$$

The resulting objects have the structure of algebras over the completed group ring

$$\Lambda_N = \varprojlim_r \mathbb{Z}_p[(\mathbb{Z}/Np^r\mathbb{Z})^\times]$$

and Hida shows the following:

Theorem 3.1.1 ([**Hid87**], Theorem 2.1)**.** *There are isomorphisms of Λ_N-algebras $\mathbb{T}_k(Np^\infty, \mathcal{O}) \cong \mathbb{T}_2(Np^\infty, \mathcal{O})$ as well as $\mathbb{T}_k^{\mathrm{ord}}(Np^\infty, \mathcal{O}) \cong \mathbb{T}_2^{\mathrm{ord}}(Np^\infty, \mathcal{O})$ for $k \geq 2$.*

Thus $\mathbb{T}^{\text{ord}}(Np^\infty, \mathcal{O})$ is independent of the weight and is by construction equipped with surjective specialization maps $\rho_{r,k} : \mathbb{T}^{\text{ord}} \to \mathbb{T}_k^{\text{ord}}(Np^r, \mathcal{O})$ for all $k \geq 2$. We call $\mathbb{T}^{\text{ord}}$ the *universal ordinary Hecke algebra of tame level N.*

3.1.2. Hida families over weight space

We now describe how $\mathbb{T}^{\text{ord}}$ interpolates classical modular forms in p-adic families parametrized by weight. Let $\Gamma = 1 + p\mathbb{Z}_p$. We may write $\Lambda_N \cong \mathbb{Z}_p[(\mathbb{Z}/Np\mathbb{Z})^\times] \times \Lambda$, where

$$\Lambda = \varprojlim_r \mathbb{Z}_p[\Gamma/\Gamma^r]$$

and Λ is isomorphic to a power series ring $\mathbb{Z}_p[[T]]$ via the identification $T \mapsto [1+p]-1$. One sees that the Λ-algebra structure of $\mathbb{T}^{\text{ord}}$ comes from the unique extension to a continuous homomorphism $\Lambda \to \mathbb{T}^{\text{ord}}$ of the map

$$\{l \in \mathbb{Z} | l \equiv 1 \mod (Np)\} \to \mathbb{T}^{\text{ord}}$$
$$l \mapsto S_l = l^{k-2}.$$

This induces a morphism of affine schemes $\text{Spec}(\mathbb{T}^{\text{ord}}) \to \text{Spec}(\mathbb{Z}_p[[T]])$ which is a finite covering map:

Theorem 3.1.2 ([**Hid86b**], Theorem 3.1)**.** *The Hecke algebra $\mathbb{T}^{\text{ord}}$ is free of finite rank over Λ.*

The interesting systems of Hecke eigenvalues are $\overline{\mathbb{Z}}_p$-points on $\mathrm{Spec}(\mathbb{T}^{\mathrm{ord}})$ living over $\overline{\mathbb{Z}}_p$-points on $\mathrm{Spec}(\Lambda)$ which are just given by continuous characters

$$\chi : \Gamma \to \overline{\mathbb{Z}}_p^{\times}.$$

If ϵ is a character of Γ modulo p^r we will write $P_{k,\epsilon}$ for the point corresponding to the character

$$\chi : \gamma \mapsto \gamma^{k-2}\epsilon(\gamma),$$

where γ is a topological generator of Γ. Explicitly, for $\gamma = 1 + p$, we get in $\mathbb{Z}_p[[T]]$ the corresponding prime element

$$P_{k,\epsilon} = (T + 1) - \epsilon(1 + p)(1 + p)^{k-2}.$$

We will call these the *classical points*. Indeed Hida shows that the systems of Hecke eigenvalues above these points correspond to classical, integral weight modular forms: a character of Γ modulo p^r acts on the congruence subgroup $\Gamma_1(Np) \cap \Gamma_0(p^r)$ via

$$\epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \epsilon(d)$$

and we write $\mathbb{T}_k^{\mathrm{ord}}(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon\mathcal{O})$ for the Hecke algebra dual to the space of cuspforms with coefficients in $\mathcal{O}$ of level $\Gamma_1(Np) \cap \Gamma_0(p^r)$, nebentypus ϵ where the norm of the normalized Fourier coefficient $|a_p(f)|_p = 1$. Hida shows:

Theorem 3.1.3 ([**Hid86a**], Theorem 1.2)**.** *For each $k \geq 2$ and ϵ of conductor p^r, the specialization $\rho_{r,k}$ induces isomorphisms sending T_l to T_l:*

$$\mathbb{T}^{\mathrm{ord}}/P_{k,\epsilon}\mathbb{T}^{\mathrm{ord}} \cong \mathbb{T}_k^{\mathrm{ord}}(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon, \mathcal{O}).$$

A *Hida family* is an irreducible component of $\mathrm{Spec}(\mathbb{T}^{\mathrm{ord}})$. We ignored here the module structure over the whole ring Λ_N and just worked with the connected component Λ. However each Hida family has a unique character ψ modulo Np attached to it and one easily sees that the specializations in Theorem 3.1.3 actually land in $\mathbb{T}_k^{\mathrm{ord}}(\Gamma_1(Np) \cap \Gamma_0(p^r), \epsilon\psi\omega^{-k}, \mathcal{O})$, where ω is the Teichmueller character.

Given any eigenform $f \in S_k(\Gamma_1(N))$ and p coprime to N, the forms $f(\tau)$ and $f(p\tau)$ have level Np and one may consider the characteristic polynomial

$$X^2 - (\alpha + \beta)X + \alpha\beta$$

of U_p acting on the space they span. If one of the eigenvalues (say α) is a p-adic unit, then $f(\tau) - \beta f(p\tau)$ is an ordinary form of level Np and U_p-eigenvalue α and by the construction above lives in a Hida family $\mathcal{H}$.

It follows in particular from Theorem 3.1.3 that *every specialization* in weight $k \geq 2$ is a classical modular form. More precisely, a Hida family corresponds to a homomorphism $\lambda : \mathbb{T}^{\mathrm{ord}} \to \mathcal{Q}$ for $\mathcal{Q}$ some finite integral domain over Λ. Then (assuming $N = 1$ for simplicity) we have:

Theorem 3.1.4 ([**Hid86a**], Theorem I). *For any $k \geq 2$ and finite order character ϵ, specializing the formal q-expansion*

$$\mathcal{F}_\lambda = \sum_{n=1}^{\infty} \lambda(T_n) q^n$$

in $\Lambda[[q]] \cong \mathbb{Z}_p[[T, q]]$ at $T = \epsilon(1+p)(1+p)^{k-2} - 1$ gives the q-expansion of an eigenform in $S_k(\Gamma_1(Np^r))$, where r is defined by $\ker(\epsilon) = 1 + p^r\mathbb{Z}_p$.

3.2. Hida families for GL(2)/F

The group GL(2)/F is associated to a PEL Shimura variety if and only if the number field F is totally real. In this context, the theory of Hida families resembles the case of elliptic modular forms ([**Hid89, Hid88**]). However, when F has at least one complex place, the relevant universal Hecke algebra is torsion over the relevant weight space, and conjecturally the image of its spectrum on weight space Spec(Λ) is of codimension the number of conjugate pairs of complex embeddings of F.

We focus here on the case when $F/\mathbb{Q}$ is an imaginary quadratic field with ring of integers $\mathcal{O}_F$. Following Taylor ([**Tay88**]) and Hida ([**Hid93, Hid94**]), we highlight some key aspects of the theory; in particular every component of the spectrum of the Hecke algebra is in this case known to have codimension exactly one (Theorem 3.2.4). We then prove the main result of this Chapter (Theorem 3.2.6) and work out an example in detail of how this can be used to prove, when true, that there are only finitely many classical automorphic forms in a family.

3.2.1. Bianchi modular forms

Let $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_F)$ be a congruence subgroup. The relevant locally symmetric spaces in this case will be quotients $\Gamma\backslash\mathbb{H}_3$ of hyperbolic 3-space, which we may view as a subset of the Hamilton quaternions $\mathbb{H}$

$$\mathbb{H}_3 = \{z = x + jy | x \in \mathbb{C}, y \in \mathbb{R}^{>0}\} \subset \mathbb{H}$$

so that the action is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = (az+b)(cz+d)^{-1}$$

with multiplication in $\mathbb{H}$. We will be primarily concerned with cuspidal automorphic forms for $\mathrm{GL}(2)/F$ which contribute to the cohomology of $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_F)$. For any $\mathcal{O}_F$-module A denote by

$$V_{k,k'}(A) = \mathrm{Sym}_k(A^2) \otimes \overline{\mathrm{Sym}_{k'}}(A^2)$$

the representation of Γ where $\gamma \in \Gamma$ acts via the complex conjugate $\overline{\gamma}$ on the second copy of the symmetric power (when $A = \mathbb{C}$ these are the irreducible finite-dimensional representations of $\mathrm{SL}_2(\mathbb{C})$). We denote by $\mathcal{V}_{k,k'}(A)$ the associated locally constant sheaf of Γ-invariant sections on $\Gamma\backslash\mathbb{H}_3$. Then by an Eichler-Shimura isomorphism, here due to Harder ([**Har87**, Section 3.6]), a form of interest π lives in a cuspidal cohomology group

$$H^1_c(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{k,k}(\mathbb{C})),$$

where we define the cuspidal subspace H_c^* as the kernel of the natural map

$$H^*(\overline{\Gamma\backslash\mathbb{H}_3}, \mathcal{V}_{k,k'}(A)) \to H^*(\partial(\overline{\Gamma\backslash\mathbb{H}_3}), \mathcal{V}_{k,k'}(A)),$$

i.e. the classes vanishing at the boundary of the Borel-Serre compactification of $\Gamma\backslash\mathbb{H}_3$. Moreover, we have from [**Har87**] or [**BW13**, Section 3] that

$$H_c^1(\mathbb{H}_3/\Gamma, \mathcal{V}_{k,k'}) = 0 \text{ if } k \neq k'. \tag{3.1}$$

It is well-known that $H^*(\Gamma\backslash\mathbb{H}_3, \mathcal{V}_{k,k'}(A)) \cong H^*(\Gamma, V_{k,k'}(A))$ and the cuspidal space may be expressed as a subspace of the relevant group cohomology as well ([**Tay88**, Section 4.1]); we will often work with these instead. The p-adic modular forms are then defined as the dual of the algebra generated by the suitable Hecke operators on cohomology with p-adic coefficients (see [**Hid94**, Section 3]).

3.2.2. Adelic modular forms and Hecke operators

In what follows, we will mainly restrict ourselves to automorphic forms for GL_2/F that contribute to the cohomology of congruence subgroups of $\mathrm{SL}_2(\mathcal{O}_F)$ when $\mathcal{O}_F$ has trivial class group $\mathcal{C}_F$. This is mostly for simplicity of exposition. We include here a definition of the adelic Hecke operators acting on the relevant cohomology and some remarks about the general setting. For a detailed definition of automorphic forms over arbitrary imaginary quadratic F we refer the reader to [**Byg98**, Section 6.2].

We consider the maximal orders of $M_2(F)$ given by

$$M_0(\mathfrak{b}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) | a, d \in \mathcal{O}_F, b \in \mathfrak{b}^{-1}, c \in \mathfrak{b}\}$$

for any ideal $\mathfrak{b} \neq 0$ of $\mathcal{O}_F$. One can check that the maximal orders $M_0(\mathfrak{a})$ and $M_0(\mathfrak{b})$ are conjugate to each other if and only if $\mathfrak{a}$ and $\mathfrak{b}$ represent the same class in $\mathcal{C}_F/\mathcal{C}_F^2$. Let now $\mathbb{A}_f$ denote the ring of finite adeles over K and U an open subgroup of $\mathrm{GL}_2(\widehat{\mathcal{O}}_F)$, where $\widehat{\mathcal{O}}_F$ is the profinite completion of $\mathcal{O}_F$. We have the following result:

Proposition 3.2.1. *The locally symmetric space*

$$Y(U) = \mathrm{GL}_2(F) \backslash (\mathrm{GL}_2(\mathbb{A}_f)/U) \times \mathbb{H}_3$$

is a disjoint union of arithmetic hyperbolic threefolds. Provided the determinant map $\det : U \to \widehat{\mathcal{O}}_F^\times$ *is surjective, we may write*

$$Y(U) = \sqcup_{i=1}^{h_F} \Gamma_i \backslash \mathbb{H}_3,$$

for $\Gamma_i = U \cap M_0(\mathfrak{b}_i)^\times$ *and ideals* $\mathfrak{b}_i$ *forming a set of representatives of the class group* $\mathcal{C}_F$ *and* $b_1 = \mathcal{O}_F$.

Proof. This follows from strong approximation, see e.g., [**Byg98**, Section 5.2]. □

Let us assume that we are working at level $\mathfrak{n}$ for $\mathfrak{n}$ an ideal of $\mathcal{O}_F$, i.e.

$$U = U_1(\mathfrak{n}) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}_F) | c, d-1 \in \mathfrak{n}\widehat{\mathcal{O}}_F, \}.$$

Elements $g \in M_2(\mathcal{O}_F)$ give rise to operators on the cohomology groups $H^i(Y(U), \mathcal{V})$ with coefficients in a local system $\mathcal{V}$ as follows: We have inclusions

$$\begin{array}{ccccc} & & U & & \\ & \nearrow & & \nwarrow & \\ U \cap gUg^{-1} & & & & U \cap g^{-1}Ug \end{array}$$

that induce finite covering maps $Y(U \cap gUg^{-1}) \to Y(U)$ and $Y(U \cap g^{-1}Ug) \to Y(U)$. The Hecke operator associated to g is the composition of the induced maps on cohomology

$$H^i(Y(U), \mathcal{V}) \longrightarrow H^i(Y(U \cap gUg^{-1}), \mathcal{V}) \xrightarrow{ad(g)} H^i(Y(U \cap g^{-1}Ug), \mathcal{V}) \xrightarrow{cor} H^i(Y(U), \mathcal{V})$$

where the middle map is induced by conjugation by g and the right hand map is corestriction. For any prime ideal $\mathfrak{l}$ coprime to the level, we get Hecke operators $T_{\mathfrak{l}}$ and $S_{\mathfrak{l}}$ by taking respectively

$$g = \begin{pmatrix} \varpi_{\mathfrak{l}} & 0 \\ 0 & 1 \end{pmatrix} \text{ or } \begin{pmatrix} \varpi_{\mathfrak{l}} & 0 \\ 0 & \varpi_{\mathfrak{l}} \end{pmatrix},$$

where $\varpi_{\mathfrak{l}}$ denotes a uniformizer at $\mathfrak{l}$.

One then has the following:

(1) The algebra generated by the Hecke operators is commutative and the Hecke action commutes with restriction maps on the level.

(2) The cohomology decomposes (in the notations of Proposition 3.2.1) as

$$H^*(Y(U), \mathcal{V}) = \bigoplus_{i=1}^{h_F} H^*(\Gamma_i \backslash \mathbb{H}_3)$$

and the Hecke operator $T_{\mathfrak{l}}$ permutes the components $\Gamma_i\backslash\mathbb{H}_3$ according to the class of $\mathfrak{l}$ in $\mathcal{C}_F$. In particular, components are fixed if and only if $\mathfrak{l}$ is principal.

(3) The systems of Hecke eigenvalues associated to non-principal ideals differ from principal ones by a character of the class group $\mathcal{C}_F$.

It follows that for many purposes it suffices to work with the principal component and subgroups of $GL_2(\mathcal{O}_F)$ even when $h_F \neq 1$.

3.2.3. The p-adic universal objects

We now discuss the relevant results on the p-adic cohomology and Hecke algebra, which go back to Taylor ([**Tay88**, Chapter 4]) for H^1 and imaginary quadratic fields and were generalized by Hida for SL_2 in [**Hid93**] and GL_2 in [**Hid94**] over arbitrary number fields.

Fix a split odd prime $p = \mathfrak{p}\overline{\mathfrak{p}}$ in F, an embedding $i : F \to \overline{\mathbb{Q}}_p$ and let $\mathcal{O}_p$ denote $\mathcal{O}_F \otimes \mathbb{Z}_p$. The p-adic modular forms will be cohomology classes with coefficients in the ring of integers of a finite extension E of $\mathbb{Q}_p$, which we denote by $\mathcal{O}$. The congruence subgroups $\Delta_i(p^r) := \Gamma_1(N) \cap \Gamma_i(p^r)$ for $i = 0, 1$ act on $V_{k,k'}(\mathcal{O})$ and $V_{k,k'}(E/\mathcal{O})$ via the embedding. For a fixed tame level N and varying $r \geq 1$ the cohomology groups

$$H^1(\Delta_1(p^r), V_{k,k'}(E/\mathcal{O})) \cong H^1(\Delta_1(p^r)\backslash\mathbb{H}_3, \mathcal{V}_{k,k'}(E/\mathcal{O}))$$

are modules over $\Delta_0(p^r)/\Delta_1(p^r) \cong (\mathcal{O}_F/p^r\mathcal{O}_F)^\times \cong (\mathcal{O}_p/p^r\mathcal{O}_p)^\times$. One again has a T_p operator acting on cohomology and we denote by H^*_{ord} the direct summand that T_p acts faithfully on. Taking the limit, we get that

$$H^1(\Delta_1(p^\infty), V_{k,k'}(E/\mathcal{O})) = \varinjlim_r H^1(\Delta_1(p^r), V_{k,k'}(E/\mathcal{O}))$$

$$H^1_{\mathrm{ord}}(\Delta_1(p^\infty), V_{k,k'}(E/\mathcal{O}) = \varinjlim_r H^1_{\mathrm{ord}}(\Delta_1(p^r), V_{k,k'}(E/\mathcal{O}))$$

will be modules over the completed group ring $\mathcal{O}[[\mathcal{O}_p^\times]]$.

We remark that all of the above can be carried through without restricting to congruence subgroups of $\mathrm{SL}_2(\mathcal{O}_F)$. In general for GL_2 one has congruence subgroups for each inner form and Hida defines nearly ordinary cohomology. See in particular [**Hid94**, p. 1298] for a relation between the ordinary and nearly ordinary cohomology. The nearly ordinary Hecke algebra is the $\mathcal{O}$-subalgebra of the endomorphisms of nearly ordinary cohomology generated by the Hecke operators. The control theorem for the nearly ordinary Hecke algebra is proved in [**Hid94**, Theorem 3.2]. The other main result we need, Theorem 3.2.4, is presented in all generality here; in particular our application is pertinent even without requiring class number one.

We denote as before $P_{k,k',\epsilon}$ the point on $\mathrm{Spec}(\mathcal{O}[[\mathcal{O}_p^\times]])$ corresponding to the arithmetic character

$$\chi_{k,k',\epsilon} : \mathcal{O}_p^\times \to \mathcal{O}^\times$$
$$x \mapsto \epsilon(x) x^k \overline{x}^{k'}$$

where ϵ is a finite order character of Γ_F the torsion-free part of $\mathcal{O}_p^\times$ of conductor p^r for some r and $k, k' \geq 0$. We also write $\chi_{k,k'}$ when ϵ is trivial. Complex conjugation acts on Γ_F which decomposes into eigenspaces $\Gamma_F^+ \times \Gamma_F^-$ and we may choose isomorphisms

$\Gamma_F^{\pm} \cong \mathbb{Z}_p$. Then up to connected components weight space $\mathcal{O}[[\mathcal{O}_p^{\times}]]$ is isomorphic to $\Lambda = \mathcal{O}[[\Gamma_F]] \cong \mathcal{O}[[X,Y]]$. The following is shown in Taylor [**Tay88**, Sections 4.4, 4.5]:

Proposition 3.2.2. *(1) The module $H^1_{\mathrm{ord}}(\Delta_1(p^\infty), V_{k,k'}(E/\mathcal{O}))$ is of cofinite type over Λ.*

(2) The natural map

$$H^1_{\mathrm{ord}}(\Delta_1(p^\infty), V_{k,k'}(E/\mathcal{O})) \to H^1_{\mathrm{ord}}(\Delta_1(p^\infty), E/\mathcal{O})[\chi_{k,k'}]$$

is a Hecke equivariant isomorphism, where the Λ-action on the right-hand side is through the unique extension of $\chi_{k,k'}$ to an endomorphism of Λ.

(3) The respective statements hold for the cuspidal subspaces as well.

Proof. The first statement follows from Nakayama's Lemma and the fact that at finite level the cohomology is finitely generated over $\mathcal{O}$. For the second statement, we use that

$$H^1_{\mathrm{ord}}(\Delta_1(p^\infty), V_{k,k'}(E/\mathcal{O}) = \varinjlim_r H^1_{\mathrm{ord}}(\Delta_1(p^r), V_{k,k'}(\mathcal{O}/\pi^r\mathcal{O}))$$

where π is a prime of $\mathcal{O}$ above p. For a fixed r, $H^1_{\mathrm{ord}}(\Gamma_1(Np^r), V_{k,k'}(\mathcal{O}/\pi^r\mathcal{O}))$ is a $\Lambda_r := \mathcal{O}/\pi^r\mathcal{O}[(\mathcal{O}/Np^r)^\times]$-module. We define a map:

$$j : V_{k,k'}(\mathcal{O}/\pi^r\mathcal{O}) \to \mathcal{O}/\pi^r\mathcal{O}$$

by projecting to the basis element of $V_{k,k'}$ on which the natural action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O})$ on $\mathrm{Sym}_k(\mathcal{O}^2) \otimes \overline{\mathrm{Sym}_{k'}}(\mathcal{O}^2)$ is multiplication by $d^k \bar{d}^{k'}$. It follows that j is a homomorphism of modules over the semi-group $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}) | c \equiv 0, d \equiv 1 \mod p^r\}$ where $\mathcal{O}/\pi^r\mathcal{O}$ is equipped with the trivial action. We thus get a Hecke-equivariant map:

$$j_* : H^q(\Gamma_1(Np^r), V_{k,k'}(\mathcal{O}/\pi^r\mathcal{O})) \to H^q(\Gamma_1(Np^r), \mathcal{O}/\pi^r\mathcal{O})$$

for all $q \geq 0$. We claim that j_* gives a map of Λ_r-modules provided the action on the right-hand side is twisted by the projection to Λ_r of $\chi_{k,k',\epsilon}$, which uniquely extends to Λ. It suffices to check this for $q = 0$ since we are working with derived functors. This reduces to checking for $x \in (\mathcal{O}/\pi^r\mathcal{O})^\times$ that j_* commutes with the action of x. For $f \in V_{k,k'}(\mathcal{O}/\pi^r\mathcal{O})^{\Gamma_1(Np^r)}$, we see x acts on the coefficient in the definition of j by multiplying by $x^k\bar{x}^{k'}$ and therefore as desired $j_*(x \cdot f) = x^k\bar{x}^{k'}(x \cdot j_*(f))$.

We now show that j_* is an isomorphism at level r on the ordinary parts and the result follows since everything is compatible under taking limits. Letting $g = \begin{pmatrix} p^r & 0 \\ 0 & 1 \end{pmatrix}$ we have

$$g \cdot V_{k,k'}(\mathcal{O}/\pi^r\mathcal{O}) \cong \mathcal{O}/\pi^r\mathcal{O} \cong g \cdot \mathcal{O}/\pi^r\mathcal{O}.$$

It follows that the middle vertical arrow in the commutative diagram is an isomorphism:

$$\begin{array}{ccccc}
H^1(\Gamma, V_{k,k'}(\mathcal{O}/\pi^r\mathcal{O})) & \xrightarrow{(ad(g^{-1}), g_*)} & H^1(\Gamma \cap g\Gamma g^{-1}, g \cdot V_{k,k'}(\mathcal{O}/\pi^r\mathcal{O})) & \xrightarrow{cor} & H^1(\Gamma, V_{k,k'}(\mathcal{O}/\pi^r\mathcal{O})) \\
\downarrow j_* & & \downarrow \cong & & \downarrow j_* \\
H^1(\Gamma, \mathcal{O}/\pi^r\mathcal{O}) & \xrightarrow{(ad(g^{-1}), g_*)} & H^1(\Gamma \cap g\Gamma g^{-1}, \mathcal{O}/\pi^r\mathcal{O}) & \xrightarrow[cor]{} & H^1(\Gamma, \mathcal{O}/\pi^r\mathcal{O})
\end{array}$$

where we write Γ for $\Gamma_1(Np^r)$. The composition of horizontal arrows is just the Hecke operator T_{p^r}. After projecting to the ordinary parts, T_{p^r} is an isomorphism. The left-hand side horizontal arrows are therefore injective and by commutativity of the left square so is $j_* : H^1_{ord}(\Gamma, V_{(k,k')}(\mathcal{O}/\pi^r\mathcal{O})) \to H^1_{ord}(\Gamma, \mathcal{O}/\pi^r\mathcal{O})$. Moreover, the corestriction maps must be epimorphisms and from commutativity of the right-hand square so is j_*. The analagous statements for cusp forms are deduced without too much difficulty. □

It follows that we again have a universal p-adic Hecke algebra by taking the limit

$$\mathbb{T}^{\mathrm{ord}} := \varprojlim_r \mathbb{T}(H^1_{c,\mathrm{ord}}(\Delta_1(p^r), E/\mathcal{O})),$$

where the algebras at finite level $\mathbb{T}(H^1_{c,\mathrm{ord}}(\Delta_1(p^r), V_{k,k'}(E/\mathcal{O}))$ are defined as the $\mathcal{O}$-subalgebras of $\mathrm{End}_{\mathcal{O}}(H^1_{c,\mathrm{ord}}(\Delta_1(p^r), V_{k,k'}(E/\mathcal{O}))$ generated by the Hecke operators. The Pontryagin dual Λ-module $M = \mathrm{Hom}_{\mathcal{O}}(H^1_{c,\mathrm{ord}}(\Delta_1(p^\infty), E/\mathcal{O}), E/\mathcal{O})$ and $\mathbb{T}^{\mathrm{ord}}$ have isomorphic quotients at arithmetic primes, and one obtains from Proposition 3.2.2 the following refinement for the universal ordinary Hecke algebra:

Corollary 3.2.3. *For any arithmetic point $P_{k,k',\epsilon} \in \mathrm{Spec}(\Lambda)$ we have that*

$$\mathbb{T}^{\mathrm{ord}}/P_{k,k',\epsilon}\mathbb{T}^{\mathrm{ord}} \cong \mathbb{T}(H^1_{c,\mathrm{ord}}(\Gamma_1(Np) \cap \Gamma_0(p^r), V_{k,k'}(E/\mathcal{O}) \otimes \epsilon)),$$

where $\ker(\epsilon) = \Gamma_F^r$. Moreover, there is again a single character of $(\mathbb{Z}/Np\mathbb{Z})^\times$ associated with each irreducible component of $\mathbb{T}^{\mathrm{ord}}$.

Thus a cuspidal automorphic form π ordinary at p lives in a Hida family $\mathcal{H}$ over weight space. Contrary to the case of modular forms, $\mathbb{T}^{\mathrm{ord}}$ cannot be flat over Λ, as we see that the specializations at $P_{k,k',\epsilon}$ for $k \neq k'$ are finite. The support S of $\mathcal{H}$ in Λ therefore has to have at least codimension one. Hida shows that S has exactly codimension one. For the reader's convenience, we give the detailed proof here, which has no assumption on the class number of F. Borrowing Hida's notations, for a congruence level Φ we denote by $Y(\Phi)$ the associated locally symmetric space. We denote $H^*_{c,ord}(Y(\Phi),\underline{M})$ the cohomology with compact support and coefficients in the locally constant sheaf associated to a Φ-module M, so that the boundary long exact sequence gives (after projecting to the ordinary part):

$$\ldots \longrightarrow H^i_{c,ord}(Y(\Phi),\underline{M}) \longrightarrow H^i_{ord}(Y(\Phi),\underline{M}) \longrightarrow H^*_{ord}(\partial Y(\Phi),\underline{M}) \longrightarrow \ldots$$

and denote by $H^1_{P,ord}(Y(\Phi),\underline{M})$ the image of $H^1_{c,ord}(Y(\Phi),\underline{M})$ in $H^1_{ord}(Y(\Phi),\underline{M})$.

Theorem 3.2.4 ([**Hid94**], Theorem 6.2.)**.** *The Pontryagin dual of the parabolic cohomology group $H^1_{P,ord}(Y(\Delta_1(p^\infty)),\underline{E/\mathcal{O}})$ is a Λ-torsion module of homological dimension one.*

Proof. The proof is a commutative algebra argument based on the following results: let $\mathcal{C}$ be the space of continuous functions on the column space $X = (\mathcal{O}_p^\times, p\mathcal{O}_p)^t$ with values in $E/\mathcal{O}$.

Writing $\Phi = \Delta_0(p^n)$ for some n, Φ acts on column vectors in X as does $\mathcal{O}_p$. We know from [**Hid93**, Proposition 2.1] that

$$H^1_{P,ord}(Y(\Delta_1(p^\infty)),\underline{E/\mathcal{O}}) \cong H^1_{P,ord}(Y(\Delta_0(p)),\underline{\mathcal{C}})$$

and we may work with the latter. We will use that:

(1) For any submodule M of $\mathcal{C}$ we have $H^0(Y(\Phi),\underline{M}) = 0$ and $H^3_{c,ord}(Y(\Phi),\underline{M}) = 0$.
(2) The boundary cohomology $H^0_{ord}(\partial Y(\Phi),\underline{\mathcal{C}})$ is a direct sum of a copy of the space of continuous functions $f : \mathcal{O}_p^\times \to E/\mathcal{O}$ for each equivalence class of cusps.

The first assertion follows from [**Hid88**, Lemma 9.2] and duality and the second is [**Hid93**, Corollary 3.14].

Let H denote the Pontryagin dual $H^1_{P,ord}(Y(\Phi), \underline{\mathcal{C}})^*$ and S the support of H in Λ. Let $\{T_1, T_2, T'\}$ denote a regular sequence in the maximal ideal of Λ and denote by $M[T]$ the submodule of M killed by $T \in \Lambda$.

First we claim that $H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}})$ is Λ-divisible, $H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1])$ is $\Lambda/T_1\Lambda$-divisible and $H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2])$ is $\Lambda/(T_1, T_2)\Lambda$-divisible. Checking for instance the second assertion amounts to showing multiplication by T_2 is surjective on $H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1])$. This follows from exactness of the sequence

$$0 \longrightarrow \mathcal{C}[T_1, T_2] \longrightarrow \mathcal{C}[T_1] \xrightarrow{T_2} \mathcal{C}[T_1] \longrightarrow 0$$

so that the long exact sequence yields

$$H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1]) \xrightarrow{T_2} H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1]) \longrightarrow H^3_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2]).$$

The claim follows since we know $H^3_{c,ord}(Y(\Phi), \underline{M}) = 0$ for any submodule M of $\mathcal{C}$.

We now show that, provided $(T_1, T_2) \notin S$,

$$H^1_{P,ord}(Y(\Phi), \underline{\mathcal{C}}) \otimes_\Lambda \Lambda/T'\Lambda \cong H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T']) \tag{3.2}$$

and

$$H^1_{P,ord}(Y(\Phi), \underline{\mathcal{C}})[T_1, T_2] \otimes_\Lambda \Lambda/T'\Lambda \cong H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2, T']). \tag{3.3}$$

For any $T \in \Lambda$, multiplication by T on $\underline{M}$ yields via the long exact sequence in cohomology:

$$0 \longrightarrow H^q_{c,ord}(Y(\Phi), \underline{M})/T \longrightarrow H^{q+1}_{c,ord}(Y(\Phi), \underline{M}[T]) \longrightarrow H^{q+1}_{c,ord}(Y(\Phi), \underline{M})[T] \longrightarrow 0$$

Taking $q = 1$, the last term vanishes by the divisibility result above if $\underline{M} = \underline{\mathcal{C}}$ or $\underline{\mathcal{C}}[T_1]$. Therefore we have isomorphisms

$$H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}}) \otimes_\Lambda \Lambda/T\Lambda \cong H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T]) \tag{3.4}$$

and

$$H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1]) \otimes_\Lambda \Lambda/T\Lambda \cong H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T]) \tag{3.5}$$

Moreover, it follows from 3.5 that the support of $H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2])^*$ is contained in the support of $H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1])^*$ and a fortiori in S. Therefore $H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2])^*$ is $\Lambda/(T_1, T_2)\Lambda$-torsion if $\{T_1, T_2\}$ is a regular sequence with $(T_1, T_2) \notin S$. However, it is also $\Lambda/(T_1, T_2)\Lambda$-torsion free since we showed the dual is divisible. This shows that $H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2]) = 0$ provided $(T_1, T_2) \notin S$.

Taking $\underline{M} = \underline{\mathcal{C}}[T_1, T_2]$ we get by virtue of $H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2]) = 0$ that

$$H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2]) \otimes_\Lambda \Lambda/T\Lambda \cong H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2, T]) \tag{3.6}$$

if $(T_1, T_2) \notin S$. On the other hand, taking $q = 0$ one has

$$H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2]) \cong H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}})[T_1, T_2],$$

so that we obtain

$$H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}})[T_1, T_2] \otimes_\Lambda \Lambda/T'\Lambda \cong H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2, T']). \tag{3.7}$$

It remains to compare $H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}})$ to the parabolic cohomology $H^1_{P,ord}(Y(\Phi), \underline{\mathcal{C}})$. From the characterization of $H^0_{ord}(\partial Y(\Phi), \underline{\mathcal{C}})$ it is an injective Λ-module and in the exact sequence

$$0 \longrightarrow H^0_{ord}(\partial Y(\Phi), \underline{\mathcal{C}}) \longrightarrow H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}}) \longrightarrow H^1_{P,ord}(Y(\Phi), \underline{\mathcal{C}}) \longrightarrow 0$$

we have a splitting of Λ-modules:

$$H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}}) \cong H^0_{ord}(\partial Y(\Phi), \underline{\mathcal{C}}) \oplus H^1_{P,ord}(Y(\Phi), \underline{\mathcal{C}}). \tag{3.8}$$

Moreover we deduce that

$$H^1_{c,ord}(Y(\Phi), \underline{\mathcal{C}}) \otimes_\Lambda \Lambda/T\Lambda \cong H^1_{P,ord}(Y(\Phi), \underline{\mathcal{C}}) \otimes_\Lambda \Lambda/T\Lambda.$$

Together with the isomorphisms 3.4 and 3.7, this proves isomorphisms 3.2 and 3.3.

We claim H has no pseudo-null (i.e. of codimension ≥ 2) submodules. If it had, there would exist $T_3 \in \Lambda$ so that $H[T_3] \neq 0$ and is $\Lambda/T_3\Lambda$-torsion. However, we have

$$H[T_3] \cong H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_3])^*$$

by taking duals in 3.2. The latter has $\Lambda/T_3\Lambda$-divisible Pontryagin dual, which is a contradiction. Therefore we may choose a regular sequence $\{T_1, T_2, T'\}$ so that $(T_1, T_2) \notin S$ and $T' \in \mathrm{Ann}_\Lambda(H)$. Taking duals, the $\Lambda/T_1\Lambda$-divisibility of $H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1])$ and the Λ-divisibility of $H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}})$ show that the sequence $\{T_1, T_2\}$ is regular for H. Moreover,

$$H/(T_1, T_2)H \cong (H/(T_1, T_2)H)[T'] \cong H^2_{c,ord}(Y(\Phi), \underline{\mathcal{C}}[T_1, T_2, T'])^*$$

from taking Pontryagin duals in 3.3 since $(T_1, T_2) \notin S$. We deduce that $H/(T_1, T_2)H$ is finite and therefore has homological dimension 3 over Λ. Since $\{T_1, T_2\}$ is regular this shows H has homological dimension one.

□

Corollary 3.2.5. *An irreducible component of the ordinary Hecke algebra* $\mathbb{T}^{\mathrm{ord}}$ *is killed by a non-zero element of* Λ *and is of homological dimension one over* Λ.

Proof. The ordinary p-adic Hecke algebra acts faithfully on

$$M = \mathrm{Hom}_{\mathbb{Z}_p}(H^1_{P,ord}(Y(\Delta_1(p^\infty)), \underline{E/\mathcal{O}}), E/\mathcal{O})$$

and can therefore be embedded into M. Moreover, M is of finite type as a module over the Hecke algebra. The result follows from Theorem 3.2.4. □

In fact the characteristic power series $\mathrm{Ch}_\Lambda(\mathbb{T}^{\mathrm{ord}})$ divides $\mathrm{Ch}_\Lambda(M)$.

3.2.4. Classical points on $\mathcal{H}$

Over $\mathbb{Q}$, we saw a Hida familiy contains a Zariski dense set of points corresponding to classical automorphic forms. One may ask if this is still true. Since the support S of the Hida family has codimension one, the question in this case amounts to:

QUESTION 1. When are there infinitely many points corresponding to classical automorphic forms on the support of the Hida family?

This question appears in the work of Calegari and Mazur ([**CM09**, Theorem 1.1. and Section 8.5]) and Taylor's thesis ([**Tay88**, Remark p.124]). One might expect that the family contains infinitely many classical automorphic forms if and only if it actually contains the whole diagonal of Spec(Λ), i.e. every specialization in parallel weight and parallel nebencharacter is classical. This can occur when the family arises via base change from $\mathbb{Q}$.

The proof of [**CM09**, Theorem 1.1] examines an explicit example and shows the support of $\mathcal{H}$ does not contain the diagonal. The claim is made that there must therefore only be finitely many classical points. The proof however fails to exclude the possibility that one might have infinitely many non-parallel nebentypi occuring on $\mathcal{H}$, whose Zariski-closure no longer has to contain the diagonal.

We are able to address this oversight by using the results of Chapter 2. As in the Manin-Mumford setting, we have a distinguished class of points: the points on Spec(Λ) that give rise to classical automorphic forms and must have parallel weight by 3.1. Denoting $\epsilon^{\pm}$ the restriction of ϵ to $\Gamma_F^{\pm}$ and choosing $(1+p)$ as a topological generator of each

$\Gamma_F^{\pm}$ we get classical points

$$P_{k,k,\epsilon} = ((X+1) - \epsilon^{+}(1+p)(1+p)^k, (Y+1) - \epsilon^{-}(1+p)(1+p)^k)$$

on Spec(Λ) or explicitly

$$X \mapsto (1+p)^k \zeta - 1$$

$$Y \mapsto (1+p)^k \zeta' - 1$$

for p-power roots of unity (ζ, ζ'). We may also normalize the weights so that $X = Y = 0$ corresponds to a given automorphic form π. We would like to show that the Zariski closure of an infinite set of such points has to be special, and that the only special subscheme of Spec(Λ) that can occur as the support of a Hida family is the diagonal. We prove:

Theorem 3.2.6. *Let $\mathcal{H}$ be the Hida family passing through an ordinary cuspidal automorphic form π, with the normalization that π corresponds to the point $(X, Y) = (0, 0)$ on weight space* Spec(Λ)*. Then the component of the support S of $\mathcal{H}$ in Λ passing through π either:*

(1) contains the diagonal $X = Y$ or

(2) contains only finitely many classical automorphic forms of weight different from π but infinitely many classical points of same weight and varying nebentypus or

(3) contains only finitely many classical points.

Moreover, when there are only finitely many classical automorphic forms on the family, the second condition can be excluded with a finite amount of computation. In Section 3.3 we work out an explicit example of this. We will deduce Theorem 3.2.6 from the following strengthening of Proposition 2.1.9:

Proposition 3.2.7. *Let $\phi \in \mathcal{O}[[X, Y]]$ be a power series so that*

$$\phi((1+p)^k\zeta - 1, (1+p)^k\zeta^n - 1) = 0$$

for an infinite set Σ of triples (k, n, ζ) in $\mathbb{Z}_p^2 \times \mu_p^\infty$. Then one of the following occurs:

(1) The power series ϕ vanishes along the diagonal.

(2) There are only finitely many weights k appearing in Σ and we may find fixed $N, K \in \mathbb{Z}_p$ and $\xi \in \mu_p^\infty$ such that for all $\zeta \in \mu_p^\infty$ we have:

$$\phi((1+p)^K\zeta - 1, (1+p)^K\xi\zeta^N - 1) = 0.$$

If, in addition to that, ϕ is irreducible and passes through the origin then $\phi(X, Y) = (X+1)^N - (Y+1)$ for some $N \in \mathbb{Z}_p$.

One may prove Proposition 3.2.7 as follows: assume that the weights appearing converge p-adically to zero, then $(1+p)^k\zeta - 1$ converges to $\zeta - 1$, and the statement in this case should follow from the "bounded away" statement in Theorem 2.1.6. More precisely, we have:

Lemma 3.2.8. *Let $\phi \in \mathcal{O}[[X,Y]]$ be a power series so that*

$$\phi((1+p)^{k_j}\zeta_j - 1, (1+p)^{k_j}\zeta_j^{n_j} - 1) = 0$$

for infinite sequences $\{\zeta_j\}$, $\{n_j\}$ and $\{k_j\}$ with $k_j \to 0$. Then ϕ is divisible (over $\mathcal{O}[\xi]$ for some fixed $\xi \in \mu_{p^\infty}$) by $\xi(X+1)^N - (Y+1)$. Moreover, if $k_j \neq 0$ infinitely often, ϕ vanishes along the diagonal.

Proof. Since $(1+p)^{k_j}\zeta_j - 1$ converges p-adically to $\zeta_j - 1$ as $j \to \infty$, by Theorem 2.1.6 there exists $N \in \mathbb{Z}_p$ so that $\phi(X, (X+1)^N(Y+1) - 1)$ vanishes at $(\zeta - 1, \xi - 1)$ for all $\zeta \in \mu_{p^\infty}$ and a fixed $\xi \in \mu_p^\infty$. That is,

$$\phi(\zeta - 1, \xi\zeta^N - 1) = 0$$

for all $\zeta \in \mu_{p^\infty}$. We conclude as in the proof of Proposition 2.1.9 that ϕ is divisible by

$$\prod_{\sigma \in \mathrm{Gal}(E[\xi]/E)} \xi(X+1)^N - (Y+1).$$

Moreover, any such irreducible factor does not vanish at $((1+p)^{k_j}\zeta_j - 1, (1+p)^{k_j}\zeta_j^{n_j} - 1)$ with $k_j \neq 0$ unless $N = \xi = 1$, which proves that in this case ϕ vanishes along the diagonal. $\square$

Proof of Proposition 3.2.7. Assume that Σ has infinitely many weights k so that we may extract a subsequence $\{k_j\}_{j \in \mathbb{N}}$ converging p-adically to K. We consider the power series

$$\Phi(X,Y) = \phi((1+p)^K(X+1) - 1, (1+p)^K(Y+1) - 1).$$

By Lemma 3.2.8, Φ and therefore ϕ must vanish along the diagonal.

If finitely many weights occur in Σ, then there is a single weight K so that setting again $\Phi(X,Y) = \phi((1+p)^K(X+1)-1,(1+p)^K(Y+1)-1)$ we have

$$\Phi(\zeta_j - 1, \zeta_j^{n_j} - 1) = 0$$

for infinite sequences $\{\zeta_j\}_{j\in\mathbb{N}}$ and $\{n_j\}_{j\in\mathbb{N}} \to N \in \mathbb{Z}_p$. It follows from Lemma 2.1.6 as in the proof of Proposition 2.1.9 that an irreducible factor of Φ vanishing at an infinite set $(\zeta_j - 1, \zeta_j^{n_j} - 1)$ must be divisible by

$$\prod_{\sigma\in\mathrm{Gal}(E[\xi]/E)} \xi(X+1)^N - (Y+1)$$

for some $\xi \in \mu_{p^\infty}$. Thus, as desired, for all $\zeta \in \mu_{p^\infty}$ we have the vanishing

$$\phi((1+p)^K\zeta - 1, (1+p)^K\xi\zeta^N - 1) = 0.$$

It also follows that a factor of the original ϕ accounting for an infinite set of such zeroes must be of the form

$$(1+p)^{-(N-1)K}\xi(X+1)^N - (Y+1).$$

However, this factor only vanishes at the origin if $\xi = 1$ and either $N = 1$ or $K = 0$, i.e. if it is of the form $(X+1)^N - (Y+1)$ for some $N \in \mathbb{Z}_p$, as claimed.

□

Proof of Theorem 3.2.6. Let D denote the component of the support S passing through the origin $(X,Y) = (0,0)$. Since D has codimension one, the ideal of definition of D in Λ is principal, generated by some $\phi \in \mathcal{O}[[X,Y]]$. If D contains infinitely many classical points, then after possibly swapping the roles of X and Y we have

$$\phi((1+p)^k\zeta - 1, (1+p)^k\zeta^n - 1) = 0$$

for an infinite set Σ of triples (k, n, ζ) in $\mathbb{Z}_p^2 \times \mu_p^\infty$. We now apply Proposition 3.2.7. $\square$

3.3. A worked out example

We conclude this Chapter by explicitly working out an example of how to apply Theorem 3.2.6. We will be working over the imaginary quadratic field $F = \mathbb{Q}(\sqrt{-2})$ and ring of integers $\mathcal{O}_F = \mathbb{Z}[\sqrt{-2}]$ of class number one. The prime 3 splits in $\mathcal{O}_F$ and we write $3 = \mathfrak{p}\overline{\mathfrak{p}}$ with $\mathfrak{p} = (1+\sqrt{-2})$. The first interesting level is at norm eleven and we set $N = (3-\sqrt{-2})$. There is a unique ordinary cuspform at level $\Gamma_0(3N)$. Via the modularity conjecture this is associated to the elliptic curve over F given by the equation

$$E : y^2 + (1+\sqrt{-2})xy + (1+\sqrt{-2})y = x^3 + (1-\sqrt{-2})x^2 - 2\sqrt{-2}x - \sqrt{-2}$$

and of conductor $9 + 3\sqrt{-2}$. Note that for a given example this can be verified by computing finitely many Hecke eigenvalues ([**DGP10**]). We show the following:

Proposition 3.3.1. *The* 3*-adic Hida family passing through the unique cuspform at level* $\Gamma_0(3N) = \Gamma_0(9-3\sqrt{-2})$ *contains only finitely many points corresponding to classical automorphic forms.*

Using Theorem 3.2.6, we only need to understand what happens at finitely many levels to prove the result. We will be working with subgroups of $\mathrm{PGL}_2(\mathcal{O}_F)$ of the appropriate level. In weight $(2,2)$, we computed the dimensions of spaces of forms of level dividing

$\Gamma_0(N) \cap \Gamma_1(9)$, which contribute to the rank of $H^1(\Gamma, \mathbb{Z}) = \Gamma^{ab}$ for $\Gamma < \Gamma_0(N) \cap \Gamma_1(9)$. We also denote by $\Gamma_H(\nu^n) < \Gamma_0(\nu^n)$ the normal subgroup of index 3^{n-1} for primes $\nu|3$, which are more convenient to work with than Γ_1, since we don't have to deal with boundary cohomology. Using the presentation (see e.g. [**FGT10**])

$$\mathrm{PGL}_2(\mathcal{O}_F) = < T, S, U, V | S^2, (TS)^3, [T, U], SU^{-1}SU, V^2, VAVA, VSVS, VUVU >,$$

where

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, U = \begin{pmatrix} 1 & \sqrt{-2} \\ 0 & 1 \end{pmatrix}, \text{ and } V = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

a MAGMA computation yields the dimensions in Table 3.1. Not all of these levels have ordinary newforms. One has the following criterion:

Lemma 3.3.2. *If the cohomology class corresponding to our ordinary automorphic form occurs first at level $\Gamma_H(\mathfrak{p}^n)$ for some $\Gamma_0(\mathfrak{p}^n) \subseteq \Gamma_H(\mathfrak{p}^n) \subseteq \Gamma_1(\mathfrak{p}^n)$ and $n > 0$, then either:*

(1) The map $\Gamma_H(\mathfrak{p}^n) \to (\mathbb{Z}/3^n\mathbb{Z})^\times$ whose kernel is $\Gamma_1(\mathfrak{p}^n)$ has non-trivial image of order divisible by 3^{n-1}.

(2) The weight $k = 2$, and $\Gamma_H(\mathfrak{p}^n) = \Gamma_0(\mathfrak{p})$.

Proof. This follows from examining the local factor of the associated cuspidal automorphic representation $\Pi = \otimes_\nu \pi_\nu$ at $\mathfrak{p}$: for ordinary forms either the conductor of $\pi_\mathfrak{p}$ exactly divides the conductor of the Nebentypus or $\pi_\mathfrak{p}$ is an unramified twist of the Steinberg representation in weight two (see e.g., [**CDT99**, Section 4]). □

Table 3.1. Dimensions for $N = (3 - \sqrt{-2})$ and $\mathfrak{p} = (1 + \sqrt{-2})$.

group Γ	$h(\Gamma)$	$h(\Gamma)^{new}$
$\Gamma_0(\mathfrak{p})$	0	0
$\Gamma_0(\overline{\mathfrak{p}})$	0	0
$\Gamma_0(\mathfrak{p}^2)$	1	1
$\Gamma_0(\overline{\mathfrak{p}}^2)$	1	1
$\Gamma_0(3)$	0	0
$\Gamma_0(N)$	0	0
$\Gamma_0(N\mathfrak{p})$	0	0
$\Gamma_0(N\overline{\mathfrak{p}})$	0	0
$\Gamma_0(\overline{\mathfrak{p}}\mathfrak{p}^2)$	2	0
$\Gamma_0(\overline{\mathfrak{p}}^2\mathfrak{p})$	2	0
$\Gamma_0(9)$	6	0
$\Gamma_0(N\mathfrak{p}^2)$	2	0
$\Gamma_0(N\overline{\mathfrak{p}}^2)$	2	0
$\Gamma_0(3N)$	1	1
$\Gamma_0(N\overline{\mathfrak{p}}\mathfrak{p}^2)$	7	1
$\Gamma_0(N\overline{\mathfrak{p}}^2\mathfrak{p})$	7	1
$\Gamma_0(9N)$	23	3
$\Gamma_H(\mathfrak{p}^2)$	1	0
$\Gamma_H(\overline{\mathfrak{p}}^2)$	1	0
$\Gamma_H(\overline{\mathfrak{p}}\mathfrak{p}^2)$	2	0
$\Gamma_H(\overline{\mathfrak{p}}^2\mathfrak{p})$	2	0
$\Gamma_H(\overline{\mathfrak{p}}^2) \cap \Gamma_0(\mathfrak{p}^2)$	10	4
$\Gamma_H(\mathfrak{p}^2) \cap \Gamma_0(\overline{\mathfrak{p}}^2)$	10	4
$\Gamma_H(9)$	14	0
$\Gamma_0(N) \cap \Gamma_H(\mathfrak{p}^2)$	2	0
$\Gamma_0(N) \cap \Gamma_H(\overline{\mathfrak{p}}^2)$	2	0
$\Gamma_0(N) \cap \Gamma_H(\overline{\mathfrak{p}}\mathfrak{p}^2)$	7	0
$\Gamma_0(N) \cap \Gamma_H(\overline{\mathfrak{p}}^2\mathfrak{p})$	7	0
$\Gamma_0(N) \cap \Gamma_H(\mathfrak{p}^2) \cap \Gamma_0(\overline{\mathfrak{p}}^2)$	33	2
$\Gamma_0(N) \cap \Gamma_H(\overline{\mathfrak{p}}^2) \cap \Gamma_0(\mathfrak{p}^2)$	31	0
$\Gamma_0(N) \cap \Gamma_H(9)$	41	0

We now turn to:

Proof of Proposition 3.3.1. Let $\mathcal{H}$ be the Hida family of tame level N passing through the unique level $\Gamma_0(3N)$ cuspidal newform π; in particular this component of the Hecke algebra has trivial character on $(\mathbb{Z}_p/N\mathbb{Z}_p)^\times$. Let S denote the support of $\mathcal{H}$ in $\Lambda = \mathbb{Z}_p[[X, Y]]$, normalized so that π corresponds to the specialization at the point $(0, 0)$. First we observe that S cannot contain the diagonal. If this were the case, the specialization at every parallel weight (k, k) for $k \geq 2$ should be an ordinary cusp form. In particular, it would follow from Corollary 3.2.3 that the specialization of $\mathcal{H}$ in weight $(4, 4)$ should be a cusp form of level $\Gamma_0(N)$. However, computations show there is no such form-see for instance [**CM09**, Lemma 8.1].

Assuming there are infinitely many arithmetic points on S, there exists $\phi \in \Lambda$ an element of the characteristic ideal of $\mathcal{H}$ vanishing at infinitely many arithmetic points in S and passing through the origin, and we may as well assume ϕ is absolutely irreducible. By Theorem 3.2.6, it must be that the specializations at arithmetic points in weight $(2, 2)$ and varying nebentypi contain infinitely many ordinary cusp forms. In fact, one has the stronger result by Proposition 3.2.7 that $\phi(\zeta - 1, \zeta^N - 1) = 0$ for all $\zeta \in \mu_{p^\infty}$ and some $N \in \mathbb{Z}_p$, after possibly switching the roles of X and Y. In particular, there exist $(\xi_1, \xi_2) \in \mu_p$ not both trivial (although one of them might be if $p|N$) so that $\phi(\xi_1 - 1, \xi_2 - 1) = 0$ and thus S contains an arithmetic point $P_{2,2,\epsilon} = ((X + 1) - \xi_1, (Y + 1) - \xi_2)$. The specialization at $P_{2,2,\epsilon}$ therefore exhibits an ordinary newform at some level $\Gamma < \Gamma_0(N) \cap \Gamma_1(9)$. Investigating Table 3.1, the only such candidate levels are $\Gamma_0(N\overline{\mathfrak{p}}\mathfrak{p}^2)$, $\Gamma_0(N\overline{\mathfrak{p}}^2\mathfrak{p})$, $\Gamma_H(\overline{\mathfrak{p}}^2) \cap \Gamma_0(\mathfrak{p}^2)$, $\Gamma_H(\mathfrak{p}^2) \cap \Gamma_0(\overline{\mathfrak{p}}^2)$, $\Gamma_0(9N)$ and $\Gamma_0(N) \cap \Gamma_H(\mathfrak{p}^2) \cap \Gamma_0(\overline{\mathfrak{p}}^2)$. All of these levels have $\Gamma_0(\mathfrak{p}^2)$ or $\Gamma_0(\overline{\mathfrak{p}}^2)$-structure, and therefore by Lemma 3.3.2 cannot be ordinary at $\mathfrak{p}$ and $\overline{\mathfrak{p}}$, a contradiction.

□

CHAPTER 4

Strengthenings beyond multiplicative Manin-Mumford

We have so far considered p-adic infinitesimal strengthenings of the Manin-Mumford Conjecture in the multiplicative case, when the ambient group variety is a product of multiplicative groups. However the classical Manin-Mumford Conjecture proven by Raynaud [**Ray83**] concerns the intersection of the image $i : C \to A$ of an algebraic curve of genus $g \geq 2$ with torsion points on an abelian variety A, which is the setting we consider here.

4.1. Infinitesimal p-adic statements for abelian varieties

One may ask if similar p-adic phenomena occur as in Chapter 2 occur for abelian varieties, namely, provided our objects are defined over p-adic rings:

(1) Is the image of C bounded away p-adically from all but finitely many torsion points provided it is not the translate of a subgroup?
(2) Can these results again be extended to curves that aren't algebraic, but are p-adic analytic objects, subject to some restrictions?

With reagards to the first question, Scanlon ([**Sca98, Sca99**]) proves the following:

Theorem 4.1.1. *Let A be an abelian variety defined over a finite extension $K/\mathbb{Q}_p$. Let $X \subseteq A$ be a subvariety defined over $\mathbb{C}_p$. Then there is a constant $C > 0$ such that for any torsion point $\zeta \in A$, either $\zeta \in X$ or $d_p(\zeta, X) \geq C$.*

Here on any affine cover $\mathcal{U} = \bigcup_{i=1}^{k} U_i$ the distance is defined so that $d_p(\zeta, X) \geq C$ if there exists a section of the ideal sheaf $f \in \mathcal{I}_X(U_i)$ with $|f(\zeta)|_p \geq C$ where $\zeta \in U_i$. The distance is, up to a bounded constant, independent of the cover. This, together with Manin-Mumford, answers the first question in the affirmative for algebraic curves.

There again is a natural p-adic infinitesimal framework in this setting which we adopt: an abelian variety A defined over a p-adic field K with good reduction at p has a smooth proper model $\mathcal{A}$ over $\mathcal{O}_K$. The generic fibre of A as a rigid analytic space has a natural reduction map

$$r : A^{an}(\mathbb{C}_p) \to \tilde{A}(\overline{\mathbb{F}}_p)$$

and the pre-image of any point is isomorphic to a unit ball $\mathbb{B}^n(\mathbb{C}_p)$, where n is the dimension of A. In particular, the kernel of reduction is isomorphic to the $\mathfrak{m}_{\mathbb{C}_p}^n$-points of the completion along the zero section of the abelian scheme $\mathcal{A}$, which is its formal Lie group. As the multiplication-by-l map is étale on $\mathcal{A}$ for any prime $l \neq p$, the torsion points of A in this residue disk are precisely those of p-power order. Thus a p-adic infinitesimal result for abelian varieties would address the following question:

Question 2. Let $\mathcal{F}_A[p^\infty] \subset \mathbb{B}^n(\mathbb{C}_p)$ denote the torsion points of the formal Lie group $\mathcal{F}_A$ of an n-dimensional abelian variety and $\mathcal{C} \subset \mathbb{B}^n(\mathbb{C}_p)$ the points on an embedded curve. When is $\mathcal{C}$ bounded away from all but finitely many points of $\mathcal{F}_A[p^\infty]$?

One has to specify what exactly is meant by a curve in this context. When $\mathcal{C}$ comes from an algebraic curve, this follows from the Manin-Mumford Conjecture and Theorem 4.1.1. Moreover, in this algebraic case we show in Section 4.2 that under some assumptions $\mathcal{C}$ avoids all non-trivial torsion points.

When the formal group is $\widehat{\mathbb{G}}_m^n$, our results in Chapter 2 answer this question for a larger class of "curves" $\mathcal{C} = \mathrm{Spec}(\mathcal{O}_F[[X_1, \ldots, X_n]]/I)$ for I of codimension two which aren't of algebraic origin (see e.g. Theorem 2.1.3), and similarly replacing $\widehat{\mathbb{G}}_m$ by a Lubin-Tate formal group. To our knowledge, for abelian varieties no such infinitesimal strengthening of Manin-Mumford is known.

More generally, let $\mathcal{O}$ denote the valuation ring of a finite extension of $\mathbb{Q}_p$. We observe that the results of Chapter 2 and Question 2 are all instances of the following p-adic infinitesimal question:

Question 3. Let $\mathcal{S}^n \subset \mathbb{B}^n(\mathbb{C}_p)$ be the torsion points of an n-dimensional connected formal Lie group $\mathcal{F}$ over $\mathcal{O}$. Is there a class of special subschemes of $\mathrm{Spec}(\mathcal{O}[[X_1, \ldots, X_n]])$ such that an irreducible component of the Zariski closure of a set of special points is a special subscheme?

We hope in future work to answer this question when the formal group $\mathcal{F}^n$ is not a copy of n one-dimensional groups.

4.2. Explicit bounds

In light of the Manin-Mumford Conjecture, one may ask if there are explicit bounds for the number of torsion points of the abelian variety lying on an embedded algebraic curve. Using his theory of p-adic integration, Coleman in [**Col85**] and [**Col87**] was able to show this in many instances. He achieves this by defining locally analytic p-adic integrals on C and A. On the abelian variety, antiderivatives of invariant differentials are logarithms, vanishing at all torsion points. Pulling back to C and using the locally analytic description of the integrals, the argument proceeds by examining their Newton polygons inside the residue disks. We show how these methods can, for a single residue disk which we will take to be the kernel of reduction, be used to often show that the curve is bounded away from all non-trivial p^∞-torsion. More precisely, the main result of this section, Theorem 4.2.6, implies the following:

let K denote a complete subfield of $\widehat{\mathbb{Q}}_p^{nr}$, R the ring of integers of K and let C be a smooth, connected one-dimensional proper scheme over R of genus $g > 1$ as well as A an abelian scheme over R.

Theorem 4.2.1. *Assume that $O \in C(K)$ is a rational point and let $i : C \to A := \mathrm{Jac}(C)$ be the Albanese morphism over R such that on generic fibers $i(O) = e$ is the origin of A. Assume furthermore:*

(1) the abelian scheme A is either superspecial or ordinary,

(2) the prime $p > 3$.

Then inside the residue disc at the identity $e \in A(R)$, there is no point on $i(U)$, the image of the residue disk of C at O, within distance $1/p$ to a non-trivial torsion point.

Let us first specify what exactly is meant by distance here. The kernel of reduction on A has the natural structure of a rigid space after base change to $\mathbb{C}_p$ and is conformal to an $n = \dim A$-dimensional open unit ball $B^n(\mathfrak{m}_{\mathbb{C}_p})$. We have n coordinate functions and for $Q = (x_1, \ldots, x_n)$ we set

$$v(Q) := \min_{1 \le i \le n} v_p(x_i),$$

where v_p is a normalized valuation. Denote by $d(P, Q)$ the induced distance function on $B^n(\mathfrak{m}_{\mathbb{C}_p})$, in particular points on the ball are close if all of their coordinates are.

Lemma 4.2.2. *This is a well-defined distance, namely it is invariant under change of coordinates* $R[[T_1, \ldots, T_n]] \cong R[[X_1, \ldots, X_n]]$ *on completed local rings.*

Proof. Indeed, writing for $a_j \in R$

$$x_i = \sum_{j=1}^{n} a_j t_j + \text{ higher order terms}$$

we get that

$$\begin{aligned} v_p(x_i) &\geq \min(v_p(\sum_{j=1}^{n} a_j t_j), \text{ higher order terms }) \\ &\geq \min_{1 \leq j \leq n} (v_p(t_j)) \\ &\geq v(t_1, \ldots, t_n). \end{aligned}$$

It follows that $v(x_1, \ldots, x_n) \geq v(t_1, \ldots, t_n)$ but since the change of coordinates is invertible, the inverse inequality holds as well and $v(x_1, \ldots, x_n)$ is independent of the choice of coordinates. □

We sketch a proof of the Theorem. The torsion points on the residue disk of $e \in A$ are exactly the p-power torsion points of the associated g-dimensional formal Lie group. Hence a torsion point P naturally sits inside an open unit ball $\mathfrak{m}_{\mathbb{C}_p}^g$ and there's a lower bound on $d(P, e)$ depending on the order of P. On the other hand, torsion points are roots of the p-adic abelian integrals defined on A. Now assume that a point $Q \in C$ is close enough to P. Our assumptions force Q to be ramified. Moreover, it follows from local analyticity of the integrals on C that the p-adic abelian integrals at Q vanish modulo p. The Newton polygon arguments for p-adic abelian integrals on C vanishing at ramified points of [**Col85**] and [**Col87**] then still essentially go through under this weaker assumption, proving a vanishing result (dependent on p) for certain differentials on the special fiber of C. We conclude using the geometry of C as in Coleman's original paper since this contradicts Riemann-Roch, provided p is large enough.

4.2.1. Proximity to torsion points

We derive some consequences from when a torsion point is close to the image of a curve $i : C \to A$ with regards to the p-adic distance on the residue disk of A at $\mathcal{O}$. The group multiplication $m : A \times A \to A$ gives rise to a map of completed local rings

$$\hat{\mathcal{O}}_{A,e} \to \hat{\mathcal{O}}_{A,e} \hat{\otimes} \hat{\mathcal{O}}_{A,e}.$$

The resulting map

$$f_A : R[[T_1, \ldots, T_n]] \to R[[T_1, \ldots, T_n]] \hat{\otimes} R[[T_1, \ldots, T_n]]$$

defines an n-dimensional formal group law $\mathcal{F}_A$ whose $\mathfrak{m}_{\mathbb{C}_p}$-points are exactly the residue disk in question. In what follows, let U denote the residue disk at O of C where $i(O) = e$. We will also sometimes use $i : U \to B^n(\mathbb{C}_p)$ for the induced map on residue disks at O.

Lemma 4.2.3. *Assume the morphism $i : C \to A$ is unramified modulo p. Let T denote a uniformizer at O of C. Then if there is a point $Q \in U$ and a non-trivial p^k-torsion* *point $P \in B^n(\mathbb{C}_p)$ of A such that $d(i(Q), P) < 1/p$, we have that*

$$v_p(T(Q)) \leq \frac{1}{p^{k-1}(p-1)}.$$

Proof. Observe that $i : C \to A$ induces a map on completed local rings

$$\hat{i} : \hat{\mathcal{O}}_{A,e} \cong R[[T_1, \ldots, T_n]] \to R[[T]] \cong \hat{\mathcal{O}}_{C,O}$$

which is explicitly given by n power series $h_j(T) := \hat{i}(T_j)$, where $1 \leq j \leq n$. Since i is unramified modulo p, writing $h_j(T) = u_j T + \ldots$ there exists at least one j with $u_j \in R^\times$. But then after possibly a linear reparametrization

$$T_m \mapsto \begin{cases} T_m + T_j & \text{if } u_m \in pR \\ T_m & \text{if } u_m \in R^\times. \end{cases}$$

for $1 \leq m \leq n$ we may assume u_m is always a unit. By Lemma 4.2.2, we may without loss of generality fix such a choice of isomorphism $\hat{\mathcal{O}}_{A,e} \cong R[[T_1, \ldots, T_n]]$. We claim that for P a torsion point of order p^k,

$$v_p(T_m(P)) \leq \frac{1}{p^{k-1}(p-1)}$$

for some $1 \leq m \leq n$. Since $d(i(Q), P) < 1/p$, it follows that the same holds for $v_p(T_m(i(Q))$ and we conclude since

$$v_p(T(Q)) = v_p(h_m(i(Q)) = v_p(T_m(i(Q))$$

where the last equality is because u_m is a unit.

We now establish the claim on $v_p(T_m(P))$. Observe that P is a p^k-torsion point of the n-dimensional formal group $\mathcal{F}_A$ whose multiplication-by-p endomorphism can be written as

$$[p]_{\mathcal{F}}(X) = pA(X) + B(X^p)$$

for power series $A(X) = X + \ldots$ and $B(0) = 0$ where for simplicity of notation we write X for $(X_1, \ldots, X_n)$ and so on. The proof proceeds by induction on k.

Base case: Let $P = (x_1, \ldots, x_n)$ be a p-torsion point of $\mathcal{F}_A$. Say the valuation of the first coordinate is the smallest. Then on the first coordinate:

$$[p]_{\mathcal{F}}(P)_1 = px_1 + p(O(x^2)) + B_1(x^p).$$

Since this has to equal zero, considering valuations yields

$$v_p(px_1) \geq v_p(x_1^p)$$

and thus $(p-1)v_p(x_1) \leq 1$, as desired.

Induction step: Suppose now P has exact order p^{k+1}. We may assume that the first coordinate of $[p]_{\mathcal{F}}(P)$ has lowest valuation. Then we must by induction have

$$v_p(px_1 + p(O(x^2)) + B_1(x^p)) \leq \frac{1}{p^{k-1}(p-1)}$$

and therefore $v_p(x_1^p) \leq \frac{1}{p^{k-1}(p-1)}$, which concludes the proof.

$\square$

Lemma 4.2.4. *Assume that there is a point $Q \in U$ such that $d(i(Q), P) < 1/p$, where $P \in B^n(\mathfrak{m}_{\mathbb{C}_p})$ is a torsion point of A. Then for any differential $\omega \in H^0(A, \Omega^1_{A/R})$ we have that*

$$\int_O^Q i^*\omega \equiv 0 \mod p,$$

where $\int_O^Q i^*\omega = \int_e^{i(Q)} \omega$ are the p-adic abelian integrals defined in [**Col85**].

Proof. This follows from the theory of p-adic integration in [**Col85**, Chapter 2]: We know that for any differential $\omega \in H^0(A, \Omega^1_{A/R})$ the integral $\lambda_\omega(P) := \int_e^P \omega$ vanishes since P is torsion and $\lambda_\omega : A(\mathbb{C}_p) \to \mathbb{C}_p$ is a group homomorphism; see [**Col85**, Theorems 2.8 and 2.11]. By [**Col85**, Theorem 2.8], the logarithms λ_ω are locally analytic and $d\lambda_\omega = \omega$. We may therefore choose local parameters $(T_1, \ldots, T_n)$ so that

$$\lambda_\omega(i(Q)) = \int_0^{(T_1(i(Q)),\ldots,T_n(i(Q)))} \omega(T_1, \ldots, T_n) dT_1 \ldots dT_n = f_w(T_1(i(Q)), \ldots, T_n(i(Q)))$$

for a convergent power series f_ω with $\frac{\partial f_\omega}{\partial T_i}(T_1, \ldots, T_n) \in R[[T_1, \ldots, T_n]]$.
Since the coefficients of f_ω are almost integral, it follows easily provided $p > 2$ that

$$d(i(Q), P) < 1/p \Rightarrow |\lambda_\omega(P) - \lambda_\omega(i(Q))|_p < 1/p$$

for any $\omega \in H^0(A, \Omega^1_{A/R})$. Thus if $d(i(Q), P) < 1/p$, by [**Col85**, Theorem 2.7] we see that

$$\int_O^Q i^*\omega = \int_e^{i(Q)} \omega \equiv 0 \mod p.$$

□

4.2.2. Vanishing modulo p of Coleman integrals on the curve

We now revisit the arguments of [**Col85**] and [**Col87**] under the weaker assumption that there exists a ramified point Q in the residue disk at O of C such that

$$\int_O^Q \omega \equiv 0 \mod p$$

for all $\omega \in i^*H^0(A, \Omega^1_{A/R})$. We will also adopt the same notations as Coleman in what follows. In particular, let $R^{nr} = W(\overline{\mathbb{F}}_p)$ denote the ring of integers of $\widehat{\mathbb{Q}_p}^{nr}$ and $\mathbb{F}$ the residue field of R. Finally let U denote the residue class of C as well as T a uniformizer at O and we use tildas for reductions modulo p of various R-modules or their elements. We define R-modules

$$H := H^1_{dR}(C/R) = \mathbb{H}^1(C, \Omega^\bullet_{C/R}),$$

the hypercohomology of the de Rham complex and

$$W = i^*H^0(A, \Omega^1_{A/R}) \subseteq H^0(C, \Omega^1_{C/R}),$$

which we may view as a submodule of H via the spectral sequence associated to hypercohomology. Note that $H \cong H^1_{cris}(\tilde{C}/R)$ and the lifts of Frobenius and Verschiebung define endomorphisms F and V of H by functoriality, making H into an F-crystal together with a submodule W.

Definition. We say that the F-crystal (H, W) is *ordinary* if $VW \equiv W \mod pH$ and *superspecial* if $FH = W + pH$, otherwise we say H is *extraordinary.*

Remark. In the case where $i : C \to A = \mathrm{Jac}(C)$ is an Albanese map, this coincides with the usual terminology for abelian varieties based on the rank of the p-torsion of the special fiber $\tilde{A}[p](\overline{\mathbb{F}})$. In this case, $W = H^0(C, \Omega^1_{C/R})$ is a free R-module of rank g. From applying F to $VH = W + pH$ we obtain that

$$VW \equiv 0 \mod pH$$

in the superspecial case. We have a Hodge filtration

$$0 \longrightarrow W \longrightarrow H \longrightarrow H^1(A, \mathcal{O}_{A/R}) \longrightarrow 0$$

lifting the filtration over $\mathbb{F}$

$$0 \longrightarrow H^{1,0}(\tilde{A}/\mathbb{F}) \longrightarrow H^1_{dR}(\tilde{A}/\mathbb{F}) \longrightarrow H^1(\tilde{A}, \mathcal{O}_{\tilde{A}/\mathbb{F}}) \longrightarrow 0$$

and the free R-module $H = H^1_{cris}(\tilde{A}/R)$ of rank $2g$ is the Dieudonné module of the p-divisible group of A ([**MM74**, Theorems 1 and 2]). It follows that $\tilde{H} = H/pH$ as a module over

$$\mathbb{D} = \mathbb{F}[F, V]/(FV = VF = 0, F\lambda = \lambda^p F, V\lambda^p = \lambda)$$

classifies the group scheme $\tilde{A}[p]$. Then A is ordinary if and only if the (covariant) Dieudonné module is isomorphic to

$$(\mathbb{D}/(F, V-1) \bigoplus \mathbb{D}/(V, 1-F))^g$$

and since F kills the tangent space modulo p, the g-dimensional piece $\mathbb{D}/(F, V-1)$ corresponds to having $VW \equiv W \mod pH$.

If A is superspecial, the Dieudonné module is isomorphic to

$$(\mathbb{D}/(F+V))^g$$

and again since $F+V=V$ on the tangent space modulo p this corresponds to $VW \equiv 0 \mod pH$.

The goal of this section is to prove the following:

Proposition 4.2.5. *Assume (H,W) is ordinary or superspecial and U is not a base point of $\tilde{W}$. If there is a point $Q \in U$ with $v_p(T(Q)) \leq \frac{1}{p-1}$ such that*

$$\int_O^Q \omega \equiv 0 \mod p$$

for all $\omega \in W$, then every differential in $\tilde{W}$ which vanishes at $\tilde{Q}$ vanishes at least $p-2$ times.

Before proving this, we show how to deduce the main theorem:

Theorem 4.2.6. *Let $i : C \to A$ be a morphism over R. Assume furthermore that:*

(1) there is a K-rational point $O \in C(K)$ such that $i(O)$ is the origin e of A

(2) the F-crystal (H,W) is either superspecial or ordinary.

(3) either i is an Albanese map and $p>3$ or $p>2g$.

Finally assume that the morphism $\tilde{C} \to i(\tilde{C})$ is unramified, where $\tilde{C}$ denotes the special fibre. Then inside the residue disc at e of A, there is no point on $i(U)$ within distance $1/p$ to non-trivial torsion points.

Note that Theorem 4.2.1 is a special case of the above as under the assumptions of Theorem 4.2.1 the morphism i is a closed immersion.

Proof of Theorem 4.2.6. Assume there is a ramified torsion point P and a point $i(Q)$ inside of the residue disk at e such that $d(i(Q), P) \leq 1/p$. Then by Lemmas 4.2.3 and 4.2.4 for suitable choices of local coordinates $v_p(T(Q)) < 1/(p-1)$ and

$$L_\omega(T(Q)) := \int_O^Q \omega \equiv 0 \mod p$$

for all $\omega \in i^* H^0(A, \Omega^1_{A/R})$. By proposition 4.2.5, every differential in $\tilde{W}$ which vanishes at $\tilde{Q}$ vanishes at least $p-2$ times. Note that we may use this result since U is not a base point of $\tilde{W}$ as we are pulling back invariant differentials on A via a morphsim that is unramified on the special fiber. We now conclude as in [**Col85**, Theorem 5.5]: Since U is not a base point of $\tilde{W}$ at lest one differential is non-vanishing at U and thus there are $\dim_{\mathbb{F}} \tilde{W} - 1$ linearly independent differentials vanishing at U and therefore vanishing $p-2$ times. Thus there is a differential vanish $p - 2 + \dim_{\mathbb{F}} \tilde{W} - 2$ times at U. But by Riemann-Roch, this quantity is bounded by $2g-2$. It follows that

$$p \leq 2g + 2 - \dim_{\mathbb{F}} \tilde{W} \leq 2g$$

Moreover if i is an Albanese map, we actually have $\tilde{W} = H^0(\tilde{C}, \Omega^1_{\tilde{C}/\mathbb{F}})$. Since $g = \dim \tilde{W} > 1$ there is a differential vanishing at U which therefore has to vanish $p-2$ times. Then there is by Riemann-Roch a function with a pole of order k at U for $2 \leq k \leq p-2$ and no other poles. Since in this case we assumed $p \geq 5$ we may produce functions with poles of coprime orders 2 and 3 at U from which one can construct functions with poles of arbitrarily large order at U. This again contradicts Riemann-Roch. $\square$

Remark. Let us explain the ordinary or superspecial assumption in Theorem 4.2.6. If H is ordinary, then $\tilde{V}$ is bijective on $\tilde{W}$. On the other hand, if H is superspecial, then a small calculation (see [**Col87**, Lemma 2]) shows $\frac{\tilde{V^2}}{p}$ is bijective on $\tilde{W}$ and this fact will be used to prove the vanishing results that follow. On the other hand, Coleman proves analogous results for ramified torsion points without that assumption (see [**Col87**, Proposition 13]) and it should follow that at least one differential in $\tilde{W}$ has to vanish $p-1$ times at U and thus a similar result as in Theorem 4.2.6 should hold if $p > 2g-1$.

Following [**Col87**], let $\mathcal{C}$ denote $\tilde{V}$ or $\frac{\tilde{V^2}}{p}$ according to whether H is ordinary or superspecial and let $q = p$, $\phi = \sigma$, $t = 1$ or $q = p^2$, $\phi = \sigma^2$ and $t = 2$, where σ is absolute Frobenius, according to whether we are in the ordinary or superspecial case; we use superscripts to denote the action of ϕ. We first examine the shape of the Newton polygons on U of the locally analytic integrals $L_\omega(T)$ for $\omega \in W$. Let $M = \min_{\omega \in W} v_p(L_\omega(0))$. Coleman shows, assuming that $M \neq \infty$ and U is not a base point of $\tilde{W}$:

Proposition 4.2.7. *The lower convex hull of the set of Newton polygons of $\{L_\omega(T) | \omega \in W\}$ has vertices*

$$(0, M) \cup \{(q^i, -i) : i \geq \max\{0, 1-M\}\},$$

Moreover, we may write:

$$L_\omega(T) = L_\omega(0) + \sum_{i=0}^{\infty} \frac{g_{\omega,i}^{\phi^i}(T^{q^i})}{p^i}$$

for power series $g_{\omega,i}(T) \in R^{nr}[[T]]$ *that encode the action of* $\mathcal{C}$ *on* W *modulo* p*:*

$$dg_{\omega,i} \equiv \mathcal{C}^i\tilde{\omega} \mod (p, T^{q-1}dT). \tag{4.1}$$

Proof. The shape of the Newton polygons stems from the action of Frobenius and Verschiebung on differentials. We first recall some general considerations from [**Col87**] and refer the reader there for details:

Regarding C as a formal scheme, the hypercohomology H may be computed using Cech cohomology for open covers $\mathscr{C}$ of C by formal affines. In fact, it suffices to consider coverings by formal completions along Zariski affine open subschemes of $\tilde{C}$, called *Zariski affinoides*. For such a $X \in \mathscr{C}$, there exists a lift of Frobenius as described in [**Col85**]:

$$\Phi_X : X \to X^\sigma.$$

If $\omega = \{(X, \omega_X)\}_{X\in\mathscr{C}}$ for formal differentials $\omega_X \in \Omega_{X/R}$ is a one-cocycle and $[\omega]$ denotes the class of ω in H then Frobenius $F[\omega]$ is given by the class of

$$\omega' = \{(X, \Phi_X^*\omega_X^\sigma)\}_{X\in\mathscr{C}}$$

in H. For an affinoid X we say $\eta \in \Omega^1_{X/R^{nr}}$ is a differential of the *second kind* on C if there exists a cover $\mathscr{C}$ such that (X, η) is part of a Cech one-cocycle for $\mathscr{C}$. We denote the R-module of such differentials on C by $\mathscr{D}_{sk}(X)$. Consider the sub-R-algebra of the rigid analytic functions $A(X)$ on X given by

$$A^0(X) := \{f \in A(X) : \|f\|_X \leq 1\}$$

for the usual norm $\|f\| = \sup_{x\in X}\{|f(x)| \text{ in } A(X)/x\}$. One can show that the de Rham cohomology may then be computed using differentials of the second kind on an affinoid X:

$$H = \mathscr{D}_{sk}(X)/dA^0(X).$$

It follows that if ν, ω are such that $[\omega] = V[\nu]$,

$$\nu = \frac{\Phi^*\omega^\sigma}{p} + df$$

for some $f \in A^0(X)$, which basically lifts to $\mathscr{D}_{sk}(X)$ the equality $FV = p$ on H.

We now turn our attention to our specific situation: we work on a residue disk U and consider the submodule $W \subseteq H$. Let $\Phi : U \to U^\sigma$ denote a lift of σ. Choose $\omega \in \mathscr{D}_{sk}(U)$ so that $[\omega] \in W$. We also pick inductively $\omega_i \in \mathscr{D}_{sk}(U)$ so that

$$[\omega_i] = V^i[\omega]$$

and $\omega_i \in p^{m_i}\mathscr{D}_{sk}(U)$, where m_i is the largest integer so that $[\omega_i] \in p^{m_i}H$. It follows from our previous considerations that there exist rigid functions $f_i \in p^{m_i}A^0(X)$ so that:

$$\omega_i = \frac{\Phi^*\omega_{i+1}^\sigma}{p} + df_i \tag{4.2}$$

Assuming H is ordinary or superspecial, we deduce from the action of V that $\tilde{\omega_{it}} \in \tilde{W}$, where $t = 1$ or 2 according to which type. Now set $\nu_i = \frac{\omega_{ti}}{p^{(t-1)i}}$. Assuming $[\omega] \in W \setminus pW$, one then has that $\tilde{\nu_i} \neq 0$. It follows from iterating 4.2 that

$$\nu_i = \frac{(\Phi^t)^*\nu_{i+1}^{\sigma^t}}{p} + dg_i \tag{4.3}$$

for $g_i \in A^0(U)$. Pick T a uniformizer at O so that $\Phi^* T^\sigma = T^p$. We get that $g_i \in R^{nr}[[T]]$. We may now deduce from $d/dT L_\omega(T) = \omega(T)/dT$ that

$$\begin{aligned} L_\omega(T) &= c + g_0(T) - g_0(0) + \frac{L^{\phi}_{\nu_1}(T^q)}{p} && \text{(by integrating 4.3)} \\ &= c + g_0(T) - g_0(0) + \frac{(g_1(T^q) - g_1(0))^{\phi}}{p} + \frac{L^{\phi^2}_{\nu_2}(T^{q^2})}{p^2} && \text{(applying 4.3 again)} \\ &= L_\omega(0) + \sum_{i=0}^{\infty} \frac{g^{\phi^i}_{\omega,i}(T^{q^i})}{p^i} && \text{(iterating as above)} \end{aligned}$$

where we rewrote $g_{\omega,i}(T) = g_i(T) - g_i(0)$. It also follows from 4.3 by expanding inside of U that

$$\begin{aligned} dg_{\omega,i} &\equiv \nu_i \mod T^{q-1}dT \\ &\equiv \mathcal{C}^i \tilde{\omega} \mod (p, T^{q-1}dT) \end{aligned}$$

as claimed. It remains to prove the claim on the shape of the Newton polygons of $\{L_\omega(T) | \omega \in W\}$. The assertion on the first vertex is clear. For any fixed $\omega = \nu_0$, we set $k_i = \operatorname{ord}_U \tilde{\nu}_i + 1$. It follows from

$$L_\omega(T) = L_\omega(0) + \sum_{i=0}^{\infty} \frac{g^{\phi^i}_{\omega,i}(T^{q^i})}{p^i}$$

and $g_{\omega,i}(T) \in TR^{nr}[[T]]$ that, provided $k_i < q - 1$, the vertices of the Newton polygon of $L_\omega(T)$ besides the first one lie among

$$\{(k_i q^i, -i) | i \geq 0\}.$$

But since $\mathcal{C}^i\tilde{\omega} = \tilde{\nu_i}$ and $\mathcal{C}$ is bijective, we see that for each $i \in \mathbb{N}$,

$$\tilde{W} = \{\tilde{\nu_i}|\omega \in W\}.$$

Since U is not a base point of $\tilde{W}$ we can find ω with $k_i = 1$ for any i, so that the lower convex hull of the vertices of Newton polygons of $\{L_\omega(T)|\omega \in W\}$ has to lie below the vertices

$$\{(q^i, -i)|i \geq 0\}.$$

It is now easy to see that for $i \geq \max\{0, 1 - M\}$ these points constitute the vertices of the lower convex hull of the vertices of $\{L_\omega(T)|\omega \in W\}$, which proves the claim. □

Using Proposition 4.2.7, we can adapt the arguments of [**Col85**, Section IV] to prove the main result of this section.

Proof of Proposition 4.2.5. Set $a = T(Q)$ so that $v_p(a) \leq \frac{1}{p-1}$. Since

$$L_\omega(a) \equiv 0 \mod p$$

for all $w \in W$, it still follows from the shape of the Newton polygons that a must have the valuation of a common root of these power series. We may not apply Proposition 4.2.7 directly since in our setup $L_\omega(0) = 0$. However, we may consider the lower convex hull of the polygons of $L_\omega(T)/T \in \hat{\mathbb{Q}}_p^{nr}[[T]]$ running over $w \in W$ which are simply:

$$(0,0) \cup \{(q^i - 1, -i) : i \geq 1\}.$$

In particular all but the first slope are the same as in Proposition 4.2.7. Considering the Newton polygon of $L_\omega(T)/T$ one deduces from

$$v_p(L_\omega(a)/a) \geq \frac{p-2}{p-1}$$

that a has to have the valuation of a root of $L_\omega(T)/T$ for all $\omega \in W$. This yields:

$$v_p(a) = \frac{1}{q^n(q-1)} \text{ for some } n \geq 0.$$

We fix this n for the rest of the proof.

Now set $C_i(\omega) = (g_{\omega,i}(0))^{\phi^i}$, which by 4.1 is congruent modulo p to $(\mathcal{C}^i(\tilde{\omega})/dT)(U))^{\phi^i}$. It follows by computing valuations of the individual terms in $L_\omega(a)$ that

$$L_\omega(a) \equiv \frac{g_{\omega,n}^{\phi^n}(a^{q^n})}{p^n} + \frac{C_{n+1}(\omega)a^{q^{n+1}}}{p^{n+1}} \mod p^{1-n}a_n, \tag{4.4}$$

where $a_n = a^{q^{n-1}}$ if $n > 0$ and $a_0 = 1$.

Step 1: We first show that for an appropriate unit $\delta \in R_{\mathbb{C}_p}^\times$, we have the congruence

$$C_n(\omega) \equiv \delta C_{n+1}(\omega) \mod p. \tag{4.5}$$

Multiplying through from the congruence 4.4 we get that

$$0 \equiv g_{\omega,n}^{\phi^n}(a^{q^n}) + \frac{C_{n+1}(\omega)a^{q^{n+1}}}{p} \mod pa_n. \tag{4.6}$$

Since $g_{\omega,n}^{\phi^n}(T) \equiv C_n(\omega)T \mod T^2$ we moreover obtain the congruence

$$C_n(\omega) \equiv -\frac{a^{q^n(q-1)}}{p}C_{n+1}(\omega) \mod a^{q^2 n}$$

and $-\frac{a^{q^n(q-1)}}{p}$ is a unit. Since $C_j(\omega) \in R^{nr}$ for $j \geq 0$, we may find an unramified unit δ such that the congruence above actually holds modulo p, which shows 4.5.

Step 2: Let now $\tilde{\omega}, \tilde{\omega}'$ be two differentials in $\tilde{W}$. Define an $\mathbb{F}$-linear map on $\tilde{W}$ by

$$D : \tilde{\omega} \mapsto \frac{\tilde{\omega}}{dT}(U).$$

The vanishing result on differentials follows from the following congruence:

$$D(\mathcal{C}^n\tilde{\omega})\mathcal{C}^n\tilde{\omega}' \equiv D(\mathcal{C}^n\tilde{\omega}')\mathcal{C}^n\tilde{\omega} \mod T^{p-2}dT \tag{4.7}$$

Indeed assume that a differential $\tilde{\nu} \in \tilde{W}$ vanishes on U. By surjectivity of $\mathcal{C}$ we may assume $\mathcal{C}^n\tilde{\omega} = \tilde{\nu}$ vanishes for some $\tilde{\omega} \in \tilde{W}$, whereby $D(\mathcal{C}^n\tilde{\omega}) = 0$. Since U is not a base point of $\tilde{W}$, there is some $\tilde{w}'$ such that $D(\mathcal{C}^n\tilde{\omega}') \neq 0$, again using that $\mathcal{C}$ is bijective. It follows from 4.7 that as desired

$$\tilde{\nu} = \mathcal{C}^n\tilde{\omega} \equiv 0 \mod T^{p-2}dT.$$

Step 3: It remains to prove the congruence 4.7 holds. We have by 4.6 that

$$\begin{aligned} C_{n+1}(\omega)g^{\phi^n}_{\omega',n}(a^{q^n}) &\equiv C_{n+1}(\omega)\frac{C_{n+1}(\omega')a^{q^{n+1}}}{p} \\ &\equiv C_{n+1}(\omega')\frac{C_{n+1}(\omega)a^{q^{n+1}}}{p} \\ &\equiv C_{n+1}(\omega')g^{\phi^n}_{\omega,n}(a^{q^n}) \mod pa_n \end{aligned}$$

and using congruence 4.5 we arrive at the congruence

$$C_n(\omega)g^{\phi^n}_{\omega',n}(a^{q^n}) \equiv C_n(\omega')g^{\phi^n}_{\omega,n}(a^{q^n}) \mod pa_n.$$

Observe that this in turn implies a congruence for the first terms of the power series themselves: we consider the following polynomial

$$B(T) := C_n(\omega)g^{\phi^n}_{\omega',n}(T) - C_n(\omega')g^{\phi^n}_{\omega,n}(T) \mod T^q.$$

Since $(a^{q^n})^q \equiv 0 \mod pa_n$, we deduce from the above that p divides $B(a^{q^n})$. Since B has coefficients in R^{nr} from considering valuations we see p divides $B(T)$ and therefore, quotienting out by T^{q-1} to account for when $n = 0$ we get

$$C_n(\omega)g^{\phi^n}_{\omega',n}(T) \equiv C_n(\omega')g^{\phi^n}_{\omega,n}(T) \mod (p, T^{q-1}).$$

Finally, by differentiating, using congruence 4.1 and the fact that

$$C_n(\omega) \equiv (\mathcal{C}^n(\tilde{\omega})/dT)(U))^{\phi^n} = D(\mathcal{C}^n\tilde{\omega})^{\phi^n} \mod p$$

we deduce the congruence

$$D(\mathcal{C}^n\tilde{\omega})^{\phi^n}(\mathcal{C}^n\tilde{\omega}')^{\phi^n} \equiv D(\mathcal{C}^n\tilde{\omega}')^{\phi^n}(\mathcal{C}^n\tilde{\omega})^{\phi^n} \mod (p, T^{q-2}dT),$$

which implies the desired result.

□

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