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Asymptotics For The Number Of Critical Points For Two Analytical Models

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Abstract

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In this work we explore a connection between some high dimensional asymptotic problems and random matrix theory. In the first part, we establish a link between the Wishart ensemble and random critical points of holomorphic sections over complex projective space and use this to establish asymptotics on the average number of them. In the second part of this work, we further explore the link between the Gaussian Elliptic Ensemble and the average number of equilibrium points for a class of random Gaussian ordinary differential equations as established in Fyodorov [18]. We use this link to establish asymptotics on the average number of stable equilibrium points for this class of random Gaussian ordinary differential equations.

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CHAPTER 1

Introduction

Connections between random matrix theory and large dimensional phenomena were known as early as 1972 from the work of May [25], where he analyzed the stability for equilibria of high dimensional ordinary differential equations arising from ecological models. Since then, random matrix theory has become an effective tool to study high dimensional problems with great success, as can be seen from the works of Auffinger, Ben Arous and Černý [3], Dedieu and Malajovich [10], Fyodorov [16], [17], [18], etc. In all of these works, it is the Gaussian Orthogonal Ensemble (GOE) that arises as the underlying random matrix model, serving as a proof that there is a link between the eigenvalues of the GOE and critical points of isotropic Gaussian fields both on $\mathbb{R}^n$ and on the sphere. The GOE is not the only such ensemble to be used in high dimensional problems. In multivariate statistics, the Wishart ensemble arises naturally as the distribution of sample covariance matrices, see Johnstone [23]. In this work, we shall add to this rich history by considering two additional models. The second of these models shares the same setting as many of the aforementioned papers, namely high dimensional spheres in $\mathbb{R}^n$. The first model takes place in the complex analog, namely complex projective space.

1.1. Complex holomorphic sections

In the first part of this work, we study the statistics of critical points of Gaussian random holomorphic sections over a complex projective space. Random holomorphic sections were originally studied in Douglas, Shiffman, and Zelditch [12], [13] as a tool to understand the vacuum selection problem in string theory. Since then, there have been a lot of results about them with most of these results focusing on the distribution of the zeroes of random holomorphic sections in the large degree limit, see e.g. [6]. We focus instead in the large dimensional limit and look at critical points instead, as originally considered Douglas, Shiffman, and Zelditch [12]. This was exactly the type of problem that was considered in Baugher [5], where bounds for the expected number of critical points (regardless of Morse index) were established. In this work, we proceed to establish a link between the statistics of these critical points and the Wishart ensemble and use this link to establish asymptotics on the exponential rate of the average number of critical points of a given Morse index, extending the results in [5]. In CHAPTER 2, we introduce our main results as well as some background material needed for understanding the rest of this part of the thesis. In CHAPTER 3 we discuss the Wishart ensemble and its large deviations needed in the proof of the main results. CHAPTER 4 is devoted to explaining the relation between the expected number of critical points and the Wishart ensemble, as well as providing the proofs for THEOREM 2.1.1 and THEOREMS 2.1.2 and 2.1.5.

It is reasonable to suspect that this link should allow us to derive asymptotics for the average number of critical points as a whole, as opposed to simply the exponential rate. By writing the average density of eigenvalues of the Wishart ensemble in terms of

Laguerre polynomials, it should be possible to derive those asymptotics. Nevertheless, we do not perform that computation here.

1.2. Equilibrium points for Gaussian ODEs on spheres

In the second part of this work, we return to the more traditional setting of spheres in $\mathbb{R}^n$. We will be interested in the dynamics of a particular family of random Gaussian ordinary differential equations. This type of ODEs were originally studied by Cugliando et al. in [9] and serve as a nice framework in which the problem of studying stability of equilibrium is analytically tractable yet still general enough to be non-trivial. In the case our ODE takes the form of gradient flow (known in the literature as relaxational dynamics), the results in Auffinger, Ben Arous and Černý [3] and Fyodorov [16] establish asymptotics on the average number of equilibrium points with any given number of stable directions. In fact, Subag [27] established almost-sure asymptotics on the total number of equibrium. To complement that work, we will be performing stability analysis for the non-gradient flow case.

The asymptotics for the average number of equilibrium points (regardless of stability) was established in Fyodorov [18]. This work complements it by establishing asymptotics on the average number of *stable* equilibria, which is more telling of the dynamics than simply equilibrium points. We do so by expanding the link between equilibrium points and the Gaussian Elliptic Ensemble(GEE) which Fyodorov established in [18].

This part of the thesis is organized as follows. In CHAPTER 5, we establish the setting we're working in and state our main results. In CHAPTER 6, we introduce the Gaussian Elliptic Ensemble and discuss some of its properties. Using properties of the logarithm potential of the ellipse, we prove a large deviation result for the eigenvalue of the Gaussian Elliptic Ensemble with the mth largest real part. The goal of CHAPTER 7 is to relate $\mathbb{E}\mathcal{N}_m(B)$ to a matrix integral involving the Gaussian Elliptic Ensemble as well as provide the proofs for the main results stated in this chapter.

1.3. Similarities between the models

At first glance, the two models seem vastly different. The first is inherently in the complex domain, while the second one unequivocally belongs to the real domain. Nevertheless, the two models have three main similarities which allow the same type of argument to work in both cases. First, the existence of a Kac-Rice formula. Second, the existence of a large symmetry group to simplify the explicit form of the Kac-Rice formula and lastly, the existence of large deviation results for the empirical distribution of the eigenvalues of the resulting ensembles that arise from the Kac-Rice formula.

Part 1

On the expected number of random holomorphic sections over a complex projective space

CHAPTER 2

Main results and background material

The goal of this chapter is to establish some necessary background in order to state the results of this part of the thesis. In the first section, we present the setting we are working on as well state the main results. The last section of this chapter presents additional background on complex projective space as well as on holomorphic sections.

2.1. Main results

We now describe the setting and our main results, which we retrieve from the author's work in Garcia [19]. We consider the line bundle $\mathcal{O}(N)$ over $\mathbb{CP}^m$ equipped with Fubini-Study metric h, induced Chern connection ∇ and $N \geq 2$. We endow the space of holomorphic sections $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ with the inner product induced by the metric, namely for two sections s_1, s_2 we set

$$\langle s_1, s_2 \rangle := \int_{\mathbb{CP}^m} h_z(s_1, s_2) v(dz),$$

where v is the Fubini-Study volume element. We view $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ as a finite dimensional Hilbert space and choose an orthonormal basis s_i^N. With this basis, we can form the Gaussian field

$$s = \sum_i c_i s_i^N, \tag{2.1.1}$$

where the c_i are independent circularly symmetric complex Gaussians with the normalized variance

$$\mathbb{E}|c_i|^2 = \frac{\mathrm{Vol}(\mathbb{CP}^m)}{\dim H^0(\mathbb{CP}^m, \mathcal{O}(N))} = \frac{N!\pi^m}{(N+m)!}.$$

With this normalization, we have that the expected value of the L^2 norm of s is one. It is clear that the distribution of s is independent of the choice of the orthonormal basis. For any Borel set $B \subset \mathbb{R}_+ = [0, \infty)$ and integer k satisfying the inequality $m \leq k \leq 2m$, we consider $\mathcal{N}_{m,k,N}(B)(s)$, the number of critical points z with Morse index k for a section s with $h_z(s,s) = ||s(z)||_h^2 \in (m+1)B$; symbolically,

$$\mathcal{N}_{m,k,N}(B) = \sum_{z:\nabla s(z)=0,\ \mathrm{Ind}(\nabla^2 s)(z)=k} \mathbb{1}_{(m+1)B}(||s(z)||_h^2),$$

where we understand $\mathrm{Ind}(\nabla^2 s)$ as the index of the real Hessian of $\log ||s(z)||_h^2$. Thus $\mathcal{N}_{m,k,N}(B)$ is an integer-valued random number if s is sampled from the Gaussian field (2.1.1). The random variable

$$\mathcal{N}_{m,N}(B) = \sum_{m \leq k \leq 2m} \mathcal{N}_{m,k,N}(B)$$

is the total number of critical points regardless of their Morse indices.

We will prove two types of asymptotics for $\mathcal{N}_{m,2m-k,N}(B)$ as the dimension m goes to infinity: with fixed k and with linearly growing k. More specifically, in the latter case we will consider a relation of the form $k(m)/m \to \gamma \in (0,1)$ as $m \to \infty$.

We now state our main results. For a given $\gamma \in (0,1)$ define s_γ by

$$\int_{s_\gamma}^{4} f_{MP}(x)dx = \gamma, \tag{2.1.2}$$

where

$$f_{MP}(x) = \frac{\sqrt{(4-x)x}}{2\pi x}$$

is the Marchenko-Pastur density function on $[0,4]$, see Pastur and Shcherbina [26] for more details on this measure.

Our first main result concerns the exponential growth rate of the expected number $\mathbb{E}\,\mathcal{N}_{m,2m-k,N}(x,\infty)$ of critical points.

Theorem 2.1.1. *Fix an integer k.*

(1) Suppose that $x \geq 0$ and let $x_N = \frac{N}{N-1}x$. If $x_N \geq 4$, then

$$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{E}\,\mathcal{N}_{m,2m-k,N}[x,\infty) = \log(N-1) - \frac{x_N}{2}\left(1-\frac{2}{N}\right) - (k+1)\int_4^{x_N} \sqrt{\frac{t-4}{4t}}\,dt.$$

$$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{E}\,\mathcal{N}_{m,2m-k,N}[0,x) = \log(N-1) - 2\left(1-\frac{2}{N}\right).$$

If $x_N \leq 4$, then

$$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{E}\,\mathcal{N}_{m,2m-k,N}[0,x) = -\infty.$$

$$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{E}\,\mathcal{N}_{m,2m-k,N}[x,\infty) = \log(N-1) - 2\left(1-\frac{2}{N}\right).$$

(2) If $k = k(m)$ such that $k/m \to \gamma \in (0,1)$, then

$$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{E}\,\mathcal{N}_{m,2m-k,N}(\mathbb{R}_+) = \log(N-1) - \left(1-\frac{2}{N}\right)\frac{s_\gamma}{2},$$

where s_γ is the number uniquely defined by the relation (2.1.2).

The above results do not include the case $k(m) = m$. However, in this case we can compute explicitly the expected value $\mathbb{E}\,\mathcal{N}_{m,m,N}(\mathbb{R}_+)$ and recover the formula in

Baugher [5]. Define the density function $p_{m,m,N}$ by

$$\mathbb{E}\left[\sum_{z:\nabla s(z)=0,\,\mathrm{Ind}(\nabla^2 s)(z)=k} f\left(\frac{1}{m+1}||s(z)||_h^2\right)\right] = \int_{\mathbb{R}_+} f(x)p_{m,m,N}(x)dx$$

for any positive continuous function f on $\mathbb{R}_+$. Note that the above sum is simply the total number of critical points of Morse index m in the case when $f := 1$. Our second main result is an explicit formula for $p_{m,m,N}$.

Theorem 2.1.2. *For any $x \geq 0$,*

$$p_{m,m,N}(x) = (N-1)^m(m+1)^2 e^{-\frac{(m+1)N}{2(N-1)}(2-\frac{2}{N}+m)x}.$$

We can draw two consequences from this explicit density.

Corollary 2.1.3. *For any $x \geq 0$,*

$$\mathbb{E}\,\mathcal{N}_{m,m,N}[x,\infty) = \frac{2(N-1)^{m+1}(m+1)}{2N-2+Nm} e^{-\frac{(m+1)N}{2(N-1)}(2-\frac{2}{N}+m)x}.$$

PROOF. Integrate the density function over $[x,\infty)$. $\square$

For $x = 0$, the above corollary recovers the formula

$$\mathbb{E}\,\mathcal{N}_{m,m,N}(\mathbb{R}_+) = \frac{2(m+1)}{2(N-1)+mN}(N-1)^{m+1} \tag{2.1.3}$$

proved in Baugher [5]. For $x > 0$, it follows from the corollary that there exist positive constants c_1 and c_2 such that $\mathbb{E}\,\mathcal{N}_{m,m,N}(x,\infty) \leq c_1 e^{-c_2 m^2 x}$, which shows that it becomes exponentially unlikely to find critical values away from 0 whose Morse index is m.

The next consequence is that we can recover the asymptotics for the exponential rate of $\mathbb{E}\,\mathcal{N}_{m,m+k,N}(\mathbb{R}_+)$ for any fixed $k > 0$.

Corollary 2.1.4. *For a fixed $k > 0$, we have*

$$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{E}\,\mathcal{N}_{m,m+k,N}(\mathbb{R}_+) = \log(N-1).$$

PROOF. According to THEOREM 1.4 of Baugher [5], the total number of critical points $\mathcal{N}_{m,m+k,N}(\mathbb{R}_+)$ decreases as k increases. Thus, given $\gamma \in (0,1)$ and $q(m)/m \to \gamma$, we have for large m,

$$\mathcal{N}_{m,2m-q(m),N}(\mathbb{R}_+) \le \mathcal{N}_{m,m+k,N}(\mathbb{R}_+) \le \mathcal{N}_{m,m,N}(\mathbb{R}_+).$$

For the right hand side, we have by (2.1.3)

$$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{E}\,\mathcal{N}_{m,m,N}(\mathbb{R}_+) = \log(N-1).$$

For the left hand side, we have by the second part of THEOREM 2.1.1,

$$\lim_{m\to\infty} \log \mathbb{E}\,\mathcal{N}_{m,2m-q(m),N}(\mathbb{R}_+) = \log(N-1) - \left(1 - \frac{2}{N}\right)\frac{s_\gamma}{2}.$$

We have $s_\gamma \to 0$ as $\gamma \to 1$, and the above limit reduces to that $\log(N-1)$. The result follows immediately. $\square$

Finally, our third and last main result concerns the total number of critical points.

Theorem 2.1.5. *As before, we let $x \geq 0$ and $x_N = \frac{N}{N-1}x$. Then:*

$$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{E}\, \mathcal{N}_{m,N}(x,\infty) = \begin{cases} \log(N-1) - \left(1 - \dfrac{2}{N}\right)\dfrac{x_N}{2} - \displaystyle\int_4^{x_N} \sqrt{\frac{t-4}{4t}}dt, & x_N \geq 4 \\ \log(N-1) - \left(1 - \dfrac{2}{N}\right)\dfrac{x_N}{2}, & x_N < 4 \end{cases}$$

2.2. Complex projective space and line bundles

In this section we recall some basic facts from complex geometry which are useful for understanding the setting of this part of the thesis.

The complex projective space $\mathbb{CP}^m$ is the quotient space of $\mathbb{C}^{m+1}\backslash\{0\}$ by the equivalence relation

$$\lambda(Z_0, ..., Z_m) \sim (Z_0, ..., Z_m), \quad \lambda \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}.$$

This is a compact complex manifold with local charts $U_i = \{[Z_0, Z_1, ..., Z_m] | Z_i \neq 0\}$ and trivializing maps $\Phi_i : U_i \to \mathbb{C}^m$ defined by

$$\Phi_i(Z) = (Z_0/Z_i, \ldots, \widehat{Z_i/Z_i}, \ldots, Z_m/Z_i).$$

We denote by $\mathcal{O}(N)$ the line bundle with the transition functions

$$\sigma_{ij} : U_i \cap U_j \to \mathbb{C}^*, \quad \sigma_{ij}(Z) = \left(\frac{Z_i}{Z_j}\right)^N.$$

The sections of this bundle correspond to homogeneous holomorphic polynomials of degree N in the variables $Z_0, ..., Z_m$. To see this, given a homogeneous holomorphic polynomial $p(Z_0, ..., Z_m)$ we define the functions f_j on U_j by $f_j(Z) = p(Z/Z_j)$. It is easy to verify that these functions glue up and yield a section on $\mathbb{CP}^m$. Indeed, on the intersection $U_i \cap U_j$, we have

$$f_i(Z)\sigma_{ij}(Z) = p\left(\frac{Z}{Z_i}\right)\left(\frac{Z_i}{Z_j}\right)^N = p\left(\frac{Z}{Z_j}\right) = f_j(Z).$$

Conversely, a section is just a collection of polynomials f_j on the charts U_j satisfying $f_i(Z)\sigma_{ij}(Z) = f_j(Z)$ on the intersection $U_i \cap U_j$, which define a homogenous polynomial in a unique way by setting $p(Z) = Z_j^N f_j(Z)$.

We equip $\mathbb{CP}^m$ with the Fubini-Study metric h and denote the corresponding Chern connection on $\mathcal{O}(1)$ by ∇. This induces canonically a connection on $\mathcal{O}(N)$, also denoted by ∇, by requiring that it satisfy Leibniz's rule on tensors of sections. More explicitly, a section $s \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$ can be written locally as $s = fe^N$, where $e^N = \otimes_{i=1}^N e$ for a trivializing local frame e for $\mathcal{O}(1)$ and a holomorphic function f on a chart of $\mathbb{CP}^m$. Then the connection ∇ can be expressed explicitly as

$$\nabla s = \sum_{j=1}^m (\partial_{z_j} f + f\,\partial_{z_j} K_N)\, dz_j \otimes e^N, \tag{2.2.1}$$

where K_N is given by

$$K_N = K_N(z, \bar{z}) = N \log(1 + |z|^2). \tag{2.2.2}$$

Since ∇ also acts on 1-forms canonically, the Hessian ∇^2 on holomorphic sections is well defined. This action can be explicitly written in local coordinates as follows. For simplicity we introduce the notation $\nabla_{z_j} f := \partial_{z_j} f + f\,\partial_{z_j} K_N$ and $\nabla^2_{z_i,z_j} f = \nabla_{z_i}(\nabla_{z_j} f)$.

In the local basis $dz_i \otimes dz_j$, we can view $\nabla^2 s$ as the $2m \times 2m$ square matrix

$$\nabla^2 s(z) = \begin{bmatrix} \nabla^2_{z_i,z_j} f & f\Theta_N \\ \\ \overline{f\Theta_N} & \overline{\nabla^2_{z_i,z_j} f} \end{bmatrix}$$

where $\Theta_N = \left\{ \partial^2_{z^i,\overline{z^j}} K_N \right\}$. Note that this matrix is not Hermitian. For this reason, when discussing critical points of a section s, it is more convenient to use the real Hessian of $\log ||s(z)||^2_h$ by viewing $\mathbb{CP}^m$ as a smooth manifold of real dimension $2m$. By a slight abuse of notation, we use $\mathrm{Ind}(\nabla^2 s)(z)$ to denote the index of this matrix. From LEMMA 7.1 of Douglas, Shiffman, and Zelditch [13], we know that in local coordinates

$$\mathrm{Ind}(\nabla^2 \log ||s(z)||^2_h) = m + \mathrm{Ind}(\nabla^2_{z_i,z_j} f \, \Theta_N^* \overline{\nabla^2_{z_i,z_j} f} - \Theta_N),$$

where Θ_N^* is the conjugate transpose of Θ_N.

CHAPTER 3

The Wishart Ensemble

In this chapter, we study the Wishart ensemble. This ensemble will arise naturally when we look at the distribution of the Hessian of random holomorphic sections. In the first section, we will give a definition of the Wishart ensemble. In the second section, we will study large deviations for eigenvalues in the Wishart distribution.

3.1. Definition of the Wishart Ensemble

Let X be a real $(m+1)\times m$ random matrix whose entries are i.i.d. Gaussians with mean zero variance $1/m$ and $W = X^T X$. We denote the law of W, the Wishart ensemble, by $\mathbb{P}_m$ and the corresponding expectation by $\mathbb{E}_m$.

The only information we will need about the Wishart ensemble is the explicit distribution of its eigenvalues. For a vector $\lambda = (\lambda_1, ..., \lambda_m)$, we define $\Delta(\lambda) = \prod_{i<j}(\lambda_i - \lambda_j)$, the Vandermonde determinant. We write the eigenvalues $\lambda = (\lambda_1, \ldots, \lambda_m)$ of W in descending order, so that the vector λ belongs to the region

$$\mathbb{R}^m_{\geq 0} = \{\lambda \in \mathbb{R}^m : \lambda_1 \geq ... \geq \lambda_m \geq 0\}.$$

Theorem 3.1.1. *The joint density function of the decreasingly ordered eigenvalues of the Wishart ensemble with respect to the Lebesgue measure on $\mathbb{R}^m_{\geq 0}$ is*

$$\frac{1}{Z_W(m)}\Delta(\lambda)\exp\left(-\frac{m}{2}\sum_{i=1}^{m}\lambda_i\right),$$

where $Z_W(m)$ is the normalizing constant given by

$$Z_W(m) = 2^m m^{-m(m+1)/2} \prod_{j=1}^{m} j! \tag{3.1.1}$$

PROOF. See THEOREM 13.3.2 in Anderson [2] for the density, and COROLLARY 2.5.9 of Anderson, Guionnet and Zeitouni [1] for the explicit formula for $Z_W(m)$. $\square$

3.2. Large deviations for the largest eigenvalue of the Wishart ensemble

We now turn to the large deviations of the largest eigenvalues of the Wishart ensemble. We will need the following large deviation principle for the law of kth largest eigenvalue under $\mathbb{P}_m$.

Theorem 3.2.1. *Under $\mathbb{P}_m$, the kth largest eigenvalue λ_k satisfies the large deviation principle (LDP) with the speed m and the good rate function kI_{MP}, where*

$$I_{MP}(x) = \int_4^x \sqrt{\frac{t-4}{4t}}dt$$

for $x \geq 4$ and ∞ otherwise.

PROOF. The case for $k = 1$ is already known, see Feral [15], pages 47 and 48. We proceed to extend this result for arbitrary m. It is obvious that I_{MP} is a good rate function. With this in mind, this theorem is equivalent to the following two assertions:

(1) $\limsup_{m\to\infty} \frac{1}{m} \log \mathbb{P}_m(\lambda_k \leq x) = -\infty$ for $0 < x < 4$.
(2) $\lim_{m\to\infty} \frac{1}{m} \log \mathbb{P}_m(\lambda_k \geq x) = -kI_{MP}(x)$ for $x \geq 4$.

For the proof, we need two previous results.

(a) Under the Wishart ensemble, the empirical measure $L_m = \frac{1}{m}\sum_i \delta_{\lambda_i}$ of the eigenvalues satisfies an LDP with speed m^2. Its rate function is minimized uniquely at the

Marchenko-Pastur distribution μ_{MP} on [0,4]

$$\mu_{MP}(dx) = \frac{1}{2\pi}\frac{\sqrt{(4-x)x}}{x}\,dx.$$

This LDP is the content of THEOREM 5.5.7 of Hiai and Petz [21].

(b) The functional

$$\phi(\mu, z) = \int_{\mathbb{R}_+} \log|z-y|\mu(dy) - \frac{z}{2}$$

defined on $\mathscr{P}(\mathbb{R}_+) \times \mathbb{R}_+$ is upper semi-continuous when we restrict it to $\mathscr{P}[0, M] \times [0, M]$ for any $M > 0$, and in fact it is continuous on $\mathscr{P}[0, r] \times [x, y]$ for $y > x > r \geq 4$, see e.g. Auffinger, Ben Arous and Černý [3]. Here $\mathscr{P}(A)$ is the space of probability measures on a set $A \subset \mathbb{R}_+$ with a metric compatible with the usual weak convergence of probability measures. The distribution μ_{MP} and the rate function I_{MP} are related through the functional by

$$\phi(\mu_{MP}, x) = -I_{MP}(x) - 1. \tag{3.2.1}$$

See Feral [15], page 48.

To prove assertion (1), we note that by definition, the inequality $\lambda_k \leq x$ for some $x < 4$ implies that $L_m[x, 4] \leq (k-1)/m$. Since $\mu_{MP}[x, 4] > 0$, there exists a closed set $C \subset \mathscr{P}(\mathbb{R}_+)$ such that $\mu_{MP} \notin C$ and $\{\lambda_k \leq x\} \subset \{L_m \in C\}$ for sufficiently large m. The LDP for L_m recalled above in (a) implies that there exists a $c > 0$ such that

$$\mathbb{P}_m(\lambda_k \leq x) \leq \mathbb{P}_m(L_m \in C) \leq Ke^{-cm^2},$$

which proves assertion (1).

To prove assertion (2), we first note that for the largest eigenvalue λ_1,

$$\lim_{M\to\infty} \limsup_{m\to\infty} \frac{1}{m} \log \mathbb{P}_m(\lambda_1 > M) = -\infty, \tag{3.2.2}$$

which is precisely LEMMA 2.6.7 of Anderson, Guionnet and Zeitouni [1]. Now we have

$$\mathbb{P}_m(\lambda_k \geq x) \leq \mathbb{P}_m(\lambda_1 > M) + \mathbb{P}_m(\lambda_k \geq x, \lambda_1 < M).$$

In view of (3.2.2), it is sufficient to show that for sufficiently large M,

$$\lim_{m\to\infty} \frac{1}{m} \log \mathbb{P}_m(\lambda_k \geq x, \lambda_1 < M) = -kI_{MP}(x). \tag{3.2.3}$$

We first prove the upper bound. We introduce variables

$$\eta_i = \frac{m}{m-k}\lambda_i$$

for $1 \leq i \leq m$ and write the density of $\mathbb{P}_m$ in terms of the η_i. On the set

$$\{x \leq \eta_k \leq \cdots \leq \eta_1 < 2M\} \supset \{x \leq \lambda_k \leq \cdots \leq \lambda_1 < M\}$$

we have $|\eta_i - \eta_j| \le 2M$, and hence

$$\begin{aligned}
\mathbb{P}_m(d\lambda) &= \frac{1}{Z_W(m)} \Delta(\lambda) \exp\left[-\frac{m}{2}\sum_{i=1}^m \lambda_i\right] d\lambda_1 \cdots d\lambda_m \\
&= \left(\frac{m-k}{m}\right)^{m(m+1)/2} \frac{1}{Z_W(m)} \Delta(\eta) \exp\left[-\frac{m-k}{2}\sum_{i=1}^m \eta_i\right] d\eta_1 \cdots d\eta_m \\
&\le \left(\frac{m-k}{m}\right)^{m(m+1)/2} \prod_{i=1}^k \prod_{j=k+1}^m (\eta_i - \eta_j) \cdot \exp\left[-\frac{m-k}{2}\sum_{i=1}^k \eta_i\right] d\eta_1 \cdots d\eta_k \times \\
&\quad \frac{(2M)^{(k-1)k/2}}{Z_W(m)} \prod_{k+1 \le i < j \le m} (\eta_i - \eta_j) \cdot \exp\left[-\frac{m-k}{2}\sum_{i=k+1}^m \eta_i\right] d\eta_{k+1} \cdots d\eta_m
\end{aligned}$$

We can also write this formula in terms of the empirical distribution of the η by defining the empirical distribution $\tilde{L}_{m-k}$ as follows:

$$\tilde{L}_{m-k} = \frac{1}{m-k} \sum_{i=1}^{m-k} \delta_{\eta_{k+i}}.$$

With this quantity defined, we can further write

$$\begin{aligned}
\mathbb{P}_m(d\lambda) = (2M)^{(k-1)k/2} \left(\frac{m-k}{m}\right)^{m(m+1)/2} \frac{Z_W(m-k)}{Z_W(m)} \cdot d\eta_1 \cdots d\eta_k \times \\
\exp\left[(m-k)\sum_{i=1}^k \phi(\tilde{L}_{m-k}, \eta_i)\right] \mathbb{P}_{m-k}(d\eta_{k+1} \cdots d\eta_m)
\end{aligned}$$

For $\epsilon > 0$, let $B_\epsilon \subset \mathscr{P}[0, M]$ be the ball of radius ϵ centered around μ_{MP} and B_ϵ^c its complement. On the set $\{x \le \eta_k \le \cdots \le \eta_1 < 2M\}$, we can bound the exponential term $\exp\left[(m-k)\sum_{i=1}^{k}\phi(\tilde{L}_{m-k},\eta_i)\right]$ from above by $(2M)^{k(m-k)}$ and thus

$$\exp\left[(m-k)\sum_{i=1}^{k}\phi(\tilde{L}_{m-k},\eta_i)\right] \leq$$
$$\exp\left[k(m-k)\sup_{\mu\in B_\epsilon, y\in[x,2M]}\phi(\mu,y)\right]\mathbb{1}_{B_\epsilon}(\tilde{L}_{m-k}) + (2M)^{k(m-k)}\mathbb{1}_{B_\epsilon^c}(\tilde{L}_{m-k}).$$

We integrate over $\{x \leq \eta_k \leq \cdots \leq \eta_1 < 2M\}$ to obtain the following upper bound for $\mathbb{P}_m(\lambda_k \geq x, \lambda_1 < M)$:

$$\left(\exp\left[k(m-k)\sup_{\mu\in B_\epsilon, y\in[x,2M]}\phi(\mu,y)\right] + (2M)^{k(m-k)}\mathbb{P}_{m-k}(\tilde{L}_{m-k}\notin B_\epsilon)\right)\times$$
$$\left(\frac{m-k}{m}\right)^{\frac{m(m+1)}{2}}(2M)^{\frac{k(k-1)}{2}}\frac{Z_W(m-k)}{Z_W(m)}$$

Two observations are in order. The first observation is that $\tilde{L}_{m-k}$ with respect to $\mathbb{P}_{m-k}$ satisfies the same LDP as L_m with respect to $\mathbb{P}_m$. In particular, this implies that for m large enough there exists a $c > 0$ for which

$$\mathbb{P}_{m-k}(\tilde{L}_{m-k}\notin B_\epsilon) \leq e^{-cm^2},$$

hence the probability $\mathbb{P}_{m-k}(\tilde{L}_{m-k}\notin B_\epsilon)$ is negligible in the limit. The second observation is that from (3.1.1), one has

$$\lim_{m\to\infty}\frac{1}{m}\log\left[\left(\frac{m-k}{m}\right)^{m(m+1)/2}\frac{Z_W(m-k)}{Z_W(m)}\right] = k.$$

In light of these two observation, we arrive at the inequality

$$\limsup_{m\to\infty} \frac{1}{m} \log \mathbb{P}_m(\lambda_k \geq x) \leq k + k \lim_{\epsilon\downarrow 0} \sup_{\mu\in B_\epsilon, y\in[x,2M]} \phi(\mu, y).$$

The second term can be computed explicitly,

$$\lim_{\epsilon\downarrow 0} \sup_{\mu\in B_\epsilon, y\in[x,2M]} \phi(\mu, z) = \sup_{y\in[x,2M)} \phi(\mu_{MP}, y) = -I_{MP}(x) - k,$$

where the first equality follows from the upper-semicontinuity of ϕ and the second equality follows from (3.2.1) and the monotonicity of I_{MP}.

To obtain the lower bound, fix $y > x > r \geq 4$ and $\epsilon > 0$, we retain the definition of the η_i as in the proof of the upper bound, and on the set

$$\left\{ y \geq \eta_1 \geq \cdots \geq \eta_k \geq \frac{m}{m-k} x \right\} = \left\{ \frac{m-k}{m} y \geq \lambda_1 \geq \cdots \geq \lambda_k \geq x \right\} \subset \{\lambda_k \geq x\}$$

we can produce the inequality

$$\begin{aligned}
\mathbb{P}_m(d\lambda) &= \frac{1}{Z_W(m)} \Delta(\lambda) \exp\left[-\frac{m}{2} \sum_{i=1}^m \lambda_i \right] d\lambda_1 \cdots d\lambda_m \\
&\geq \frac{(m-k)!}{m!} \left(\frac{m-k}{m} \right)^{m(m+1)/2} \frac{Z_W(m-k)}{Z_W(m)} \prod_{1\leq i<j\leq k} |\eta_i - \eta_j| \cdot d\eta_1 \cdots d\eta_k \times \\
&\quad \mathbb{1}_{B_\epsilon \cap \mathscr{P}[0,r]}(\tilde{L}_{m-k}) \exp\left[k(m-k) \inf_{\mu\in B_\epsilon\cap\mathscr{P}[0,r], z\in[x,y]} \phi(\mu, z) \right] \mathbb{P}_{m-k}(d\eta_{k+1} \cdots d\eta_m),
\end{aligned}$$

where by $B_\epsilon \cap \mathscr{P}[0,r]$, we mean the set of measures in B_ϵ whose support is contained in $[0,r]$. By integrating over $\left\{y \geq \eta_1 \geq \cdots \geq \eta_k \geq \frac{m}{m-k}x\right\}$, we obtain

$$\begin{aligned}
\mathbb{P}_m(\lambda_k \geq x) \geq \mathbb{P}_m\left(y \geq \eta_1 \geq \cdots \geq \eta_k \geq \frac{m}{m-k}x\right) \\
\geq \frac{(m-k)!}{m!}\left(\frac{m-k}{m}\right)^{m(m+1)/2}\frac{Z_W(m-k)}{Z_W(m)}\int \prod_{1\leq i<j\leq k}|\eta_i-\eta_j|d\eta_1\cdots d\eta_k \times \\
\exp\left[k(m-k)\inf_{\mu\in B_\epsilon\cap\mathscr{P}[0,r],z\in[x,y]}\phi(\mu,z)\right]\mathbb{P}_{m-k}(\tilde{L}_{m-k}\in B_\epsilon\cap\mathscr{P}[0,r]),
\end{aligned}$$

where the integral is over the set

$$\left\{y \geq \eta_1 \geq \cdots \geq \eta_k \geq \frac{m}{m-k}x\right\}.$$

The integral is bounded away from zero and from above, so it will have no effect in the limit. The factor $\mathbb{P}_{m-k}(\tilde{L}_{m-k} \in B_\epsilon \cap \mathscr{P}[0,r])$ converges to one by the previously mentioned LDP, hence it too will not affect the limit. It follows that in the limit the inequality becomes

$$\liminf_{m\to\infty}\frac{1}{m}\log\mathbb{P}_m(\lambda_k \geq x) \geq k + k\lim_{\epsilon\downarrow 0}\inf_{\mu\in B_\epsilon\cap\mathscr{P}[0,r],z\in[x,y]}\phi(\mu,z).$$

We use the continuity of ϕ and (3.2.1) to obtain

$$\lim_{\epsilon\downarrow 0}\inf_{\mu\in B_\epsilon\cap\mathscr{P}[0,r],z\in[x,y]}\phi(\mu,z) = -I_{MP}(y) - 1.$$

Finally, we let $y \to x$ and use the continuity of I_{MP} to obtain our desired result. $\square$

CHAPTER 4

Expected number of critical points and the Wishart ensemble

In this chapter, we first establish the connection between the Wishart ensemble and critical points of holomorphic sections. In the first section, we make the connection explicit. Following that, we provide proofs of the remaining theorems.

4.1. Relating the Wishart ensemble to critical points

In this section we relate $\mathbb{E}\,\mathcal{N}_{m,2m-k,N}(B)$ to the $(k+1)$th largest eigenvalue of an $(m+1)\times(m+1)$ Wishart matrix.

Theorem 4.1.1. *For a Borel set $B \subset \mathbb{R}_+$,*

$$\mathbb{E}\,\mathcal{N}_{m,2m-k,N}(B) = \frac{2(N-1)^{m+1}}{N}\,\mathbb{E}_{m+1}\left[e^{-(1-\frac{2}{N})\frac{m+1}{2}\lambda_{k+1}};\ \lambda_{k+1} \in \frac{N}{N-1}B\right]. \tag{4.1.1}$$

The proof of this identity is based on the following Kac-Rice formula adapted to our setting.

Proposition 4.1.2. *Let $\rho_{\nabla s(z)}$ denote the probability density function of $\nabla s(z)$ as a (random) vector in $\mathbb{C}^m$ (see (2.2.1)). Then $\mathbb{E}\,\mathcal{N}_{m,2m-k,N}(B)$ equals*

$$\int_{\mathbb{CP}^m} \rho_{\nabla s(z)}(0)\mathbb{E}[|\det \nabla^2 s(z)|\mathbb{1}_{(m+1)B}(||s(z)||_h^2)\mathbb{1}_{\mathrm{Ind}\nabla^2 s(z)=2m-k}|\nabla s(z)=0]v(dz).$$

PROOF. See THEOREM 4.4 of Douglas, Shiffman, and Zelditch [12]. $\square$

Remark 4.1.3. *In general $\rho_{\nabla s(z)}$ depends on our choice of s_i^N. Nevertheless, its value $\rho_{\nabla s(z)}(0)$ at the origin is independent of the choice.*

By $SU(m+1)$-invariance, the integrand in the above Kac-Rice formula is independent of z, thus the z-integration can be replaced by the multiplication of $\mathrm{vol}(\mathbb{CP}^m)$ and we need to evaluate the expectation at the point $z = 0$. For this purpose, we write $s(z) = f(z)e^N$ in local coordinates near the point $z = 0$. We have $\nabla_{z_i} f = \partial_{z_i} f := \partial_i f$ at $z = 0$.

Lemma 4.1.4. *The covariance of f and its first and second derivatives at $z = 0$ are given as follows.*

$$\begin{aligned}
\mathbb{E}[f(0)\overline{f(0)}] &= 1, \\
\mathbb{E}[f(0)\overline{\partial_i f(0)}] &= 0, \\
\mathbb{E}[f(0)\overline{\partial_i \partial_j f(0)}] &= 0, \\
\mathbb{E}[\partial_i f(0)\overline{\partial_j f(0)}] &= N\delta_{ij}, \\
\mathbb{E}[\partial_i f(0)\overline{\partial_j \partial_k f(0)}] &= 0, \\
\mathbb{E}[\partial_i \partial_j f(0)\overline{\partial_k \partial_l f(0)}] &= N(N-1)(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}).
\end{aligned}$$

PROOF. The Gaussian field defined in (2.1.1) is uniquely determined by its covariance kernel

$$\mathbb{E}[s(x) \otimes \overline{s(y)}] = \frac{N!\pi^m}{(N+m)!}\Pi_{N,m}(x,y).$$

Here $\Pi_{N,m}$ is the kernel of the projection from $L^2(\mathbb{CP}^m, \mathcal{O}(N))$ into $H^0(\mathbb{CP}^m, \mathcal{O}(N))$. Note that this kernel is independent of our choice of an orthonormal basis s_i^N in (2.1.1).

In local coordinates, it can be explicitly written as

$$\frac{N!\pi^m}{(N+m)!}\Pi_{N,m}(x,y) = (1 + z \cdot \bar{w})^N e^N(z) \otimes \overline{e^N(w)},$$

where z and w are the (inhomogeneous) coordinates of x and y. The covariances in the statement follow by straightforward computations. □

As immediate consequences of LEMMA 4.1.4, we see that $\rho_{\nabla f(0)}(0) = 1/(N\pi)^m$ and that both the matrix $\partial^2_{ij} f(0)$ and $f(0)$ are independent of $\partial_k f(0)$, hence also independent of $\nabla s(0)$. From (2.2.2) we have $\partial^2_{z_i,\overline{z_j}} K_N(0) = N\delta_{ij}$, hence from (2.2.1) we have

$$\det \nabla^2 s(0) = \det(YY^* - N^2|f(0)|^2 I_m),$$

where the matrix $Y = \left\{\partial^2_{ij} f(0)\right\}$ and I_m is the $m \times m$ identity matrix. Obviously the value of the determinant depends only on the eigenvalues of YY^*. Therefore we need to study the distribution of the eigenvalues of YY^*, which is a Hermitian random matrix.

Proposition 4.1.5. *The law of the eigenvalues of $W = YY^*/mN(N-1)$ is identical with the law of the eigenvalues under the Wishart ensemble.*

PROOF. The natural Lebesgue measure on $\mathrm{Sym}(m, \mathbb{C})$ as a real vector space is

$$dH = \prod_{i \le j} \mathrm{Re}\, dH_{ij}\, \mathrm{Im}\, dH_{ij}.$$

From the last covariance identification in LEMMA 4.1.4 the density function of Y with respect to the Lebesgue measure dH is

$$\frac{1}{2^m (N(N-1)\pi)^{\frac{m(m+1)}{2}}} e^{-\frac{1}{2N(N-1)}\mathrm{Tr}(HH^*)}. \tag{4.1.2}$$

Define the map $\Phi : U(m) \times \mathbb{R}^m_{\geq 0} \to \mathrm{Sym}(m, \mathbb{C})$ by

$$\Phi(U, \lambda) = U\mathrm{diag}(\sqrt{\lambda})U^T,$$

where $\mathrm{diag}(\sqrt{\lambda})$ is the diagonal matrix whose entries are $\sqrt{\lambda_1}, ..., \sqrt{\lambda_m}$ and U^T is the transpose of U. By Takagi's factorization (see COROLLARY 4.4.4 of Horn and Johnson [22]), almost every $X \in \mathrm{Sym}(m, \mathbb{C})$ can be written uniquely as

$$X = U\mathrm{diag}\left(\sqrt{\lambda(XX^*)}\right)U^T,$$

where U is a unitary matrix and $\lambda_i(XX^*)$ are the eigenvalues of XX^* in decreasing order. A straightforward computation shows that the image of the Lebesgue measure dH under Φ becomes $\Phi_*(dH) = \Delta(\lambda)\, d\lambda\, dU$, where dU is the properly normalized Haar measure on $U(m)$. Note that the Jacobian in this case is $\Delta(\lambda)$, a function of λ alone. On the other hand, the exponent in (4.1.2) is

$$\frac{1}{N(N-1)}\mathrm{Tr}(YY*) = \frac{1}{N(N-1)}\sum_{i=1}^{m}\lambda_i(YY^*) = m\sum_{i=1}^{n}\lambda_i(W).$$

By passing from $\mathrm{Sym}(m, \mathbb{C})$ to $U(m) \times \mathbb{R}^m_{\geq 0}$, we see from (4.1.2) that the density functions for the distribution of the eigenvalues of $W = YY^*/N(N-1)$ must be a constant multiple of $\Delta(\lambda)\exp\left[-\frac{m}{2}\sum_{i=1}^{m}\lambda_i\right]$. Comparing this with the density function of the eigenvalues under the Wishart ensemble in LEMMA 3.1.1 we obtain the result immediately. □

Summarizing what we have proved so far, from the Kac-Rice formula in PROPOSITION 4.1.2 we conclude that $\mathbb{E}\,\mathcal{N}_{m,2m-k,N}(B)$ equals $\frac{\mathrm{Vol}(\mathbb{CP}^m)m^m(N-1)^m}{\pi^m}$ times

$$\mathbb{E}\left[\mathbb{1}_{(m+1)B}(|f(0)|^2)\mathbb{1}_{[\lambda_{k+1},\lambda_k]}\left(\frac{N|f(0)|^2}{(N-1)m}\right)\prod_{i=1}^{m}\left|\lambda_i - \frac{N|f(0)|^2}{(N-1)m}\right|\right],$$

where $\lambda_i = \lambda_i(W)$ with W obeying the Wishart ensemble and $f(0)$ is, according to LEMMA 4.1.4, a standard complex Gaussian random variable independent of W. It remains to identify this with (4.1.1). For this purpose, we note that $\frac{N|f(0)|^2}{(N-1)m}$ is exponentially distributed with mean $\frac{N}{(N-1)m}$. Thus the mean is

$$\frac{m(N-1)}{NZ_W(m)}\int_{\frac{N(m+1)}{m(N-1)}B}\int\prod_{i=1}^{m}|\lambda_i - x|\Delta(\lambda)e^{-\frac{m}{2}(1-\frac{2}{N})x}e^{-\frac{m}{2}(\sum_{i=1}^{m}\lambda_i+x)}d\lambda dx, \tag{4.1.3}$$

where the inner integral with respect to λ is over the set

$$\{\lambda_1 > ... > \lambda_k > x > \lambda_{k+1} > ... > \lambda_m > 0\}.$$

This domain suggests we treat x as if it is another λ. More precisely, introduce the new variables $\mu_i = \lambda_i$ for $1 \le i \le k$, $\mu_{k+1} = x$, and $\mu_i = \lambda_{i-1}$ for $k+2 \le i \le m$. For the Vandermonde polynomial we have $\Delta(\mu) = \Delta(\lambda)\prod_{i=1}^{m}|\lambda_i - x|$. In terms of the new variables μ, the integral (4.1.3) becomes

$$\frac{m(N-1)}{NZ_W(m)}\int_{\mathbb{R}_{\ge 0}^{m+1}}\mathbb{1}_{\frac{(N-1)(m+1)}{mN}B}(\mu_{k+1})e^{-(1-\frac{2}{N})\frac{m}{2}\mu_{k+1}}\exp\left[-\frac{m}{2}\sum_{i=1}^{m+1}\mu_i\right]\Delta(\mu)d\mu.$$

Comparing this with LEMMA 3.1.1, this is exactly the expectation with respect to $\mathbb{P}_{m+1}$ up to a constant. We will omit the identification of the constant stated in the theorem, it being a straightfoward computation using Selberg's integral formula for $Z_W(m)$. This completes the proof THEOREM 4.1.1, our main result of this section.

An immediate consequence of THEOREM 4.1.1 is that $\mathbb{E}\,\mathcal{N}_{m,q+1,N}(\mathbb{R}_+)$ is decreasing in q in the range $m \leq q < 2m$, agreeing with THEOREM 1.4 of Baugher [5]. We can also obtain a formula for the total number of critical points. We first define the expecteed density p_{m+1} of the empirical distribution of the eigenvalues of the Wishart ensemble; namely, for any bounded continuous function f,

$$\mathbb{E}_{m+1}\left[\frac{1}{m+1}\sum_{i=1}^{m+1} f(\lambda_i)\right] = \int_{\mathbb{R}_+} f(x)p_{m+1}(x)dx$$

Summing over k in (4.1.1), we obtain the following corollary.

Corollary 4.1.6.

$$\mathbb{E}\,\mathcal{N}_{m,N}(B) = \frac{2(m+1)(N-1)^{m+1}}{N}\int_{\frac{N}{N-1}B} e^{-(1-\frac{2}{N})\frac{m+1}{2}x}p_{m+1}(x)dx.$$

4.2. Proof of the main results

In this section we prove our main results stated in CHAPTER 2. In the first subsection, we finish off the proof of THEOREM 2.1.1 by proving a small lemma. In the second subsection, we present the proof of THEOREM 2.1.2.

4.2.1. Proof of the asymptotics

THEOREMS 4.1.1 and 3.2.1 together with Varadhan's lemma (see THEOREM 4.3.1 of Dembo and Zeitouni [11]) imply the first part of THEOREM 2.1.1. The second part of THEOREM 2.1.1 is a straightforward corollary of the following lemma.

Lemma 4.2.1. *For any $\epsilon > 0$, $\gamma \in (0, 1)$ and $\frac{k(m)}{m} \to \gamma$, there exists a constant $C = C(\epsilon)$ such that*

$$\mathbb{P}_m(\lambda_{k(m)} \notin (s_\gamma - \epsilon, s_\gamma + \epsilon)) \le e^{-Cm^2},$$

where s_γ is defined as in (2.1.2).

PROOF. This is an immediate consequence of the large deviation principle for $L_m = \frac{1}{m}\sum_{i=1}^{m} \delta_{\lambda_i}$ with respect to $\mathbb{P}_m$ whose rate function is minimized at the Marchenko-Pastur distribution μ_{MP} (see THEOREM 5.5.7 of Hiai and Petz [21]). To see this, we use the fact that

$$\mathbb{P}_m(\lambda_{k(m)} > s_\gamma + \epsilon) = \mathbb{P}_m\left(L_m(s_\gamma + \epsilon, \infty) \ge \frac{k(m)}{m}\right).$$

Since $\mu_{MP}(s_\gamma + \epsilon, \infty) < \mu_{MP}(s_\gamma, \infty) = \gamma$, there must exist a positive constant C such that for large m

$$\mathbb{P}_m\left(L_m(s_\gamma + \epsilon, \infty) \ge \frac{k(m)}{m}\right) \le \exp(-Cm^2)$$

An analogous argument can be made for $\mathbb{P}_m(\lambda_{k(m)} < s_\gamma - \epsilon))$, which we leave to the reader. $\square$

4.2.2. Computation of the density

THEOREM 2.1.2 is equivalent to the statement that for any Borel set B,

$$\mathbb{E}\,\mathcal{N}_{m,m,N}(B) = \int_B (N-1)^m (m+1)^2 e^{-\frac{(m+1)N}{2(N-1)}(2-\frac{2}{N}+m)x} dx$$

The crux of the proof lies in the following

Lemma 4.2.2. *The distribution of the smallest eigenvalue λ_m of the Wishart ensemble given by*

$$\mathbb{P}_m\left(\frac{m}{2}\lambda_m \geq x\right) = e^{-mx}.$$

PROOF. This is THEOREM 4.2 of Edelman [14] but we provide a short proof here. We have

$$\mathbb{P}_m\left(\frac{m}{2}\lambda_m \geq x\right) = \frac{1}{m! Z_W(m)} \int_{\frac{2x}{m}}^{\infty} \ldots \int_{\frac{2x}{m}}^{\infty} \Delta(\lambda) \exp\left(-\frac{m}{2}\sum_{i=1}^{m} \lambda_i\right) d\lambda.$$

Making a change of variable $\mu = \lambda - \frac{2x}{m}$ we see that the probability must be of the form of a constant times e^{-mx}, hence the result.. $\square$

Returning to the proof of THEOREM 2.1.2, we recall from (4.1.1) that

$$\mathbb{E}\,\mathcal{N}_{m,m,N}(B) = \frac{2(N-1)^{m+1}}{N}\mathbb{E}_{m+1}\left[e^{-(1-\frac{2}{N})\frac{m+1}{2}\lambda_{m+1}}; \lambda_{m+1} \in \frac{N}{N-1}B\right].$$

LEMMA 4.2.2 allows us to write

$$\begin{aligned}\mathbb{E}_{m+1}\left[e^{-(1-\frac{2}{N})\frac{m+1}{2}\lambda_{m+1}}; \lambda_{m+1} \in \frac{N}{N-1}B\right] &= \int_{\frac{(m+1)N}{2(N-1)}B} (m+1) e^{-(1-\frac{2}{N})x} e^{-(m+1)x} dx \\ &= \int_B \frac{(m+1)^2 N}{2(N-1)} e^{-\frac{(m+1)N}{2(N-1)}(2-\frac{2}{N}+m)u} du,\end{aligned}$$

where the second equality follows from the change of variables

$$u = \frac{2(N-1)}{(m+1)N}x.$$

Since this is true for any Borel set B, we obtain the desired result.

4.2.3. Asymptotics on the average number of critical points

Finally, we present the proof of THEOREM 2.1.5. To simplify the notation, we introduce

$$\psi(t) = \log(N-1) - \left(1 - \frac{2}{N}\right)\frac{t}{2}.$$

We first consider the case $x_N \geq 4$. We have the following inequalities:

$$\begin{aligned}\frac{2}{N}\mathbb{E}_{m+1}[e^{(m+1)\psi(\lambda_1)}; \lambda_1 \geq x_N] &\leq \frac{2(m+1)}{N}\int_{x_N}^{\infty} e^{(m+1)\psi(t)} p_{m+1}(t)dt \\ &\leq \frac{2(m+1)}{N} e^{(m+1)\psi(x_N)}\mathbb{P}_{m+1}(\lambda_1 \geq x_N).\end{aligned}$$

By COROLLARY 4.1.6, the middle expression is $\mathbb{E}\,\mathcal{N}_{m,N}(x,\infty)$. For the right hand side, THEOREM 3.2.1 yields

$$\lim_{m\to\infty}\frac{1}{m}\log\left[\frac{2(m+1)}{N}e^{(m+1)\psi(x_N)}\mathbb{P}_{m+1}(\lambda_1 \geq x_N)\right] = \psi(x_N) - I_{MP}(x_N).$$

For the left hand side, we apply Varadhan's lemma (THEOREM 4.3.1 of Dembo and Zeitouni [11]) in conjunction with THEOREM 3.2.1 to obtain

$$\lim_{m\to\infty}\frac{1}{m}\log\left[\frac{2}{N}\mathbb{E}_{m+1}[e^{(m+1)\psi(\lambda_1)}; \lambda_1 \geq x_N]\right] = \psi(x_N) + I_{MP}(x_N).$$

The use of Varadhan's lemma is justified because ψ is bounded from above and thus the tail condition in THEOREM 4.3.1 of Dembo and Zeitouni [11]) is satisfied.

We now consider the case $x_N < 4$. We can use the same inequality we used in the case $x_N \geq 4$ for the upper bound. Unfortunately, the lower bound given by this inequality is not sharp enough. To remedy this defect, we use a different inequality

$$\frac{2}{N}e^{(m+1)\psi(x_N+\epsilon)}\mathbb{P}_{m+1}(L_{m+1}[x_N,\infty) > 0) \leq \frac{2(m+1)}{N}\int_{x_N}^{\infty} e^{(m+1)\psi(t)}p_{m+1}(t)dt,$$

which holds for any positive ϵ. The LDP on L_m guarantees that

$$\mathbb{P}_m(L_m[x_N,\infty) > 0) \to 1$$

since the rate function for this LDP is minimized at the Marchenko-Pastur distribution on [0,4], which assigns positive measure to $[x_N,\infty)$. Hence,

$$\lim_{m\to\infty}\frac{1}{m}\log\left(\frac{2}{N}e^{(m+1)\psi(x_N+\epsilon)}\mathbb{P}_{m+1}(L_{m+1}[x_N,\infty) > 0)\right) = \psi(x_N+\epsilon).$$

Since ϵ is arbitrary and ψ is continuous, we are done.

Part 2

On the number of equilibria with a given number of unstable directions

CHAPTER 5

Differential equations on spheres

In this chapter, we introduce the setting of this part of the thesis. In the first section, we set out to define the notation needed to state the main results. In the next section, we present the main results of this part of the thesis.

5.1. Gaussian vector fields on spheres

The standard setup is to consider a first order ODE

$$\frac{dx}{dt} = F(x)$$

where F is a random vector field on $S^{N-1}(\sqrt{N})$ and attempt to describe the behavior of the possible solutions as $N \to \infty$. One natural starting point is to count the number $\mathcal{N}_{tot}$ of equilibrium points and to study the large-dimensional asymptotics of this quantity. This is the content of Fyodorov's work in [17]. In this part of the thesis we classify the equilibrium points by stability. For an equilibrium point σ and a neighborhood U around σ, we choose coordinates on U and the tangent space TU such that $\sigma = 0 \in \mathbb{R}^{N-1}$ and write F locally as a function $F : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ denoted by

$$F(x) = (c_1(x), \dots, c_{N-1}(x)).$$

We say σ is an equilibrium point with m unstable directions if the Jacobian matrix $JF(\sigma) := (\partial_i c_j(0))$ has exactly m eigenvalues with non-negative real part. While the explicit formula for $JF(\sigma)$ depends on the coordinates chosen, its eigenvalues do not. Our focus on this part of the thesis will be on the number $\mathcal{N}_m$ of equilibria with m unstable directions and its related large-dimensional asymptotics. In the case of a gradient flow (known in the literature as relaxational dynamics), this problem has been studied in great detail, as can be found in Auffinger, Ben Arous and Černý [3] and Fyodorov [16]. It is the purpose of this work to obtain the asymptotics in the non-relaxational case. We compute the exponential rate of $\mathbb{E}\mathcal{N}_m$ under very general conditions as the dimension goes to infinity. The methods in this paper will follow the tried-and-true approach of relating the problem to a matrix integral through the Kac-Rice formula, then invoking a large deviation principle (LDP) to obtain the asymptotics. To be more precise, let us make our assumptions on F explicit.

Classically, F is viewed as a map $F : S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N \to \mathbb{R}^N$ by identifying $S^{N-1}(\sqrt{N})$ with the usual $(N-1)$ dimensional sphere in $\mathbb{R}^N$ centered at 0 with radius $\sqrt{N}$ and for $x \in S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N$,

$$T_x S^{N-1}(\sqrt{N}) = \{v \in \mathbb{R}^N : \langle v, x \rangle = 0\}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$. With this framework, the vector fields considered in Fyodorov [17] take the form:

$$F(x) = -\lambda(x)x + f(x) + h$$

where $h = (h_1, ..., h_N)$ is an N-dimensional Gaussian vector with covariance structure

$$\mathbb{E}[h_i h_j] = \sigma^2 \delta_{ij}$$

for some $\sigma > 0$, δ_{ij} the usual Kronecker delta and f is an N-dimensional smooth Gaussian field with covariance kernel

$$\mathbb{E}[f_i(x) f_j(y)] = \delta_{ij} \Phi_1 \left(\frac{\langle x, y \rangle}{N} \right) + \frac{x_j y_i}{N} \Phi_2 \left(\frac{\langle x, y \rangle}{N} \right)$$

where Φ_1 and Φ_2 are smooth functions satisfying

$$0 < \Phi_1(1) < \Phi_1'(1), -\Phi_1(1) \leq \Phi_2(1) \leq \Phi_1'(1). \tag{5.1.1}$$

The Lagrange multiplier λ is chosen so that the vector belongs to $T_x S^{N-1}(\sqrt{N})$. Explicitly,

$$\lambda(x) = \frac{1}{N} \langle x, f(x) + h \rangle.$$

We also make the added assumption that h is independent of f. Before stating our results, we define two quantities which will play an important role in our analysis:

$$\tau = \frac{\Phi_2(1)}{\Phi_1'(1)}, \ b^2 = \frac{\sigma^2 + \Phi_1(1)}{\Phi_1'(1)}.$$

The restrictions given by (5.1.1) imply $-1 < \tau \leq 1$ and $b^2 + \tau \geq 0$. We additionally require that $b^2 + \tau > 0$ and restrict ourselves to the non-gradient case $\tau \neq 1$.

5.2. Asympotics for $\mathbb{E}\mathcal{N}_m$

In our first main result, we compute the exponential rate of the expected number $\mathbb{E}\mathcal{N}_m$ of critical points with m unstable directions:

Theorem 5.2.1. *For $b < 1$ and $-1 < \tau < 1$, we have:*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathcal{N}_m = \log \frac{1}{b} - \frac{(1-b^2)(1+\tau)}{2(b^2+\tau)}.$$

This result says that there exists a curve Γ given explicitly by

$$\Gamma = \left\{ (b,\tau) : \log \frac{1}{b} - \frac{(1-b^2)(1+\tau)}{2(b^2+\tau)} = 0 \right\}$$

such that if (b, τ) are not in the interior of Γ then we have exponentially abundant equilibria with m unstable directions and otherwise the probability of finding such equilibria is exponentially small. We remark that the case $b > 1$ is the "topologically trivial" case with only two equilibrium points in the limit (see Fyodorov [17]), so we will omit the analysis of this case.

We are also interested in the case when we have a diverging number of unstable directions. Let U_τ denote the uniform distribution on the ellipse

$$E_\tau = \left\{ (x,y) : \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \le 1 \right\},$$

$\gamma \in (0,1)$, and define $s_\gamma \in (-1-\tau, 1+\tau)$ to be the unique number such that

$$U_\tau(\operatorname{Re} z \ge s_\gamma) = \gamma.$$

Theorem 5.2.2. *Let $m(N)$ be a sequence of integers which satisfy $\frac{m(N)}{N} \to \gamma \in (0,1)$. Then,*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathcal{N}_{m(N)} = \log \frac{1}{b} - \frac{1-b^2}{2(b^2+\tau)(1+\tau)} s_\gamma^2.$$

Note that this quantity is maximized at $s_\gamma = 0$ which occurs at $\gamma = 1/2$. This allows us to recover estimates on the average total number of equilibria, since THEOREM 5.2.2 implies

$$\mathbb{E}[\mathcal{N}_{m(N)}] \leq \mathbb{E}[\mathcal{N}_{tot}] \leq (N+1)\mathbb{E}[\mathcal{N}_{m(N)}]$$

for N large and $m(N)/N \to 1/2$. Thus, both $\mathbb{E}[\mathcal{N}_{tot}]$ and $\mathbb{E}[\mathcal{N}_{m(N)}]$ have the exponential rate given by $\log \frac{1}{b}$ agreeing with PROPOSITION 2.5 of Fyodorov [16].

Our next result details the relationship between the value of the Lagrange multiplier and critical points. For a Borel set $B \subset \mathbb{R}$, define $\mathcal{N}_m(B)$ to be the number of equilibria with m unstable directions whose Lagrange multiplier has values in B.

Theorem 5.2.3. *For $-\infty \leq c < d \leq \infty$ and $\mathcal{N}_{m-1} = \mathcal{N}_{m-1}(c,d)$, we have:*

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathcal{N}_{m-1} = \begin{cases} -\infty & \text{if } d < (1+\tau)\sqrt{\Phi_1'(1)} \\ \log \frac{1}{b} - \frac{2(1-b^2)\Phi_1'(1)c^2}{(b^2+\tau)(1+\tau)} - mI_\tau\left(\frac{1}{\sqrt{\Phi_1'(1)}}c\right) & \text{if } c > (1+\tau)\sqrt{\Phi_1'(1)} \\ \log \frac{1}{b} - \frac{(1-b^2)(1+\tau)}{2(b^2+\tau)} & \text{else} \end{cases}$$

where I_τ is defined in LEMMA 6.2.1 below. Hence, the probability of finding equilibria with m unstable directions and a Lagrange multiplier less than $(1+\tau)\sqrt{\Phi_1'(1)}$ becomes exponentially small. Since the function

$$c \mapsto \frac{(1-b^2)c^2}{2\Phi_1'(1)(b^2+\tau)(1+\tau)} + mI_\tau\left(\frac{1}{\sqrt{\Phi_1'(1)}}c\right)$$

is increasing, unbounded and attains zero at $(1+\tau)\sqrt{\Phi_1'(1)}$, there is a unique point $z_0 > (1+\tau)\sqrt{\Phi_1'(1)}$ for which it is equal to $\log\left(\frac{1}{b}\right)$. The probability of finding equilibria with m unstable directions and a Lagrange multiplier larger than z_0 also becomes exponentially small. By the symmetry of the problem (and more explicitly by THEOREM 7.0.1), we have that $\mathcal{N}_m(B) = \mathcal{N}_{N-m}(-B)$ so we can make analogous statements about equilibria with m stable directions.

CHAPTER 6

On the Gaussian Elliptic Ensemble

In this chapter, we analyze the Gaussian Elliptic Ensemble (GEE). In the first section, we define the GEE and discuss ways in which we can talk about a density for the eigenvalues, even though formally there isn't one. In the following section we point out some properties of the logarithm potential function on an ellipse. In that last section, we use the results from the previous section to prove a large deviation result for the eigenvalues of this ensemble.

6.1. Definition of the GEE

We define the GEE as an $N \times N$ random matrix X whose entries are mean zero Gaussian random variables with covariance structure

$$\mathbb{E}[X_{ij}X_{lk}] := \mathbb{E}[X_{ij}X_{lk}] = \frac{1}{N}(\delta_{il}\delta_{jk} + \tau\delta_{ik}\delta_{jl}), 1 < \tau \leq 1$$

We can write the density of this measure against the Lebesgue measure dX on the space of real $N \times N$ matrices:

$$\mathbb{P}_N(dX) = \frac{1}{Z_N(\tau)} \exp\left(-\frac{N}{2(1-\tau^2)} \mathrm{Tr}(XX^T - \tau X^2)\right) dX$$

where

$$Z_N(\tau) = 2^{N/2}\pi^{N(N+1)/2}(1+\tau)^{N(N+1)/4}(1-\tau)^{N(N-1)/4}N^{\frac{N^2}{2}}.$$

We shall also think of the eigenvalues of X as ordered by decreasing real parts i.e., we will denote them by $\lambda_1(X), \ldots, \lambda_N(X)$ with $\lambda_1(X)$ having the largest real part, with the understanding that if we have complex eigenvalues, we list the ones with positive imaginary part first. Almost surely, this is a well-defined ordering. When there is no room for confusion, we will usually drop the X from the $\lambda(X)$ to ease the notation.

The purpose of this chapter is to state some properties of the GEE as well as to prove a large deviation principle for the eigenvalue with the mth largest real part. We first require a formula for the joint distribution of the eigenvalues. Since a matrix distributed like the GEE will have real eigenvalues with positive probability, the joint distribution of the eigenvalues will not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{C}^N$. Nevertheless, we can write out formulas if we restrict ourselves to the sets

$$S_k = \{X \text{ has exactly } k \text{ real eigenvalues}\}.$$

If $A \in S_k$, we can write the eigenvalues of A as $\sigma_1, ..., \sigma_k, x_1 \pm iy_1, ..., x_{\frac{N-k}{2}} \pm iy_{\frac{N-k}{2}}$. This suggests defining the measure

$$\mu^{(N,k)}(d\sigma, dx, dy) = 2^{(N-k)/2} \prod_{i=1}^{k} d\sigma_i \prod_{j=1}^{\frac{N-k}{2}} dx_j dy_j$$

on the set

$$\{(\sigma, x, y) \in \mathbb{R}^k \times \mathbb{R}^{N-k} : \sigma_1 > ... > \sigma_k, x_1 > ... > x_{\frac{N-k}{2}}, y_i \geq 0 \, \forall i\}.$$

If we define the measure $\mathbb{P}_{N,k}$ by $\mathbb{P}_{N,k}(V) = \mathbb{P}_N(\lambda(X) \in V, X \in S_k)$ for any Borel set $V \subset \mathbb{C}^N$, then $\mathbb{P}_{N,k}$ is absolutely continuous with respect to $\mu^{(N,k)}$ and the density is given by

$$\mathbb{P}_{N,k}(d\sigma, dx, dy) = \prod_{i=1}^{k} e^{-\frac{N}{2(1+\tau)}\lambda_i^2} \prod_{j=1}^{\frac{N-k}{2}} e^{-\frac{N}{1+\tau}(x_j^2-y_j^2)} \mathrm{erfc}\left(\sqrt{\frac{2N}{1-\tau^2}}y_j\right) \times$$
$$\frac{1}{K_N(\tau)}|\Delta(\sigma, x \pm iy)|\mu^{(N,k)}(d\sigma, dx, dy)$$

where

(6.1.1) $$K_N(\tau) = 2^{N(N+1)/4}(1+\tau)^{N/2}N^{\binom{N+1}{2}/2}\prod_{j=1}^{N}\Gamma(j/2),$$

$\Delta(\sigma, x \pm iy)$ is the Vandermonde polynomial

$$|\Delta(\sigma, x \pm iy)|^2 = \prod_{u,v\in S, u\neq v} |u - v|, S = \{\sigma_1, ..., \sigma_k, x_1 \pm iy_1, ..., x_{\frac{N-k}{2}} \pm iy_{\frac{N-k}{2}}\}$$

and erfc is the complementary error function

$$\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt.$$

See Lehmann and Sommers [24].

We will find it convenient to rewrite the integrand in a more compact way, namely as

$$\frac{1}{K_N(\tau)}|\Delta(\lambda)| \exp\left(-\frac{N}{2(1+\tau)}\sum_{j=1}^{N}\lambda_j^2\right)\prod_{j=1}^{N}\sqrt{\mathrm{erfc}\left(\sqrt{\frac{2N}{1-\tau^2}}|\mathrm{Im}\,\lambda_j|\right)}$$

with the understanding that $\lambda_j = x_j + iy_j$ if λ is complex, and $\lambda_j = \sigma_j$ if it's real. The disadvantage of this form is that it obscures the dependence of k and the symmetry obtained from the fact that the complex eigenvalues come in pairs.

We find it useful to get rid of the ordering of the eigenvalues by their real parts. We can do this by simply replacing our region of integration by $\{(\sigma, x, y) : y_i \geq 0 \, \forall i\}$ and dividing by the factor of $k! \left(\frac{N-k}{2}\right)!$

6.2. The logarithm potential on an ellipse

In this chapter, we recall some properties of the logarithmic potential for the uniform distribution U_τ on the ellipse E_τ

$$E_\tau = \left\{(x,y) : \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1\right\}, -1 < \tau < 1.$$

The logarithmic potential of U_τ is defined as a function $\phi_\tau : \mathbb{R}^2 \to \mathbb{R}$ explicitly given by

$$\phi_\tau(x,y) = \int_{E_\tau} \log|x+iy-w| \; U_\tau(dw).$$

We summarize the properties of ϕ_τ we will need in the lemma below.

Lemma 6.2.1. *On the set $\{x \geq 1+\tau, y \geq 0\}$:*

(1) $\phi_\tau(x,0) + \frac{1}{2} - \frac{x^2}{2(1+\tau)} = -I_\tau(x)$ *where*

$$I_\tau(x) := \begin{cases} \frac{1}{2(1+\tau)}x^2 - \frac{x(x-\sqrt{x^2-4\tau})}{4\tau} - \log(\frac{x+\sqrt{x^2-4\tau}}{2}) & \text{if } \tau \neq 0 \\ -\log x + \frac{1}{2}x^2 - \frac{1}{2} & \text{if } \tau = 0 \end{cases}$$

(2) $\partial_x \phi_\tau(x,y) \leq \frac{x}{1+\tau}$

(3) $\partial_y \phi_\tau(x,y) \leq \frac{y}{1-\tau}$

PROOF. In the case $\tau = 0$, we can compute $\phi_0(x,y) = \frac{1}{2}\log(x^2+y^2)$ from which we can verify all the results instantly. Henceforth, we shall assume $\tau \neq 0$.

From page 9 of Bell et. al [7], we have for $z = x + iy$:

$$\partial_x \phi_\tau(x,y) - i\partial_y \phi_\tau(x,y) = \int_{E_\tau} \frac{1}{z-w} U_\tau(dw) = \frac{1}{2\tau}\left(z - \sqrt{z^2 - 4\tau}\right).$$

In particular, this implies

$$\partial_x \phi_\tau(x,0) = \frac{1}{2\tau}\left(x - \sqrt{x^2 - 4\tau}\right).$$

By integration, we obtain

$$\phi_\tau(x,0) - \phi_\tau(1+\tau,0) = \frac{1}{2(1+\tau)}x^2 - \frac{1+\tau}{2} - I_\tau(x).$$

By LEMMA 5.3.12 of Hiai and Petz [21], we know that $\phi_\tau(1+\tau,0) = \frac{\tau}{2}$ thus yielding the first statement. Statements (2) and (3) follow from the proof of LEMMA 5.3.12 of Hiai and Petz [21]. $\square$

Since $y \mapsto \phi_\tau(x,y)$ is an even function of y for all x, we have the following corollary to LEMMA 6.2.1:

Corollary 6.2.2. *Define* $\Psi_\tau(x,y) = \phi_\tau(x,y) - \frac{x^2}{2(1+\tau)} - \frac{y^2}{2(1-\tau)}$. *For* $x \geq 1+\tau$, $y \in \mathbb{R}$ *and* $\tau \in (-1,1)$,

$$\sup_{u \geq x, v \geq y} \Psi_\tau(u,v) = \Psi_\tau(x,y) \leq \Psi_\tau(x,0).$$

6.3. Large deviation principle for $\lambda_m(X)$

In this section, we establish a result of large deviation type for $\lambda_m(X)$.

Theorem 6.3.1. *Under $\mathbb{P}_N$, the quantity $\lambda_m \cdot \mathbb{1}_{\mathbb{R}}(\lambda_m)$ satisfies a large deviation principle with speed N and good rate function mI_τ where*

$$I_\tau(x) = \begin{cases} \infty & \text{if } x < 1+\tau \\ \frac{1}{2(1+\tau)}x^2 - \frac{x(x-\sqrt{x^2-4\tau})}{4\tau} - \log(\frac{x+\sqrt{x^2-4\tau}}{2}) & \text{if } x \geq 1+\tau, \tau \neq 0 \\ -\log x + \frac{1}{2}x^2 - \frac{1}{2} & \text{if } x \geq 1+\tau, \tau = 0. \end{cases}$$

In order to prove Theorem 6.3.1, we will first need an exponential tightness result:

Lemma 6.3.2 (Exponential tightness from the right)**.** *The following limit holds:*

$$\lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N \left(|\mathrm{Re}\, \lambda_1(X)| \geq M \text{ or } \max_j \mathrm{Im}\, \lambda_j(X) > M \right) = -\infty.$$

PROOF. We shall only prove

$$\lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N \left(\max_i \mathrm{Im}\, \lambda_i > M \right) = -\infty$$

since an analogous argument will provide a proof for the statement involving $|\mathrm{Re}\, \lambda_1|$. We shall make use of the following four inequalities:

(1) There exists $\alpha > 0$ such that for $|z|$ for sufficient large and $w \in \mathbb{C}$,

$$|z-w||\bar{z}-w| \exp\left[-\frac{|w|^2}{1+|\tau|}\right] \leq (|z|+|w|)^2 \exp\left[-\frac{|w|^2}{1+|\tau|}\right] \leq \alpha^2 \exp\left[\frac{|z|^2}{2(1+|\tau|)}\right].$$

(2) For any $z = x + iy \in \mathbb{C}$,

$$\exp\left[-\frac{N}{2(1+\tau)}(z^2+\bar{z}^2)\right] \operatorname{erfc}\left[\sqrt{\frac{2N}{1-\tau^2}}|y|\right] \leq \frac{\exp\left[\frac{-N|z|^2}{1-\tau}\right]}{\sqrt{\frac{2N}{1-\tau^2}y^2} + \sqrt{\frac{2N}{1-\tau^2}y^2 + \frac{4}{\pi}}} \tag{6.3.1}$$

$$\leq \exp\left[\frac{-N|z|^2}{1+|\tau|}\right]. \tag{6.3.2}$$

(3) For any $y \in \mathbb{R}$,

$$\operatorname{erfc}\left[\sqrt{\frac{2N}{1-\tau^2}}|y|\right] \leq \sqrt{\pi}\operatorname{erfc}\left[\sqrt{\frac{2(N-2)}{1-\tau^2}}|y|\right] \operatorname{erfc}\left[\sqrt{\frac{4}{1-\tau^2}}|y|\right] \times \left(\sqrt{\frac{4y^2}{1-\tau^2}} + \sqrt{\frac{4y^2}{1-\tau^2}+2}\right)$$

(4) There exists a C such that for all $y \in \mathbb{R}$,

$$\sqrt{\pi}\frac{\sqrt{\frac{4y^2}{1-\tau^2}} + \sqrt{\frac{4y^2}{1-\tau^2}+2}}{\sqrt{\frac{4}{1-\tau^2}y^2} + \sqrt{\frac{4}{1-\tau^2}y^2 + \frac{4}{\pi}}} \leq C^{-2}$$

We shall establish an inequality for $\mathbb{P}_{N,k}(|\mathrm{Im}\,\lambda_l| > M)$ for an arbitrary l since

$$\mathbb{P}_N\left(\max_j \mathrm{Im}\,\lambda_j > M\right) \leq \sum_k \sum_l \mathbb{P}_{N,k}(|\mathrm{Im}\,\lambda_l| > M).$$

We use this to obtain the following bound for $\mathbb{P}_{N,k}(|\mathrm{Im}\,\lambda_l| > M)$:

$$\int \exp\left(-\frac{N}{2(1+\tau)}\sum_{j=1}^m \lambda_j^2\right) \prod_{j=1}^N \sqrt{\operatorname{erfc}\left(\sqrt{\frac{2N}{1-\tau^2}}|\mathrm{Im}\,\lambda_j|\right)} \frac{|\Delta(\lambda)|}{K_N(\tau)} d\mu^{(N,k)} \leq$$

$$\frac{\alpha^{N+k-2}}{C^{N-k-2}} \frac{K_{N-2}(\tau)}{K_N(\tau)} \int_{-\infty}^{\infty}\int_M^{\infty} 2y_l \exp\left(-\frac{(N-k+2)(x_l^2+y_l^2)}{2(1+|\tau|)}\right) \mathbb{P}_{N-2,k}(S_k)\, dx_l dy_l$$

where the first integral is over $\{\text{Im } \lambda_l > M\}$. The inequality follows from the fact that we can split the exponential by

$$\exp\left(-\frac{N}{2(1+\tau)}\lambda_j^2\right) = \exp\left(-\frac{N-2}{2(1+\tau)}\lambda_j^2\right)\exp\left(-\frac{2}{2(1+\tau)}\lambda_j^2\right)$$

and use inequality (3) to split the erfc term into three parts, whose first factor coupled with the first factor of the exponential and the appropiate factors from the Vandermonde polynomial yields the density for $\mathbb{P}_{N-2,k}$. We then use inequality (2) with $z = \sqrt{\frac{2}{N}}\lambda_j$ for $j = 1, ..., N$ to get rid of our remaining erfc factors. This also allows us to bound the term coming from the last factor on the right hand side of inequality (3) by C^2. We then use inequality (1) to bound the remaining terms from the Vandermonde polynomial with the exception of $|\lambda_l - \bar{\lambda}_l|$, which is equal to $2y_l$. Finally, we invoke inequality (2) again to deal with the remaining erfc factors involving λ_l. After some simplification, the inequality follows. Without loss of generality, we can assume $C \leq 1$ and hence $C^{N-k-2} \geq C^{N-2}$ and $\alpha^{N+k-2} \geq \alpha^{2N-2}$. We can sum over l then over k to obtain:

$$\begin{aligned}
\mathbb{P}_N\left(\max_j \text{Im } \lambda_j(X) > M\right) &\leq C^{2-N}\alpha^{2N-2}\frac{K_{N-2}(\tau)}{K_N(\tau)}N^2\int_{-\infty}^{\infty}\int_M^{\infty} 2ye^{-\frac{(N+2)}{2(1+|\tau|)}(x^2+y^2)}dydx \\
&\leq \frac{2(1+|\tau|)N^2}{N-k+2}\frac{\alpha^{2N-2}}{C^{N-2}}\frac{K_{N-2}(\tau)}{K_N(\tau)}e^{-\frac{N-k+2}{2(1+|\tau|)}M^2}\int_{-\infty}^{\infty} e^{-\frac{(N-k+2)}{2(1+|\tau|)}x^2}dx \\
&= \sqrt{\frac{2\pi(1+|\tau|)}{N-k+2}}C^{2-N}\alpha^{2N-2}\frac{K_{N-2}(\tau)}{K_N(\tau)}N^2e^{-\frac{N-k+2}{2(1+|\tau|)}M^2}.
\end{aligned}$$

where N^2 factors comes from the fact that the number of terms is bounded by N^2. The final exponential term will yield the desired asymptotics. □

Finally, we'll need the two more preliminary results.

(a) Define $\mathcal{P}(\mathbb{C})$ to be the space of probability measures on $\mathbb{C}$ which are invariant under complex conjugation endowed with a metric compatible with the usual weak convergence of measures. The empirical measure of the eigenvalues $L_N = \frac{1}{N}\sum_{j=1}^{N}\delta_{\lambda_j}$ with respect to $\mathbb{P}_N$ satisfies an LDP of speed N^2 on $\mathcal{P}(\mathbb{C})$ whose rate function is minimized at the uniform distribution U_τ on the ellipse E_τ. The proof of this LDP for the case $\tau = 0$ can be found in Ben Arous and Zeitouni [8] but the same argument extends to $\tau \in (-1, 1)$.

(b) The functional

$$\Psi(\mu, x + iy) = \int_{\mathbb{C}} \log|x + iy - z| d\mu(z) - \frac{x^2}{2(1+\tau)} - \frac{y^2}{2(1-\tau)}$$

defined on $\mathcal{P}(\mathbb{C}) \times \mathbb{C}$ is upper-semicontinuous when restricted to $\mathcal{P}(B_M) \times B_M$ for any $M > 0$, and in fact is continuous when restricted to $\mathscr{P}(B_M) \times (B_M \cap \{z : \operatorname{Re} z > x\})$ for any $x > 1 + \tau$. The distribution U_τ is related to the rate function I_τ by

$$I_\tau(x) = -\Psi(U_\tau, x) - \frac{1}{2} \tag{6.3.3}$$

as per (1) of LEMMA 6.2.1 since $\Psi(U_\tau, x) = \Psi_\tau(x, 0)$.

With these preparations in order, we can begin the proof of THEOREM 6.3.1.

PROOF OF THEOREM 6.3.1. It is obvious that I_τ is a good rate function. Our theorem will follow if we can prove the following equalities:

(1) $\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N(\lambda_m \in [0, x)) = -\infty$ for $0 < x < 1 + \tau$.

(2) $\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N(\lambda_m \in [x, \infty)) = -mI_\tau(x)$ for $x > 1 + \tau$.

To prove the first equality, we note that by definition, $\operatorname{Re} \lambda_m(X) < x$ for some $x < 1 + \tau$ is equivalent to $L_N(z : x \le \operatorname{Re} z < 1 + \tau) \le \frac{m-1}{N}$. Since $\mu_\tau[x, 1 + \tau) > 0$, there exist constants $K, \kappa > 0$ such that if $x < 1 + \tau$ then

$$\mathbb{P}_N(\mathrm{Re}\, \lambda_m(X) < x) \leq \mathbb{P}_N\left(L_N(z : x \leq \ \mathrm{Re}\, z < 1 + \tau) \leq \frac{m-1}{N}\right) \leq K e^{-\kappa N^2}.$$

This inequality implies the result.

We now turn to the proof of the second equality. In view of LEMMA 6.3.2, the second equality is equivalent to the following equality for sufficiently large M satisfying $1 + \tau < x < M$:

$$\lim_{N \to \infty} \frac{1}{N} \log \mathbb{P}_N\left(\lambda_m(X) \in [x, M], |\mathrm{Re}\, \lambda_1(X)| \leq M, \max_j \mathrm{Im}\, \lambda_j(X) \leq M\right) = -m I_\tau(x).$$

We will estimate this by decomposing $\mathbb{P}_N$ into a sum of the $\mathbb{P}_{N,k}$ for admissible k. To that end, fix k for the time being, introduce new variables $\tilde{\lambda}_j = \sqrt{\frac{N}{N-m}}\lambda_j$ for $1 \leq j \leq N$. On the larger set of

$$\left\{\tilde{\lambda}_m \in [x, 2M], \max_j(\mathrm{Im}\tilde{\lambda}_j, |\mathrm{Re}\tilde{\lambda}_1|) \leq 2M\right\} \supset \left\{\lambda_m \in [x, M], \max_j(\mathrm{Im}\, \lambda_j, |\mathrm{Re}\lambda_1| \leq M\right\}$$

we have the $|\tilde{\lambda}_j - \tilde{\lambda}_i| \leq 4\sqrt{2}M$ and hence we obtain the following upper bound for $\mathbb{P}_{N,k}(d\lambda)$:

$$\frac{1}{K_N}|\Delta(\lambda)|\exp\left(-\frac{N}{2(1+\tau)}\sum_{j=1}^{N}\lambda_j^2\right)\prod_{j=1}^{m}\sqrt{\operatorname{erfc}\left(\sqrt{\frac{2N}{1-\tau^2}}|\operatorname{Im}\lambda_j|\right)}d\mu^{(N,k)}(\lambda)$$

$$\leq \sum_l \prod_{1\leq i<j\leq m}|\tilde{\lambda}_j-\tilde{\lambda}_i|\prod_{j=1}^{m}e^{-\frac{N-m}{2(1+\tau)}\tilde{\lambda}_j^2}\sqrt{\operatorname{erfc}\sqrt{\frac{2(N-m)}{1-\tau^2}}|\operatorname{Im}\tilde{\lambda}_j|}\,\mu^{(m,l)}(d\tilde{\lambda}_1,...,d\tilde{\lambda}_m)\times$$

$$\frac{K_{N-m}}{K_N}\frac{(k-l)!\left(\frac{N-m+l-k}{2}\right)!}{k!\left(\frac{N-k}{2}\right)!}\left(\frac{N-m}{N}\right)^{\frac{N(N+1)}{4}}\prod_{i=1}^{m}\prod_{j=m+1}^{N}|\tilde{\lambda}_i-\tilde{\lambda}_j|\mathbb{P}_{N-m,k-l}(d\tilde{\lambda}_{m+1},...,d\tilde{\lambda}_N)$$

$$\leq (4\sqrt{2}M)^{\frac{m(m-1)}{2}}\frac{K_{N-m}}{K_N}\left(\frac{N-m}{N}\right)^{\frac{N(N+1)}{4}}\sum_l\frac{(k-l)!\left(\frac{N-m+l-k}{2}\right)!}{k!\left(\frac{N-k}{2}\right)!}\mu^{(m,l)}(d\tilde{\lambda}_1,...,d\tilde{\lambda}_m)\times$$

$$\exp\left[(N-m)\sum_{i=1}^{m}\Psi(\tilde{L}_{N-m},\tilde{\lambda}_i)\right]\mathbb{P}_{N-m,k-l}(d\tilde{\lambda}_{m+1},...,d\tilde{\lambda}_N)$$

where

$$\tilde{L}_{N-m}=\frac{1}{N-m}\sum_{j=m+1}^{N}\delta_{\tilde{\lambda}_j(X)}.$$

and we've omitted the dependence on τ on K_N for the sake of space. The factor of $1/(k!(\frac{N-k}{2})!)$ arises from removing the ordering on the eigenvalues by real parts and the factor of $(k-l)!(\frac{N-m+l-k}{2})!$ comes from ordering the last $N-m$ eigenvalues by real parts with the assumption that $k-l$ of them are real. For $\epsilon>0$, let $\mathbb{B}_\epsilon\subset\mathscr{P}(B_M)$ be the ball of radius ϵ around U_τ and $\mathbb{B}_\epsilon^c$ its complement. On the set

$$\{\tilde{\lambda}_m\in[x,2M],\max_j\operatorname{Im}\tilde{\lambda}_j(X)\leq 2M\},$$

we have

$$\exp\left[(N-m)\sum_{i=1}^{m}\Psi(\tilde{L}_{N-m},\tilde{\lambda}_i)\right]\leq(4\sqrt{2}M)^{m(N-m)},$$

we can further bound the exponential factor according by whether $\tilde{L}_{N-m}$ is in $\mathbb{B}_\epsilon$ or not, i.e,

$$\exp\left[(N-m)\sum_{i=1}^{m}\Psi(\tilde{L}_{N-m},\lambda_i)\right]\leq\exp\left[m(N-m)\sup_{\mu\in\mathbb{B}_\epsilon,x\leq\operatorname{Re}z\leq M}\Psi(\mu,z)\right]$$
$$+(4\sqrt{2}M)^{m(N-m)}\mathbb{1}_{\mathbb{B}_\epsilon^c}(\tilde{L}_{N-m}).$$

We can now perform the integral with respect to $\tilde{\lambda}_1,\ldots,\tilde{\lambda}_m$. Since we have removed all appearances of the variables of integration from the integrand and we are integrating over a compact region, the integral is finite for any l, and since l belongs to a finite set, we can bound it by a constant γ independent of N, l, and k. For any set A, the inequality $\mathbb{P}_{N-m,k-l}(A)\leq\mathbb{P}_{N-m}(A)$ allows us to replace $\mathbb{P}_{N-m,k-l}$ by $\mathbb{P}_{N-m}$.

We continue by summing over admissible l. Since we've eliminated any dependencies on l, we can replace the sum over l by a factor of k. By integrating over the remaining portion of our domain of integration and using the fact that

$$\frac{(k-l)!\left(\frac{N-m+l-k}{2}\right)!}{k!\left(\frac{N-k}{2}\right)!}\leq 1,$$

we show the following upper bound:

$$\mathbb{P}_{N,k}\left(\lambda_m \in [x,M], |\mathrm{Re}\,\lambda_1| \leq M, \max_j \mathrm{Im}\,\lambda_j \leq M\right) \leq$$
$$\left\{\exp\left[m(N-m) \sup_{\mu\in\mathbb{B}_\epsilon, x\leq \mathrm{Re}\,z\leq M} \Psi(\mu,z)\right] + (4\sqrt{2}M)^{m(N-m)}\mathbb{P}_{N-m}(\tilde{L}_{N-m}\in\mathbb{B}_\epsilon^c)\right\}\times$$
$$k\gamma(4\sqrt{2}M)^{\frac{m(m-1)}{2}}\frac{K_{N-m}(\tau)}{K_N(\tau)}\left(\frac{N-m}{N}\right)^{\frac{N(N+1)}{4}}$$

The only dependence on k on the right hand side comes from the factor k, so when we sum over k we can just replace it with $N(N-1)/2$. At this we point, we make the following two observations: the first is that $\tilde{L}_{N-m}$ satisfies the same LDP as L_N. This implies that there exists $c>0$ such that

$$\mathbb{P}_{N-m}(\tilde{L}_{N-m}\notin\mathbb{B}_\epsilon) \leq e^{-cN^2}$$

hence the second term on the right hand side of our upper bound is negligible in the limit. The second observation is that from (6.1.1) we have

$$\lim_{N\to\infty}\frac{1}{N}\log\left[\frac{K_{N-m}(\tau)}{K_N(\tau)}\left(\frac{N-m}{N}\right)^{\frac{N(N+1)}{4}}\right] = \frac{m}{2}. \tag{6.3.4}$$

We use these two observations to establish the following inequality:

$$\limsup_{N\to\infty}\frac{1}{N}\log\mathbb{P}_N\left(\lambda_m\in[x,M], \max_j(|\mathrm{Re}\,\lambda_1|, \mathrm{Im}\,\lambda_j)\leq M\right) \leq \frac{m}{2} + m\sup_{\substack{\mu\in\mathbb{B}_\epsilon\\ x\leq \mathrm{Re}\,z<M}}\Psi(\mu,z).$$

The second term on the right hand side can by computed explicitly,

$$\lim_{\epsilon\downarrow 0} \sup_{\substack{\mu\in\mathbb{B}_\epsilon \\ x\leq \mathrm{Re}\, z<M}} \Psi(\mu,z) = \Psi(U_\tau,x) = -I_\tau(x) - \frac{1}{2}$$

where the first equality follows from upper semi-continuity of Ψ and COROLLARY 6.2.2 and the second equality follows from (6.3.3). This proves the upper bound for the equality (2) stated at the beginning of the proof.

To obtain the lower bound, we fix $y > x > r > 1+\tau$ and $\epsilon,\delta > 0$. We first need a lower bound analogous to (6.3.1). We can obtain one if we restrict ourselves to $|\mathrm{Im}\, z| \leq \delta$ and N large enough:

$$\exp\left[-\frac{N-m}{2(1+\tau)}(z^2+\bar{z}^2)\right] \mathrm{erfc}\left[\sqrt{\frac{2(N-m)}{1-\tau^2}}|\mathrm{Im}\, z|\right] \geq \frac{\beta}{\sqrt{N}} \exp\left[\frac{-(N-m)|z|^2}{1-\tau}\right]$$

for some positive constant $\beta < 1$ depending on δ. Retaining the previous notation from the upper bound, we further define $\mathbb{B}_\epsilon \cap \mathscr{P}(B_r)$ to mean the set of measures in $\mathbb{B}_\epsilon$ whose support is contained in the ball $B_r \subset \mathbb{C}$ of radius r. On the set

$$\left\{\tilde{\lambda}_m(X) \in \left[\sqrt{\frac{N}{N-m}}x, y\right], \mathrm{Im}\, \tilde{\lambda}_j(X) \leq \delta\, , |\mathrm{Re}\, \lambda_1(X)| \leq y, |\lambda_j(X)| \leq r\, \forall j\right\}$$

which is a subset of $\left\{\lambda_m(X) \in [x,M], \max_j \mathrm{Im}\, \lambda_j(X), |\mathrm{Re}\, \lambda_1(X)| \leq M\right\}$, we can find a lower bound on the density $\mathbb{P}_{N,k}(d\lambda)$ by noting that its equal to the following quantity:

$$\sum_l \prod_{1\le i<j\le m} |\tilde\lambda_j - \tilde\lambda_i| \prod_{j=1}^m e^{-\frac{N-m}{2(1+\tau)}\tilde\lambda_j^2} \sqrt{\operatorname{erfc}\left(\sqrt{\frac{2(N-m)}{1-\tau^2}}|\mathrm{Im}\,\tilde\lambda_j|\right)}\, \mu^{(m,l)}(d\tilde\lambda_1,...,d\tilde\lambda_m)\times$$

$$\frac{K_{N-m}(\tau)}{K_N(\tau)} \frac{(k-l)!\left(\frac{N-m+l-k}{2}\right)!}{k!\left(\frac{N-k}{2}\right)!} \left(\frac{N-m}{N}\right)^{\frac{N(N+1)}{4}} \prod_{i=1}^m \prod_{j>m}^N |\tilde\lambda_i - \tilde\lambda_j| \mathbb{P}_{N-m,k-l}(d\tilde\lambda_{m+1},...,d\tilde\lambda_N)$$

$$\ge \sum_l \left(\frac{N-m}{N}\right)^{\frac{N(N+1)}{4}} \frac{(k-l)!\left(\frac{N-m+l-k}{2}\right)!}{k!\left(\frac{N-k}{2}\right)!} \prod_{1\le i<j\le m} |\tilde\lambda_j - \tilde\lambda_i|\, \mu^{(m,l)}(d\tilde\lambda_1,...,d\tilde\lambda_m)\times$$

$$\frac{K_{N-m}(\tau)}{K_N(\tau)} \mathbb{1}_{\mathbb{B}_\epsilon\cap\mathcal{P}(B_r)}(\tilde L_{N-m}) \exp\left[m(N-m) \inf_{\substack{\mu\in\mathbb{B}_\epsilon\cap\mathscr{P}(B_r)\\ x\le\mathrm{Re}\,z\le y, |\mathrm{Im}\,z|<\delta}} \Psi(\mu,z)\right]\times$$

$$\left(\frac{\beta}{\sqrt{N}}\right)^{\frac{m-l}{2}} \mathbb{P}_{N-m,k-l}(d\tilde\lambda_{m+1},...,d\tilde\lambda_N)$$

We now point out two quantities which will end up becoming negligible in the limit. We first proceed by integrating out the $\tilde\lambda_1,\ldots,\tilde\lambda_m$ variables which yields a finite quantity since we are integrating over a bounded region and the integrand is bounded. Moreover, this quantity is bounded both from above and from below by constants independent of N so this term will be neglible in the limit. The second quantity which is also negligible in the limit is the one appearing in the following limit which holds for $l \le m$ and all k:

$$\lim_{N\to\infty} \frac{1}{N} \log\left[\frac{(k-l)!\left(\frac{N-m+l-k}{2}\right)!}{k!\left(\frac{N-k}{2}\right)!}\right] = 0. \tag{6.3.5}$$

Since the inequality

$$\sum_k \sum_l \mathbb{P}_{N-m,k-l}(A) \geq \mathbb{P}_{N-m}(A)$$

is true for any Borel set A, we can use (6.3.4) and (6.3.5) to obtain the following lower bound:

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{P}_N \left(\lambda_m \in [x, M]\right) \geq \frac{m}{2} + m \lim_{\epsilon \downarrow 0} \inf_{\substack{\mu \in \mathbb{B}_\epsilon \cap \mathscr{P}(B_r) \\ x \leq \mathrm{Re}\, z \leq y, |\mathrm{Im}\, z| < \delta}} \Psi(\mu, z)$$

By continuity of Ψ and COROLLARY 6.2.2, we obtain:

$$\lim_{\epsilon \downarrow 0} \inf_{\substack{\mu \in \mathbb{B}_\epsilon \cap \mathscr{P}(B_r) \\ x \leq \mathrm{Re}\, z \leq y, |\mathrm{Im}\, z| < \delta}} \Psi(\mu, z) = \Psi(U_\tau, y + \delta i).$$

Finally, we take $\delta \to 0$ then $y \to x$ and use the continuity of Ψ coupled with (6.3.3) to obtain the desired lower bound for equality (2) stated at the beginning of the proof.

□

CHAPTER 7

Expected number of equilibria and the Gaussian Elliptic Ensemble

In this chapter, we establish the connection between the GEE and the expected number of equilibria. In the first section, we make this connection explicit through THEOREM 7.0.1. In the second section, we provide a proof of this theorem. In the last section, we use THEOREM 7.0.1 to prove THEOREM 5.2.1 and THEOREM 5.2.2.

We now relate $\mathcal{N}_m$ to the eigenvalue with mth largest real part of an $N \times N$ GEE matrix.

Theorem 7.0.1. *For a Borel set $B \subset \mathbb{R}$, we have:*

$$\mathbb{E}\mathcal{N}_m(B) = 2\sqrt{\frac{1+\tau}{b^2+\tau}}b^{1-N}\mathbb{E}_N\left[\exp\left(-\frac{N(1-b^2)}{2(b^2+\tau)(1+\tau)}\lambda_m^2(X)\right)\mathbb{1}_B\left(\sqrt{\Phi_1'(1)}\lambda_m(X)\right)\right].$$

The proof of THEOREM 7.0.1 will follow from two results. The first relates $\mathbb{E}\mathcal{N}_m(B)$ to a matrix integral through the Kac-Rice formula adapted to our setting.

Theorem 7.0.2. *For a matrix A and nonnegative integer m, set*

$$i_m(A) = \begin{cases} 1 & \text{if } A \text{ has exactly } m \text{ eigenvalues with nonnegative real part} \\ 0 & \text{if else} \end{cases}$$

then $\mathbb{E}\mathcal{N}_m(B)$ *is equal to* $\frac{2\sqrt{N-1}^N}{2^{N/2}\Gamma(N/2)}\frac{b^{1-N}}{\sqrt{b^2+\tau}}$ *multiplied by*

$$\int_{-\infty}^{\infty} \mathbb{1}_{\sqrt{\frac{N}{N-1}}B}\left(\sqrt{\Phi_1'(1)}\lambda\right) e^{-\frac{(N-1)\lambda^2}{2(b^2+\tau)}} \mathbb{E}_{N-1}\left[\left|\det\left(X-\lambda I\right)\right| i_m(X-\lambda I)\right] d\lambda$$

The second result relates the complicated integral against $\mathbb{P}_{N-1}$ appearing in THEOREM 7.0.2 to a simpler one against $\mathbb{P}_N$.

Lemma 7.0.3. *For any bounded Borel measurable function* f *on* $\mathbb{R}$*, we have*

$$\int_{-\infty}^{\infty} f(t\sqrt{N-1}) \exp\left(-\frac{N-1}{2(1+\tau)}t^2\right) \mathbb{E}_{N-1}\left[\left|\det\left(X-tI\right)\right| i_m(X-tI)\right] dt$$
$$= \frac{\Gamma(N/2)\sqrt{2}^N\sqrt{1+\tau}}{\sqrt{N-1}^N}\mathbb{E}_N[\mathbb{1}_{\mathbb{R}}(\lambda_{m+1}(X))f(\sqrt{N}\cdot\lambda_{m+1}(X))].$$

Given these two results, the proof of THEOREM 7.0.1 goes as follows.

PROOF OF THEOREM 7.0.1. If we apply LEMMA 7.0.3 to the function

$$f(x) = \mathbb{1}_{\sqrt{N}B}\left(\sqrt{\Phi_1'(1)}x\right) \exp\left(-\frac{1-b^2}{2(1+\tau)(b^2+\tau)}x^2\right)$$

then we can use the resulting equality to simplify the formula in THEOREM 7.0.2 to recover the expression on the right of THEOREM 7.0.1 and hence conclude the result.

$\square$

We relegate the proof THEOREM 7.0.2 to the next section and finish the current subsection with a proof of LEMMA 7.0.3.

PROOF OF LEMMA 7.0.3. We first remark that $\Delta(\lambda(X), t) = |\det(X - tI)|\Delta(\lambda(X))$. Next, note that the factor $i_m(X - tI) = 1$ if and only if we have the following inequality:

$$\text{Re } \lambda_1 > ... > \text{Re } \lambda_m > t > ... > \text{Re } \lambda_{N-1};$$

otherwise it is 0. These two remarks suggest that t can fit in nicely as a (real) eigenvalue of a larger GEE matrix. If we restrict to the case of only k real eigenvalues and if we relabel t as λ_m and $\lambda_j := \lambda_{j+1}$ for $j \geq m$ then we can rewrite $d\mu^{(N-1,k)}dt = d\mu^{(N,k+1)}$ and expand the left hand side of LEMMA 7.0.3 as $\frac{K_N(\tau)}{K_{N-1}(\tau)}$ multiplied by

$$\sum_k \int f(\lambda_m \sqrt{N-1}) \frac{|\Delta(\lambda)|}{K_N(\tau)} e^{-\frac{N-1}{1+\tau} \sum_{j=1}^N \frac{\lambda_j^2}{2}} \prod_{j=1}^N \sqrt{\text{erfc}\left(\sqrt{\frac{2(N-1)}{1-\tau^2}} |\text{Im } \lambda_j|\right)} \mu^{(N,k+1)}(d\lambda)$$

where the integral is over the appropiate domain. The factor to the right of f looks exactly like the density for $\mathbb{P}_{N,k+1}$ except with an implicit factor of $\mathbb{1}_{\mathbb{R}}(\lambda_m)$ since we are mandating that λ_m be real and the fact that we have $N-1$ instead of N scattered in the density. We can remedy the latter issue by performing a substitution $\lambda := \sqrt{\frac{N}{N-1}}\lambda$. Following the substitution, we obtain the left hand side of LEMMA 7.0.3 is equivalent to the following expression:

$$\frac{K_N(\tau)}{K_{N-1}(\tau)} \sqrt{\frac{N}{N-1}}^{N+\binom{N}{2}} \mathbb{E}_N[\mathbb{1}_{\mathbb{R}}(\lambda_{m+1}(X)) f(\sqrt{N} \cdot \lambda_{m+1}(X))].$$

A simple algebra computation using (6.1.1) reveals the leading constant is exactly as stated in the lemma.

$\square$

7.1. A Kac-Rice formula for the average number of equilibria

The proof of THEOREM 7.0.2 will be broken up into a series of steps.

Our first step is to invoke the traditional Kac-Rice formula. In order to do so, we will establish some notation. Given an equilibrium point, we choose coordinates in a neighborhood around σ so that we can write $\sigma = 0$ and $F(0)$ as a random vector in $\mathbb{R}^{N-1}$. We define $\rho_{F(\sigma)}$ to be the density function for the random vector $F(0)$. This depends on the choice of coordinates, but its value at 0 does not. Through the use of local coordinates, the classical Kac-Rice formula (see e.g. THEOREM 6.2 in Azaïs and Wschebor [4]) yields the following formula for $\mathbb{E}\mathcal{N}_m(B)$:

$$\mathbb{E}\mathcal{N}_m(B) = \int_{S^{N-1}(\sqrt{N})} \mathbb{E}[|\det JF(\sigma)| i_m(JF(\sigma))\mathbb{1}_B(\lambda(\sigma))|F(\sigma)=0]\rho_{F(\sigma)}(0)d\sigma. \tag{7.1.1}$$

The second step is to exploit the large symmetry group of the sphere, the orthonormal group $O(N)$, and its relationship with the integrand. It will allow us to reduce the integral in (7.1.1) to the integrand evaluated at the point

$$\mathbf{n} = (0, ..., 0, \sqrt{N}) \in S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N$$

times a factor of $\mathrm{vol}(S^{N-1}(\sqrt{N}))$.

Lemma 7.1.1. *The function*

$$\sigma \mapsto \mathbb{E}\left[|\det JF(\sigma)| \mathbb{1}_B(\lambda(\sigma)) i_m(JF(\sigma))|F(\sigma)=0\right] \rho_{F(\sigma)}(0)$$

is invariant under the standard $O(N)$ action on $S^{N-1}(\sqrt{N})$ and hence is constant.

PROOF. The crux of the argument is that the only probabilistic portion of JF comes from f, h, and the partial derivatives of f with respect to the ambient $\mathbb{R}^N$ variables.

To formalize this statement, we define $j : S^{N-1}(\sqrt{N}) \to \mathbb{R}^N$ to be the usual embedding. For $x \in S^{N-1}(\sqrt{N}) \subset \mathbb{R}^N$, define $\mathrm{proj}_x : \mathbb{R}^N \to T_x S^{N-1}(\sqrt{N})$ to be the standard projection. Then

$$JF(x) = \mathrm{proj}_x \circ J_{euc}F_x \circ dj_x$$

where $J_{euc}F|_x := J_{euc}F(x) := \left(\frac{\partial F_j}{dx_i}(x)\right)$ is the Jacobian of F (viewed as a function from $\mathbb{R}^N$ to $\mathbb{R}^N$) at x and dj_x is the differential of j at x. There is a simple relationship between the $O(N)$ action and the functions proj_x and dj_x. For any $g \in O(N)$,we have the following two equalities: $g\,\mathrm{proj}_x\, g^{-1} = \mathrm{proj}_{gx}$ and $dj_{gx} = g\, dj_x\, g^{-1}$. We can use this to the following expression for $JF(gx)$:

$$\begin{aligned} JF(gx) &= \mathrm{proj}_{gx} \cdot J_{euc}F|_{gx} \cdot dj_{gx} \\ &= g\,\mathrm{proj}_x \left(g^{-1} \cdot J_{euc}F|_{gx} \cdot g\right) dj_x\, g^{-1}. \end{aligned}$$

We aim to prove that $(F(x), J_{euc}F|_x) = (g^T F(gx), g^{-1} \cdot J_{euc}F|_{gx} \cdot g)$ in distribution. The lemma will follow from this claim since conditioning on $F(gx) = 0$ is equivalent to $g^T F(gx) = 0$ and by orthogonality of g, $g^T = g^{-1}$. To obtain the required equality, we first write out $J_{euc}F|_x$ and $g^{-1} \cdot J_{euc}F|_{gx} \cdot g$ in terms of the ambient $\mathbb{R}^N$ coordinates:

$$
\begin{aligned}
(J_{euc}F|_x)_{ij} &= -\frac{\partial \lambda}{\partial x_j}(x)x_i - \lambda(x)\delta_{ij} + \frac{\partial f_i}{\partial x_j}(x) \\
\left(g^{-1} \cdot J_{euc}F|_{gx} \cdot g\right)_{ij} &= -(g^T \nabla \lambda(gx))_j x_i - \lambda(gx)\delta_{ij} - \left(g^T \cdot J_{euc}f|_{gx} \cdot g\right)_{ij},
\end{aligned}
$$

where $\nabla \lambda$ is the gradient of λ. We can rewrite the expressions involving λ in terms of h, f and its derivatives as follows:

$$
\begin{aligned}
\lambda(gx) &= \frac{1}{N}\langle gx, f(gx) + h\rangle = \frac{1}{N}\langle x, g^T f(gx) + g^T h\rangle \\
\frac{\partial \lambda}{\partial x_j}(x) &= \frac{1}{N}\left(f_j(x) + h_j\right) + \langle x, j^{th} \text{ column of } J_{euc}f|_x\rangle \\
g^T F(gx) &= \lambda(gx)x + \frac{1}{N}\left(g^T f(gx) + g^T h\right) \\
\left(g^T \nabla \lambda(gx)\right)_j &= \frac{1}{N}\left(g^T f_j(gx) + g^T h_j\right) + \langle x, j^{th} \text{ column of } g^T \cdot J_{euc}f|_{gx} \cdot g\rangle
\end{aligned}
$$

From Equation (3.17) in Fyodorov [17], we know that

$$
(f(x), J_{euc}f|_x) = \left(g^T f(gx), g^T \cdot J_{euc}f|_{gx} \cdot g\right)
$$

in distribution. Since h is Gaussian, then $h = g^T h$ in distribution and thus by independence, we have

$$
(f(x), J_{euc}f|_x, h) = \left(g^T f(gx), g^T \cdot J_{euc}f|_{gx} \cdot g, g^T h\right)
$$

in distribution which implies what was desired. $\square$

The third step is to employ explicit coordinates around $\mathbf{n}$ to write down a formula for JF and F. Let $B_{\sqrt{N}}$ denote the ball centered around $0 \in \mathbb{R}^{N-1}$ of radius $\sqrt{N}$ and define the map $P_N : B_{\sqrt{N}} \to S^{N-1}(\sqrt{N})$ by

$$P_N(x_1, ..., x_{N-1}) = \left(x_1, ..., x_{N-1}, \sqrt{N - |x|^2}\right)$$

where $|x|^2 := \sum_{i=1}^{N-1} x_i^2$. With these coordinates, it is easy to compute formulas for $dj_{\mathbf{n}}$ and $\text{proj}_{\mathbf{n}}$, yielding:

$$JF(\mathbf{n}) = \left(J_{euc}F_{ij}\right)_{i=1,j=1}^{N-1,N-1} = \left(\frac{\partial f_j}{\partial x_i}(\mathbf{n}) - \lambda(\mathbf{n})\delta_{ij}\right)_{i=1,j=1}^{N-1,N-1}$$

Note that the indices go up to $N-1$ and not up to N. Next, we compute $F(0)$ in coordinates as a vector in $\mathbb{R}^{N-1}$. We will compute it by establishing a choice of basis vectors for $T_{\mathbf{n}}S^{N-1}(\sqrt{N})$. Our basis vectors $\{v_i\}$ will be the pushforward of the basis in $\mathbb{R}^{N-1}$ through our map P_N i.e., $v_i = dP_N(e_i)$ where $\{e_i\}$ is the standard basis vectors for $\mathbb{R}^{N-1}$. In this basis, we can write $F(0)$ as

$$F(0) = (f_i(\mathbf{n}) + h_i)_{i=1}^{N-1}$$

and $\lambda(\mathbf{n})$ as

$$\lambda(\mathbf{n}) = \frac{f_N(\mathbf{n}) + h_N}{N}$$

We also remark that conditioning on $F(\mathbf{n}) = 0 \in \mathbb{R}^N$ is the same as conditioning on $F(0) = 0 \in \mathbb{R}^{N-1}$.

Our fourth step is to relate our random matrix integral to the Gaussian Elliptic Ensemble. To that end, we now make four assertions which we leave to the reader to verify:

(1) $\frac{\partial f_i}{\partial x_j}(\mathbf{n})$ is independent of $f_N(\mathbf{n})$ for $i, j \leq N-1$.

(2) $F(0)$ is independent of $JF(\mathbf{n})$.

(3) $f_N(\mathbf{n}) + h_N$ is a mean zero Gaussian with variance $\Phi_1(1) + \Phi_2(1) + \sigma^2$.

(4) For $1 \leq i, j, n, m \leq N-1$, we have:

$$\frac{N}{(N-1)\Phi_1'(1)}\mathbb{E}[\partial_j f_i(\mathbf{n})\partial_n f_m(\mathbf{n})] = \frac{1}{N-1}\left(\delta_{in}\delta_{jm} + \frac{\Phi_2(1)}{\Phi_1'(1)}\delta_{im}\delta_{jn}\right).$$

Through the use of assertion (4) and the formula for $JF(\mathbf{n})$, we can write $JF(\mathbf{n})$ in terms of the Gaussian Elliptic Ensemble:

$$\sqrt{\frac{N}{(N-1)\Phi_1'(1)}}JF(\mathbf{n}) = X - ZI$$

in distribution, where X has the law $\mathbb{P}_{N-1}$ and Z is a Gaussian random variable independent of X with mean 0 and variance given by

$$\frac{\sigma^2 + \Phi_1(1) + \Phi_2(1)}{(N-1)\Phi_1'(1)} = \frac{b^2 + \tau}{N-1}.$$

By the independence of h from f, we know that $F(0)$ consists of $N-1$ independent mean zero Gaussian random variables with variance $\Phi_1(1) + \sigma^2$ and hence

$$\rho_{F(\mathbf{n})}(0) = (2\pi(\Phi_1(1) + \sigma^2))^{-(N-1)/2}.$$

Summarizing all the steps we have taken, we can conclude the following expression for $\mathbb{E}\mathcal{N}_m(B)$:

$$\mathbb{E}\mathcal{N}_m(B) = \mathbb{E}\left[|\det(X - ZI)|\mathbb{1}_{\sqrt{\frac{N}{N-1}}B}\left(\sqrt{\Phi_1'(1)Z}\right)\right]\frac{\mathrm{vol}(S^{N-1}(\sqrt{N}))}{\sqrt{2\pi b^2}^{N-1}}\left(\frac{N-1}{N}\right)^{\frac{N-1}{2}}.$$

Finally, using the formula for the volume of a sphere,

$$\mathrm{vol}\left(S^{N-1}(\sqrt{N})\right) = \frac{2\pi^{N/2}}{\Gamma(N/2)}\sqrt{N}^{N-1}$$

and the explicit density function of Z, we obtain the expression on the right hand side of THEOREM 7.0.2. This completes the proof.

7.2. Proof of the asymptotics for $\mathbb{E}\mathcal{N}_m$

In this section, we prove the main results stated in SECTION 1. THEOREM 6.3.1 and THEOREM 7.0.1, coupled with Varadhan's lemma (see THEOREM 4.3.1 of Dembo and Zeitouni [11]) yields THEOREM 5.2.3. Setting $c = -\infty$ and $d = \infty$ in THEOREM 5.2.3 results in THEOREM 5.2.1. Finally, THEOREM 5.2.2 is a trivial corollary of the following lemma:

Lemma 7.2.1. *Define $m(N)$ to be a sequence integers such that $\frac{m(N)}{N} \to \gamma \in (0,1)$ and let $\epsilon > 0$. Then, there exists a constant $c := c(\epsilon) > 0$ such that*

$$\mathbb{P}_N\left(\mathrm{Re}\,\lambda_{m(N)} \notin (s_\gamma - \epsilon, s_\gamma + \epsilon)\right) \leq \exp(-cN^2).$$

PROOF. This is an immediate consequence of the fact that L_N satisfies a large deviation principle with speed N^2 whose rate function is minimized at U_τ. The proof of this LDP for the case $\tau = 0$ can be found in Ben Arous and Zeitouni [8] but the same argument extends to $\tau \in (-1, 1)$.

We first break up the left hand side of the equality as follows:

$$\mathbb{P}_N(\mathrm{Re}\, \lambda_{m(N)} \notin (s_\gamma - \epsilon, s_\gamma + \epsilon)) = \mathbb{P}_N(\mathrm{Re}\, \lambda_{m(N)} < s_\gamma - \epsilon) + \mathbb{P}_N(\mathrm{Re}\, \lambda_{m(N)} > s_\gamma + \epsilon).$$

To estimate the first term, we let $L_N := L_N(X_N) := \frac{1}{N}\sum_{i=1}^N \delta_{\lambda_i(X_N)}$ be the empirical distribution of the eigenvalues of a matrix X_N with law $\mathbb{P}_N$. Given the aforementioned LDP, we have:

$$\mathbb{P}_N(\mathrm{Re}\, \lambda_{m(N)} > s_\gamma + \epsilon) = \mathbb{P}_N\left(L_N(z : \mathrm{Re}\, z > s_\gamma + \epsilon) \geq \frac{m(N)}{N}\right) \leq \frac{1}{2}\exp(-cN^2)$$

for some $c > 0$ since $U_\tau(z : \mathrm{Re}\, z > s_\gamma + \epsilon) < \gamma$. Similarly,

$$\mathbb{P}_N(\mathrm{Re}\, \lambda_{m(N)} < s_\gamma - \epsilon) = \mathbb{P}_N\left(L_N(z : \mathrm{Re}\, z > s_\gamma - \epsilon) \leq \frac{m(N)-1}{N}\right) \leq \frac{1}{2}\exp(-cN^2).$$

$\square$

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