NORTHWESTERN UNIVERSITY

The Brink–Schwarz Superparticle in the Batalin–Vilkovisky Formalism

A DISSERTATION

SUBMITTED TO THE GRADUATE SCHOOL

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

DOCTOR OF PHILOSOPHY

Field of Mathematics

By

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EVANSTON, ILLINOIS

June 2019

ABSTRACT

The Brink–Schwarz superparticle is a one-dimensional analogue of the Green–Schwarz superstring. In this thesis, we use the Batalin–Vilkovisky formalism to study the superparticle. After proving a vanishing result for its Batalin–Vilkovisky cohomology, we explain the sense in which the superparticle exhibits general covariance in the world-line. Using techniques from rational homotopy theory, we then show how to patch local choices of the light-cone gauge condition together, and define the path integral in this setting.

Acknowledgments

The mathematical results contained in this thesis are the product of my collaboration with Ezra Getzler, and are the content of our joint papers. I would like to thank Professor Getzler for his guidance during my time as a graduate student at Northwestern and for introducing me to the mathematical richness of the Batalin–Vilkovisky formalism.

Dedication

This thesis is dedicated to Jenny and Barry Pohorence, whose unwavering support has been as crucial to its completion as any mathematical theory.

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CHAPTER 1

A brief history of the superparticle

The Brink–Schwarz superparticle [**3**] was introduced as a toy model for the Green–Schwarz superstring. The latter is a two-dimensional sigma model

$$\Sigma^2 \to X$$

where X is a superspace with underlying vector space ten-dimensional Minkowski space. This model describes a string moving in superspace; the magic number ten is required to achieve consistency during quantization (a similar constraint in the case of the bosonic string forces us to restrict attention to 26-dimensional space-time). The main advantage of the Green–Schwarz formulation of the superstring over the Ramond/Neveu–Schwarz superstring is that, in the former, one may choose X so that the model exhibits manifest space-time supersymmetry. Its main disadvantage is that there is no known method for quantizing the Green–Schwarz model in a Lorentz-covariant manner (one reason this is, indeed, a disadvantage is that computing scattering amplitudes is simpler when one can take advantage of manifest Lorentz symmetry of the quantization). We hope that some of the techniques introduced in this thesis may lead to progress towards obtaining a Lorentz covariant quantiztion in which one can do such computations.

Before considering quantization, we need to understand how supersymmetry and Lorentz symmetry are exhibited in the classical Green–Schwarz superstring. To do so, we choose X so that the theory exhibits both space-time Lorentz covariance and supersymmetry. The former requires that the odd part of the superspace X transforms as a spin representation of the Lorentz

Lie algebra $\mathfrak{so}(9,1)$. The latter requires, at the very least, agreement in the number of propagating bosonic and fermionic degrees of freedom. As in bosonic string theory, these bosonic degrees of freedom correspond to transverse modes of the string. This feature allows us to study the interplay between Lorentz covariance and space-time supersymmetry by considering the simpler case of a one-dimensional sigma model with the same target X.

Start by considering the bosonic portion of such a sigma model, written in the first-order formalism as

$$\int dt \left(p_\mu \partial x^\mu - \tfrac{1}{2} e \eta^{\mu\nu} p_\mu p_\nu\right)$$

where x is the coordinate of the particle, p is the corresponding momentum, e is the world-line einbein, and η is the ten-dimensional Minkowski space inner-product. As in the Polyakov action in string theory, the incorporation of the einbein produces a model which is invariant under worldline reparameterization. Additionally, as one would expect, this symmetry comes paired with a constraint; the mass-shell condition on the momentum of the particle

$$\eta^{\mu\nu} p_\mu p_\nu = 0.$$

This leaves eight of the ten bosonic directions in X as freely propagating degrees of freedom and indicates that we need to choose an eight-dimensional spin representation of $\mathfrak{so}(9,1)$. Unfortunately, no such representation exists; the smallest spin representation is the sixteen-dimensional Majorana–Weyl representation. Physicists devised a clever way to circumvent this dimensional mismatch which turns out to be responsible for much fo the richness of the superparticle model (see the introduction of [**8**]). Let θ^a be a fermionic Majorana–Weyl spinor and consider the following action

$$\int dt \left(p_\mu \partial x^\mu - \tfrac{1}{2} e \eta^{\mu\nu} p_\mu p_\nu - \tfrac{1}{2} p_\mu \gamma^\mu_{ab} \theta^a \partial \theta^b\right)$$

where γ^{μ}_{ab} are the ten-dimensional gamma matrices. The additional term involving θ^a is the remnant of the WZW topological term in the Green–Schwarz action; in one-dimension there is no topological content.

Using the mass shell constraint along with the identity

$$\gamma^{\mu}_{ab}\gamma^{\nu\, bc} + \gamma^{\nu}_{ab}\gamma^{\mu\, bc} = 2\eta^{\mu\nu}\delta^a_c$$

we see that this action is invariant under the odd symmetry

$$\delta\theta^a = p_\mu \gamma^{\mu\, ab}\kappa_b.$$

The mass shell condition implies that the rank of the matrix $p_\mu\gamma^{\mu\, ab}$ is eight, hence eight of the sixteen fermionic degrees of freedom are redundant. The remaining eight fermionic degrees of freedom are freely propagating and match up with the eight bosonic propagating degrees of freedom described above.

Another consequence of the degeneracy of $p_\mu\gamma^{\mu\, ab}$ is that the gauge parameters κ_a themselves contain redundancies. That is, we are free to perform the transformation

$$\delta\kappa_a = p_\mu\gamma^{\mu}_{ab}\kappa^b_2$$

without affecting the transformation rule for θ. Since the transformation rule for κ is identical to that for θ, we see that the same redundancies are present in κ_2. As we will explain, in the Batalin-Vilkovisky formalism this never ending cycle of redundancies requires us to introduce an infinite number of so-called "ghost for ghost" fields. These additional fields act as negative degrees of freedom and effectively cancel the eight redundant fermionic degrees of freedom in θ. One of the key technical results of this thesis implies that, while an infinite number of additional fields are incorporated to the Batalin-Vilkovisky formulation of the superparticle model, in aggregate we are left with an equal number of propagating bosonic and fermionic degrees of freedom.

CHAPTER 2

The Batalin Vilkovisky formalism

2.1. Classical gauge theory

2.1.1. Batalin-Vilkovisky formalism in finite dimensions: a toy model

Gauge theory studies the critical locus of a function S, called the *action*, and its quotient by the action of a symmetry group of the model. The aim of the Batalin-Vilkovisky formalism is to model stacky critical locii as the zeroth cohomology of a differential graded supermanifold. We work in the setting where S is a function on a supermanifold X and $\mathfrak{g} \to TX$ is a Lie algebroid acting on X. Let x^i be coordinates on X and $\xi^i_a(x)$ be the vector fields corresponding to a local frame of $\mathfrak{g}$.

We introduce the differential graded supermanifold $\mathcal{X} = \mathrm{CE}^*(X, \mathfrak{g})$ whose coordinates are x^i in degree zero and the exterior algebra on c^a, the shifted duals to the frame ξ_a, in positive degrees. The differential acts on the degree zero coordinates by

$$\delta x^i = c^a \xi^j_a \partial_j x^i.$$

In particular, the degree zero cohomology is identified with the functions on the leaf space of the Lie algebroid $\mathfrak{g}$. In the case where $\mathfrak{g}$ is the action Lie algebroid associated to a Lie algebra, this identification of the quotient with the zeroth cohomology of the Chevalley-Eilenberg complex is known in physics as the BRST method.

Forgetting, for now, the Chevalley-Eilenberg differential, we form the shifted cotangent bundle of $\mathcal{X}$, $\mathcal{M} = T^*[-1]\mathcal{X}$. In addition to the coordinates on $\mathcal{X}$, we denote the dual coordinates to x^i and c^a by x^+_i and c^+_a. These dual coordinates have degree $-1 - \deg(x^i)$ and $-1 - \deg(c^a)$ respectively, as well as having opposite parity. We now describe how to construct a differential on $\mathcal{M}$ whose zeroth cohomology is the desired subquotient. To motivate what follows, note that the critical locus of S in X is defined by the equations

$$\frac{\partial S}{\partial x^i} = 0$$

while functions on the quotient of X by the action of $\mathfrak{g}$ are given by the kernel of the Chevalley-Eilenberg differential δ acting on x^i. A first attempt is to simply piece these two observations together and define a differential on $\mathcal{M}$ which acts by the Chevalley-Eilenberg differential on the coordinates in non-negative degrees and by

$$x_i^+ \mapsto \frac{\partial S}{\partial x^i}$$

on the dual coordinates to x^i. A simple examples quickly shows that this will not produce a differential. Let $X = \mathbb{R}^2$, let S equal $|x|^4 - 2|x|^2$, and $\mathfrak{g}$ be the action Lie algebroid of $\mathbb{R}$ acting by rotations. Applying the above differential to x_1^+ followed by the Chevalley-Eilenberg differential gives

$$x_1^+ \mapsto 4(x_1^3 + x_1 x_2^2 - x_1) \mapsto 4c(x_2 - x_1^2 x_2 - x_2^3) \neq 0.$$

This failure of the differential to square to zero is easily mended: notice that if we add the terms

$$x_2^+ x_1 c - x_1^+ x_2 c$$

to the action then the Chevalley-Eilenberg differential is defined by the formula

$$\delta x_i = \frac{\partial S}{\partial x_i^+},$$

in a sense the dual of the equation defining the differential on the dual coordinates x_i^+. These terms also add new terms to the differentials of x_i^+ and c^+, and one can easily check that the formulas

$$x_i \mapsto \frac{\partial S}{\partial x_i^+} \qquad\qquad x_i^+ \mapsto \frac{\partial S}{\partial x_i} \qquad\qquad c^+ \mapsto \frac{\partial S}{\partial c}$$

do define a differential.

Looking back at the general setting, to define a differential on $\mathcal{M}$, begin by adding terms to the action of the form $x_i^+ \delta x^i$ and $c_a^+ \delta c^a$. Equivalently, we see that this modification to S allows us to express the action of the Chevalley-Eilenberg differential as

$$\delta x^i = \frac{\partial S}{\partial x_i^+} \qquad\qquad \text{and} \qquad\qquad \delta c^a = \frac{\partial S}{\partial c_a^+}.$$

Making use of fact that $\mathcal{M}$ is naturally endowed with the structure of a -1-shifted symplectic supermanifold, we can observe that the differential defined in our example above can be written as

$$Z^I \mapsto (S, Z^I)$$

where Z^I are coordinates on $\mathcal{M}$ and $(\,,)$ is the 1-shifted Poisson bracket associcted to the symplectic structure on $\mathcal{M}$, called the *Batalin-Vilkovisky anti-bracket* or sometimes just the *anti-bracket*. A simple application of the Jacobi identity for the Poisson bracket shows that this formula will define a differential if the action S satisfies the following, known as the *classical master equation*

$$(S, S) = 0.$$

We can summarize the procedure above as follows. First, we form the graded supermanifold $\mathcal{M}$ and add terms to the action consisting of the dual coordinates multiplied by the action of the Chevalley-Eileberg differential on the coordinates. Next, we check if the modified action satisfies the classical master equation. If not, we must incorporate additional terms into the action, which will be quadratic in the dual coordinates, to ensure that the classical master equation is satisfied.

The result will be a new action which agrees with the original action when the dual coordinates are all set to zero.

2.1.2. Higher gauge symmetry

In interesting cases, we may find further symmetries acting on $\mathcal{M}$ itself which preserve the critical locus of the modified action S. In these cases, we need to repeat the above procedure to incorporate these new symmetries into our model. To see how one may detect when further symmetries are present, we return to our original setting of a supermanifold X and action $S \in \mathcal{O}(X)$. If we introduce the shifted cotangent bundle $M = T^*[-1]X$ we see that, since the action is only a function of the base variables x^i, it trivially satisfies the classical master equation on M. We can think of this as the zeroth step in the iterative procedure described above. It follows that S defines a differential s on $\mathcal{O}(M)$ whose degree zero cohomology will be functions on the critical locus of S. Suppose there is a Lie algebroid $\mathfrak{g} \to TX$ which preserves S. If ξ_a are vector fields on X corresponding to a local frame of $\mathfrak{g}$ then we see that

$$s(x_i^+ \xi_a x^i) = \frac{\partial S}{\partial x^i} \xi_a x^i = \xi_a S = 0$$

by hypothesis. In other words, we have found closed elements of $\mathcal{O}(M)$ in degree -1. It is easy to check that such en element will be exact precisely when $\xi_a x^i$ is a linear combination of the equations defining the critical set of S:

$$\xi_a^i = f^{ij} \frac{\partial S}{\partial x^j}.$$

where f^{ij} is graded skew-symmetric. Since these types of vector fields will always preserve S, we do not consider them true symmetries of the model. The conclusion is that symmetries of S correspond to cohomology classes of the differential graded supermanifold (M, s) in degree -1.

This description gives us, at least in theory, a computational way of determining if a theory has residual gauge symmetry. At the zeroth stage, one looks for cohomology in degree -1. If such classes are present, one introduces the Chevalley-Eilenberg coordinates c^a of $\mathcal{X}$ and their duals, modifies the action, and tries to solve the master equation once more. Note that the additional term $x_i^+ \delta x^i = x_i^+ \xi_a x^i c^a$ ensures that there is a term in the modified differential

$$c_a^+ \mapsto x_i^+ \xi_a x^i$$

which trivializes the cohomology classes found in step zero. Once this has been accomplished and the degree -1 cohomology vanishes, in stage one we search for cohomology in degree -2. If any such classes exist, we need to repeat this process. If this iterative procedure terminates or converges, then we will have successfully resolved the critical locus of S modulo $\mathfrak{g}$ in the sense that all the negative degree cohomology of the resolution is trivial. In finite dimensions, this convergence is always possible when X is a manifold (as opposed to a supermanifold); see [**4**].

2.1.3. Variational calculus and general covariance in the Batalin–Vilkovisky formalism

We now extend the Batalin-Vilkovisky formalism described above to one-dimensional field theory, the setting of the superparticle. For this, we must adopt the formalism of variaitonal calculus.

In the finite-dimensional setting, we studied differential graded supermanifolds $\mathcal{M}$ which could be realized as -1-shifted cotangent bundles and whose differentials were Hamiltonian vector fields

$$s = (S, -).$$

In field theory, we study sections of bundles over a "world-manifold" Σ whose fibers are the differential graded supermanifolds described above. We will focus on the case when Σ is one-dimensional with coordinate t and work locally on Σ. Let the fibers of our bundle be modeled on

$\mathcal{M} = T^*[-1]\mathcal{X}$ as above. We introduce the notation φ^a for the coordinates on $\mathcal{X}$ and φ_a^+ for their dual coordinates on the fibers of $\mathcal{M}$. To align with the physics terminology, we call the degree of these fiber coordinates the *ghost number* and denote it by gh. Sections of ghost number zero are usually referred to as the classical fields of the theory. Those of ghost numbers one, two, three, etc. are called ghosts, ghosts for ghosts, and so on. Sections of negative ghost number are referred to as anti-fields, and sometimes more specifically as anti-fields for the respective dual fields in non-negative ghost number. For instance, a section with ghost number -1 is just an anti-field, while one with ghost number -2 is an anti-field for a ghost. Physicists also have special words to describe the parity of coordinates; even parity coordinates are called bosonic while odd parity coordinates are called fermionic.

When Σ is one-dimensional the jet-bundle has a relatively simple description. Fiber-wise coordinates are given by derivatives of the coordinates φ^a and φ_a^+ along the base, denoted by

$$\partial^k \varphi^a, \qquad\qquad \partial^k \varphi_a^+$$

for $k \geq 0$. In order to deal with graded supermanifolds with coordinates in ghost numbers unbounded from above or below, we will need to take some special care in specifying what we mean by functions on the jet bundle. The bosonic fiber coordinates with ghost number zero play a special role since they describe an ordinary manifold which we will denote M. We may consider the algebra generated by the fiber-wise coordinates above as a sheaf over M, and denote this algebra by $\mathcal{A}$. The algebra $\mathcal{A}$ is filtered by the ghost number of the anti-fields. Let $F^k\mathcal{A}$ be the ideal generated by monomials

$$\partial^{l_1}\varphi_{a_1}^+ \cdots \partial^{l_n}\varphi_{a_n}^+$$

such that $\mathrm{gh}(\varphi_{a_1}^+) + \cdots + \mathrm{gh}(\varphi_{a_n}^+) + k \leq 0$. The subspaces $F^k\mathcal{A}$ define a decreasing filtration of $\mathcal{A}$, with $F^0\mathcal{A} = \mathcal{A}$ and $F^i\mathcal{A} \cdot F^j\mathcal{A} \subset F^{i+j}\mathcal{A}$. Denote by $\widehat{\mathcal{A}}$ the completion of $\mathcal{A}$ with respect to this filtration:

$$\widehat{\mathcal{A}} = \varprojlim_k \mathcal{A}/F^k\mathcal{A}.$$

The replacement for functions in the finite-dimensional setting is local functionals. These can be written as integrals of local densities, and when Σ is one-dimensional, the space of local densities and the space of local functions are both isomorphic to $\widehat{\mathcal{A}}$, the former identically so and the latter via multiplication by the local frame dt. The differential along the base acts on these two spaces via the operator $dt\,\partial$

$$\widehat{\mathcal{A}} \xrightarrow{dt\,\partial} \widehat{\mathcal{A}} \cdot dt.$$

We ignore subtleties of boundary conditions so that the space of local functionals, $\mathcal{F}$, is isomorphic to the kernel of $\widehat{\mathcal{A}}$ by the image of this differential

$$\mathcal{F} \cong \widehat{\mathcal{A}}/\partial\widehat{\mathcal{A}}.$$

We find it easier to work with the ring $\widehat{\mathcal{A}}$ and then pass to the quotient $\mathcal{F}$, where there is no well-defined notion of multiplication. Denote the image of an element $f \in \widehat{\mathcal{A}}$ in $\mathcal{F}$ by $\int f$. The following notation will be convenient. Denote the partial derivatives on the jet bundle by

$$\partial_{k,a} = \frac{\partial}{\partial(\partial^k \varphi^a)} \qquad\qquad \partial_k^a = \frac{\partial}{\partial(\partial^k \varphi_a^+)}.$$

In the special case when $k = 0$ we simply write ∂_a and ∂^a.

Definition 2.1.4. The *Soloviev bracket* is defined on $\mathcal{A}$ by the formula

$$(\!(f, g)\!) = \sum_a (-1)^{(p(f)+1)p(\varphi^a)} \sum_{k,l=0}^{\infty} \left(\partial^l(\partial_{k,a} f)\partial^k(\partial_l^a g) + (-1)^{p(f)}\partial^l(\partial_k^a f)\partial^k(\partial_{l,a} g)\right)$$

where $p(-)$ denotes the parity of an element of $\mathcal{A}$.

The Soloviev bracket, and its extension to $\widehat{\mathcal{A}}$, satisfies the axioms for a 1-shifted graded Lie superalgebra: it is graded supersymmetric

$$(\!(f,g)\!) = -(-1)^{(p(f)+1)(p(g)+1)}(\!(f,g)\!),$$

and satisfies the Jacobi relation

$$(\!(f,(\!(g,h)\!))\!) = (\!((\!(f,g)\!),h)\!) + (-1)^{(p(f)+1)(p(g)+1)}(\!(g,(\!(f,h)\!))\!).$$

Furthermore, it is linear over ∂:

$$\partial(\!(f,g)\!) = (\!(\partial f,g)\!) = (\!(f,\partial g)\!).$$

Most importantly, one can check that the Soloviev bracket descends to the usual Batalin-Vilkovisky anti-bracket on $\mathcal{F}$, defined by

$$\left(\int f,\int g\right) = \sum_a (-1)^{(p(f)+1)p(\varphi^a)}\int\big((\delta_a f)(\delta^a g) + (-1)^{p(f)}(\delta^a f)(\delta_a g)\big)$$

where δ_a and δ^a are the variational derivatives

$$\delta_a = \sum_{k=0}^{\infty}(-\partial)^k\partial_{k,a} \qquad\qquad \delta^a = \sum_{k=0}^{\infty}(-\partial)^k\partial_k^a.$$

An evolutionary vector field on $\mathcal{A}$ is a graded derivation that commutes with ∂; such a vector field has the form

$$X = \mathrm{ev}(X^a\partial_a + X_a\partial^a) := \sum_{k=0}^{\infty}\left(\partial^k X^a\partial_{k,a} + \partial^k X_a\partial_k^a\right).$$

We only consider evolutionary vector fields since they are the ones that descend to vector fields on $\mathcal{F}$.

Definition 2.1.5. The *Hamiltonian vector field* X_f of an element $f \in \mathcal{A}$ is the evolutionary vector field

$$\mathsf{X}_f = \sum_a \sum_{k=0}^{\infty} (-1)^{(p(f)+1)p(\varphi^a)} \left(\partial^k(\delta_a f)\partial_k^a + (-1)^{p(f)}\partial^k(\delta^a f)\partial_{a,k} \right).$$

In field theory the classical master equation for a bosonic local functional S of ghost number zero is the following

$$\left(\int S, \int S \right) = 0. \tag{2.1}$$

The Batalin-Vilkovisky differential corresponding to an action functional S is the evolutionary vector field $s = \mathsf{X}_S$. The vector field s has ghost number 1 and is indeed a differential precisely when S satisfies the classical master equation.

We now explore the definition of global covariance in the Batalin-Vilkovisky formalism, following the account in [**5**]. Consider the element $\mathsf{D} \in \mathcal{A}$ defined by

$$\mathsf{D} = \varphi_a^+ \partial \varphi^a.$$

The Hamiltonian vector field X_D generates reparameterizations of Σ. Moreover, this vector field acts trivially on $\mathcal{F}$ since $\int \mathsf{D}$ is in the center of $\mathcal{F}$,

$$\left(\int \mathsf{D}, \int f \right) = 0$$

for any $\int f \in \mathcal{F}$. A global covariant field theory is one in which X_D, and hence, reparameterizations, acts trivially on the cohomology of s in a coherent way. To make this precise, introduce an auxiliary bosonic variable u of ghost number 2 and consider the extended space of local functionals $\mathcal{F}[[u]]$.

Definition 2.1.6. A *global covariant field theory* is a solution $\int S_u \in \mathcal{F}[[u]]$ to the following equation

$$\frac{1}{2}\left(\int S_u, \int S_u\right) = -u\int \mathsf{D}. \tag{2.2}$$

To unpack this definition, write a solution of (2.2) as a series

$$S_u = \sum_{n=0}^{\infty} u^n S_n.$$

The constant term in (2.2),

$$\left(\int S_0, \int S_0\right) = 0,$$

is simply the classical master equation for S_0. This means that we may consider S_0 as a classical action in the Batalin-Vilkovisky formalism with associated Batalin-Vilkovisky differential s.

The linear term in u tells us that

$$\left(\int S_0, \int S_1\right) = \int \mathsf{D},$$

or in other words, $\int \mathsf{D}$ is exact under the Batalin-Vilkoviski differential $s\int S_1 = \int \mathsf{D}$.

2.1.7. The Thom–Whitney normalization

In the case of the superparticle, it is difficult to write down closed form solutions S_u to (2.2) which are defined for all values of the momentum p_μ. To overcome this difficulty, we use a tool from rational homotopy theory to patch local solutions together into a global one.

Let X be a manifold with cover

$$\mathcal{U} = \{U_\alpha\}_{\alpha \in I}.$$

The nerve $N_k\mathcal{U}$ of the cover is the sequence of manifolds indexed by $k \geq 0$

$$N_k\mathcal{U} = \bigsqcup_{\alpha_0\ldots\alpha_k \in I^{k+1}} U_{\alpha_0\ldots\alpha_k},$$

where

$$U_{\alpha_0\ldots\alpha_k} = U_{\alpha_0} \cap \cdots \cap U_{\alpha_k}.$$

Denote by $\epsilon : N_0\mathcal{U} \to X$ the map which on each summand U_α equals the inclusion $U \hookrightarrow X$.

In order to globalize (2.2), we have to replace the manifold X by a sequence of manifolds of the form $\{N_k\mathcal{U}\}$. To do this, we will use the formalism of simplicial and cosimplicial objects, and we now review their definition.

Let Δ be the category whose objects are the totally ordered sets

$$[k] = (0 < \cdots < k), \quad k \in \mathbb{N},$$

and whose morphisms are the order-preserving functions. A simplicial manifold $M_\bullet$ is a contravariant functor from Δ to the category of manifolds. (We leave open here whether we are working in the smooth, analytic or algebraic setting.) Here, M_k is the value of $M_\bullet$ at the object $[k]$, and $f^* : M_\ell \to M_k$ is the action of the arrow $f : [k] \to [\ell]$ of Δ. The arrow $d_i : [k] \to [k+1]$ which takes $j < i$ to j and $j \geq i$ to $j+1$ is known as a face map, while the arrow $s_i : [k] \to [k-1]$ which takes $j \leq i$ to j and $j > i$ to $j-1$ is known as a degeneracy map.

An example of a simplicial manifold is the Čech nerve $N_\bullet\mathcal{U}$ of the cover $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$. The face map $\delta_i = d_i^* : N_{k+1}\mathcal{U} \to N_k\mathcal{U}$ corresponds to the inclusion of the open subspace

$$U_{\alpha_0\ldots\alpha_{k+1}} \subset N_{k+1}\mathcal{U}$$

into the open subspace

$$U_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_{k+1}} \subset N_k\mathcal{U},$$

and the degeneracy map $\sigma_i = s_i^* : N_{k-1}U \to N_kU$ corresponds to the identification of the open subspace

$$U_{\alpha_0\dots\alpha_k} \subset N_k\mathcal{U}$$

with the open subspace

$$U_{\alpha_0\dots\alpha_i\alpha_i\dots\alpha_{k+1}} \subset N_{k+1}\mathcal{U}.$$

Any simplicial map $f^* : M_\ell \to M_k$ is the composition of a sequence of face maps followed by a sequence of degeneracy maps. In particular, we see that in the case $M_\bullet = N_\bullet\mathcal{U}$ of the nerve of a cover, all of these maps are local embeddings.

A covariant functor $F^\bullet$ from Δ to a category $\mathcal{C}$ is called a cosimplicial object of $\mathcal{C}$. These arise as the result of applying a contravariant functor to a simplicial space: for example, given a cover $\mathcal{U}$ of M from the previous section, applying the local sections functor of the sheaf $\mathcal{F}(-)$ to the simplicial graded supermanifold $N_\bullet\mathcal{U}$, we obtain the cosimplicial graded Lie superalgebra

$$\mathcal{F}(N_\bullet\mathcal{U})$$

with the Batalin–Vilkovisky antibracket.

We now generalize the classical master equation of Batalin–Vilkovisky theory to a Maurer–Cartan equation for the cosimplicial graded Lie superalgebra $\mathcal{F}(N_\bullet\mathcal{U})$. We use a construction introduced in rational homotopy theory by Sullivan [**12**] (see also Bousfield and Guggenheim [**2**]), the Thom–Whitney normalization.

Let Ω_k be the free graded commutative algebra with generators $\{t_i\}_{i=0}^k$ of degree 0 and $\{dt_i\}_{i=0}^k$ of degree 1, and relations

$$t_0 + \cdots + t_k = 1$$

and $dt_0 + \cdots + dt_k = 0$. There is a unique differential δ on Ω_k such that $\delta(t_i) = dt_i$, and $\delta(dt_i) = 0$.

The differential graded commutative algebras Ω_k are the components of a simplicial differential graded commutative algebra $\Omega_\bullet$ (that is, contravariant functor from Δ to the category of differential graded commutative algebras): the arrow $f : [k] \to [\ell]$ in Δ acts by the formula

$$f^* t_i = \sum_{f(j)=i} t_j, \quad 0 \le i \le n.$$

The Thom–Whitney normalization $\Omega_\bullet \otimes_\Delta V^\bullet$ of a cosimplicial superspace is the equalizer of the maps

$$\prod_{k=0}^{\infty} \Omega_k \otimes V^k \underset{1\otimes f_*}{\overset{f^*\otimes 1}{\rightrightarrows}} \prod_{k,\ell=0}^{\infty} \prod_{f:[k]\to[\ell]} \Omega_k \otimes V^\ell$$

If the superspaces V^k making up the cosimplicial superspace are themselves graded V^{k*}, the Thom–Whitney totalization of $V^{\bullet *}$ is the product superspace

$$\|V\|^n = \prod_{k=0}^{\infty} \Omega_\bullet^k \otimes_\Delta V^{\bullet, n-k}.$$

The Thom–Whitney normalization takes cosimplicial 1-shifted graded Lie superalgebras to 1-shifted graded Lie superalgebras. The reason is simple: if L^k is a 1-shifted graded Lie superalgebra, then so is $\Omega_k \otimes L^k$, with differential δ and antibracket

$$[\alpha_1 \otimes x_1, \alpha_2 \otimes x_2] = (-1)^{j_2 p(x_1)+1} \alpha_1\alpha_2 [x_1, x_2],$$

where $\alpha_\ell \in \Omega_k^{i_\ell}$ and $x_\ell \in L^{k,j_\ell}$. The Thom–Whitney totalization $\|L\|$ is a subspace of the product of 1-shifted graded superalgebras $\Omega_k \otimes L^k$, and this subspace is preserved by the differential and by the antibracket. Furthermore, the construction of $\|\mathcal{F}(N_\bullet \mathcal{U})\|$ behaves well under refinement of covers, see [**6**].

The analogue of the classical master equation (2.1) in the global setting is the Maurer–Cartan equation for the differential graded 1-shifted Lie superalgebra $\|\mathcal{F}(N_\bullet\mathcal{U})\|$:

$$\delta \textstyle\int S + \frac{1}{2}\left(\int S, \int S\right) = 0. \tag{2.3}$$

Here, S is a consistent collection of elements $S^j_{\alpha_0\ldots\alpha_k} \in \Omega^j_k \otimes \mathcal{F}^{-j}(U_{\alpha_0\ldots\alpha_k})$ of total degree 0 which satisfies the sequence of Maurer–Cartan equations

$$\delta \textstyle\int S^{j-1} + \frac{1}{2}\displaystyle\sum_{i=0}^{j}\left(\textstyle\int S^i, \int S^{j-i}\right) = 0.$$

The analogue of the definition of global covariance is the curved Mauer-Cartan equation for the differential graded 1-shifted Lie superalgebra $\|\mathcal{F}(N_\bullet\mathcal{U})[[u]]\|$

$$\delta \textstyle\int S_u + \frac{1}{2}\left(\int S_u, \int S_n\right) = u \int \mathsf{D}.$$

2.2. Quantization

We will review quantization in the Batalin-Vilkovisky formalism. For convenience, we restrict attention to the finite-dimensional setting. The path integral defines the expression

$$\int_X dx \left[e^{iS(x)/\hbar}\right] \tag{2.4}$$

as an assymptotic series in $\hbar$. When $S(x)$ has the form

$$S(x) = \langle x, Ax\rangle + I(x)$$

where $I(x)$ contains higher degree polynomial terms in the coordinate x, this series involves the inverse of the operator A. If this operator is degenerate, then one would first need to choose a decomposition of the coordinates into the kernel of A and its complement in order to construct the path integral. Unfortunately, such a decomposition is usually not natural when taking into account symmetries of the theory. The BRST and Batalin-Vilkovisky formalisms for quantization allow us to perform this type of path integral in the case the A is degenerate due to symmetry of S under the action of a Lie algebra/algebroid $\mathfrak{g}$.

As described in section 2.1.1 we first replace the supermanifold X with the -1-shifted cotangent bundle to the Chevalley-Eilenberg complex for $\mathfrak{g}$, which we denoted by $\mathcal{M}$. After extending S to a solution of the classical master equation, we saw that the zeroth cohomology of the resulting differential graded supermanifold was a model for the quotient of the critical locus of S by the action of $\mathfrak{g}$. By specializing to the case of trivial $\mathfrak{g}$ we see that this cohomology is not the space we wish to perform the analog of the integral (2.4) over. Indeed, the path integral (2.4) depends on the behavior of S not just on the critical locus, but also along a formal neighborhood of the critical locus [**9**]. The correct prescription, which we now describe, is to perform the path integral over a Lagrangian subsupermanifold of $\mathcal{M}$.

2.2.1. Lagrangians and integration

We follow the presentation of integration of half-forms in odd symplectic supermanifolds given in [**7**]. Let V be a superspace with even subspace V_0 and odd subspace V_1. An endomorphism

$$A : V \to V$$

may be written in block form

$$A = \begin{pmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{pmatrix}$$

where $A_{00} : V_0 \to V_0$, $A_{11} : V_1 \to V_1$, $A_{01} : V_1 \to V_0$, and $A_{10} : V_0 \to V_1$. Since we are interested in integration, we study the analog of the determinant for superspaces, called the Berezinian. In particular, for purely even superspaces, $\mathrm{Ber}(A) = \mathrm{Det}(A)$. For a general superspace, the Berezinian is a rational function

$$\mathrm{Ber}(A) = \frac{\mathrm{Det}(A_{00})}{\mathrm{Det}(A_{11} - A_{10}A_{00}^{-1}A_{01})} = \frac{\mathrm{Det}(A_{00} - A_{01}A_{11}^{-1}A_{10})}{\mathrm{Det}(A_{11})}.$$

Like the determinant, the Berezinian satisfies $\mathrm{Ber}(\mathrm{Id}) = 1$ and $\mathrm{Ber}(AB) = \mathrm{Ber}(A)\,\mathrm{Ber}(B)$. Furthermore, denoting the parity-reversed supertranspose of A as

$$A^\circ : \Pi V^* \to \Pi V^*,$$

we have $\mathrm{Ber}(A^\circ) = \mathrm{Ber}(A)^{-1}$; see [**7**].

V is an odd symplectic superspace if it comes equipped with a non-degenerate bilinear pairing ω satisfying

- $\omega(v, w) = 0$ unless v has opposite parity to w;
- $\omega(v, w) = -(-1)^{p(v)p(w)}\omega(w, v) = -\omega(v, w)$.

A polarization of an odd symplectic superspace V is a decomposition

$$V = L \oplus M$$

where L and M are Lagrangian subsuperspaces for ω. We see that ω induces a natural isomorphism $M \cong L^\circ$. A polarization allows us to decompose an endomorphism $A : V \to V$ into the block form

$$A = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$$

where $P : L \to L$, $S : M \to M$, $Q : M \to L$, and $R : L \to M$. The key result we will use about endomorphisms of such superspaces is the following.

Proposition 2.2.2. *Let A be an endomorphism of an odd symplectic superspace V preserving the symplectic form ω. Then* $\mathrm{Ber}(A) = \mathrm{Ber}(P)^2$.

We write $\mathrm{Ber}^{1/2}(A) = \mathrm{Ber}(P)$.

Now, since a -1-shifted cotangent bundle of a supermanifold is naturally endowed an odd symplectic form, we see that such a supermanifold is locally modelled on odd symplectic superspaces. The line bundle of half-forms on $\mathcal{M} = T^*[-1]\mathcal{X}$ is defined by the transition functions $\mathrm{Ber}^{-1/2}$. We denote this line bundle by $\Omega^{1/2}(\mathcal{M})$. From the local discussion above, we see that a section of $\Omega^{1/2}(\mathcal{M})$ may be integrated over a Lagrangian subsupermanifold $\mathcal{L} \subset \mathcal{M}$. Integrals of ths type are called Batalin-Vilkovisky integrals and denoted by

$$\int_{\mathcal{L}} \sigma$$

where σ is a section of $\Omega^{1/2}$. The Lagrangian subsupermanifold $\mathcal{L}$ is referred to as the *gauge fixing Lagrangian*, or *gauge fixing* for short.

The following is the main result regarding integration of half-forms over Lagrangian subsupermanifolds. For details, see [**7**].

Proposition 2.2.3. *There is a second order differential operator Δ defined on sections of $\Omega^{1/2}(\mathcal{M})$ which squares to zero and is related to the anti-bracket as follows:*

$$(f, g) = (-1)^{p(f)}[[\Delta, f], g].$$

Integration of a Δ-closed section σ of $\Omega^{1/2}(\mathcal{M})$ over a Lagrangian subsupermanifold $\mathcal{L} \subset \mathcal{M}$ is invariant under the Hamitlonian flow Φ_t of the Lagrangian:

$$\int_{\mathcal{L}} \sigma = \int_{\Phi_t^*(\mathcal{L})} \sigma.$$

Suppose that $\sigma = e^{iS/\hbar} d\varphi$ where $d\varphi$ is a non-vanishing section of $\Omega^{1/2}(\mathcal{M})$ satisfying $\Delta d\varphi = 0$, and S is an $\hbar$-dependent function on $\mathcal{M}$. We see that $\Delta\sigma = 0$ is equivalent to the *quantum master equation*:

$$(S, S) - i\hbar\Delta S = 0.$$

In the limit $\hbar \to 0$, this recovers the classical master equation.

All canonical transformations (diffeomorphisms preserving the anti-bracket) are generated by Hamiltonian flows. The invariance of the Batalin-Vilkovisky integral under canonical transformations was the motivation for defining Lagragian subsupermanifolds of simplicial -1-shifted symplectic supermanifolds, see [**7**]. In the case of a simplicial -1-shifted symplectic supermanifold originating as the nerve of a cover $\mathcal{U}$, such a Lagrangian subsupermanifold is a collection of Lagranian subsupermanifolds of each open set in the cover together with coherent families of canonical transformations relating these subsuperanifolds to one another on intersections. This generalizes the case of an ordinary Lagrangian subsupermanifold, which can be expressed as a collection of Lagrangian subsupermanifolds on each open set which agree identically on intersections. In a later section, we outline how to construct an example of such a Lagrangian for the superparticle and explain how it leads to a Lorentz-covariant gauge fixing.

CHAPTER 3

The classical superparticle and covariance

We now apply the classical Batalin-Vilkovisky formalism to the superparticle model described in section 1. First, we study the gauge symmetries of the model and how to extend the classical action to a solution of the classical master equation. We then study the classical BV cohomology and prove a technical vanishing result. This result is then used to show that the superparticle is a covariant field theory in the sense of [**5**].

3.1. Solving the classical master equation

The solution to the classical master equation was originally found by Lindström et al. [**10**]. We first define the mathematical context we will be working in and give a useful characterization of this solution. After doing so, we explain how one may derive this solution from the classical action and its gauge symmetries.

3.1.1. A characterization of the solution

As the Green–Schwarz supestring is a supersymmetric generalization of the bosonic string, the Brink-Schwarz sueprparticle is a supersymmetric generalization of the free particle. For reasons related to consistency of superstrings, we will focus on ten-dimensional spacetime. Let $V = \mathbb{R}^{9,1}$ be ten-dimensional Minkowski space with basis $\{v_\mu\}_{0\leq\mu\leq 9}$ and inner product

$$\langle v_\mu, v_\nu \rangle = \eta_{\mu\nu}.$$

In particular, $\eta^{00} = -1$. The free particle has physical fields x^μ, and Lagrangian density $S = \frac{1}{2}\eta_{\mu\nu}\partial x^\mu \partial x^\nu$. For technical reasons, we prefer to work in the first-order formalism of this theory, which has additional fields for the momentum of the particle p_μ and Lagrangian density

$$S = p_\mu \partial x^\mu - \tfrac{1}{2}\eta^{\mu\nu} p_\mu p_\nu .$$

In order to have a theory with local reparameterization invariance, we may couple the particle to "gravity" on the world-line, represented by a nowhere-vanishing 1-form field e. Of course, the gravitational field in dimension 1 has no dynamical content; the introduction of the einbein e is reminiscent of the introduction of the world-sheet metric in the Polyakov action for the bosonic string. The modified Lagrangian density for the particle is

$$S_{[0]} = p_\mu \partial x^\mu - \tfrac{1}{2} e \eta^{\mu\nu} p_\mu p_\nu .$$

The associated BV differential is

$$s_{[0]} = \operatorname{ev}\left((\partial x^\mu - \eta^{\mu\nu} e p_\nu) \frac{\partial}{\partial p^{+\mu}} - \partial p_\mu \frac{\partial}{\partial x^+_\mu} - \tfrac{1}{2}\eta^{\mu\nu} p_\mu p_\nu \frac{\partial}{\partial e^+} \right) .$$

The variation $s_{[0]} e^+ = \frac{1}{2}\eta^{\mu\nu} p_\mu p_\nu$ may be recognized as the one-dimensional stress-energy tensor.

The local gauge symmetries of this model correspond to cohomology classes of $s_{[0]}$ at ghost number -1:

$$s_{[0]}(\partial e^+ - \eta^{\mu\nu} x^+_\nu p_\nu) = 0.$$

This cohomology class is killed by the introduction of a ghost field c, with ghost number 1, transforming as a scalar on the world-line, and the addition to the Lagrangian density of the term

$$S_{[1]} = (\partial e^+ - \eta^{\mu\nu} x^+_\mu p_\nu) c .$$

This adds the following terms to the differential:

$$s_{[1]} = \mathrm{ev}\left(\eta^{\mu\nu} c x_\nu^+ \frac{\partial}{\partial p^{+\mu}} + (\partial e^+ - \eta^{\mu\nu} x_\mu^+ p_\nu)\frac{\partial}{\partial c^+} - \eta^{\mu\nu} c p_\nu \frac{\partial}{\partial x^\mu} - \partial c \frac{\partial}{\partial e}\right).$$

Note that this is not the usual way of introducing ghosts for reparameterization invariance via Lie derivatives. In that prescription we introduce a ghost field c which transforms as a vector field on the world-line and modify the Lagrangian density so that the additional terms in the BV differential acted by Lie derivative on the fields x^μ, p_μ, e, and c itself. Our approach leads to a simplified solution of the classical master equation and is equivalent to this more typical model via canonical transformation. Effectively, we are using the einbein to convert the vector fields appearing in the Lie algebra of the diffeomorphism group of the world-line to scalars. For more details on the relation between the two descriptions and their relation to world-line diffeomorphism invariance, see [**5**].

The sum $S_{[0]} + S_{[1]}$ is the solution of the classical master equation for the free particle. We now consider the addition of fermionic fields to this model. Recall some properties of Majorana–Weyl spinors in signature $(9,1)$: for further details see the Appendix of [**6**]. The spin group $\mathrm{Spin}(9,1)$ is the universal cover of the identity component of $\mathrm{SO}(9,1)$. It has two real irreducible sixteen-dimensional representations: the left and right-handed Majorana–Weyl spinors $\mathbb{S}_+$ and $\mathbb{S}_-$. The γ-matrices $\gamma^\mu : \mathbb{S}_\pm \longrightarrow \mathbb{S}_\mp$ satisfy the relations

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}.$$

The Lie algebra of the group $\mathrm{Spin}(9,1)$ is spanned by the quadratic expressions

$$\gamma^{\mu\nu} = \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu).$$

There is a non-degenerate symmetric bilinear form $\mathsf{T}(\alpha,\beta)$ on $\mathbb{S} = \mathbb{S}_+ \oplus \mathbb{S}_-$, which vanishes on $\mathbb{S}_\pm \otimes \mathbb{S}_\pm$ and places $\mathbb{S}_\pm$ in duality with $\mathbb{S}_\mp$. We have

$$\mathsf{T}^\mu(\alpha,\beta) = \mathsf{T}(\gamma^\mu\alpha,\beta) = \mathsf{T}(\alpha,\gamma^\mu\beta).$$

In particular, we see that

$$\mathsf{T}(\gamma^{\mu\nu}\alpha,\beta) = -\mathsf{T}(\alpha,\gamma^{\mu\nu}\beta).$$

Hence, the pairing $\mathsf{T}(\alpha,\beta)$ is invariant under the action of the Lie group $\mathrm{Spin}(9,1)$, and, in particular, $\mathbb{S}_- \cong (\mathbb{S}_+)^*$ as a representation of $\mathrm{Spin}(9,1)$.

To obtain the superparticle, we adjoin to the free particle a series of fields θ_n for $n \geq 0$ of ghost number n, which are left-handed Majorana–Weyl spinors if n is even, and right-handed Majorana–Weyl spinors if n is odd. Additionally, these fields are "fermionic" in the sense that the parity of θ_n is the opposite of the parity of n. As functions on the world-line, these fields transform as scalars.

The position and momentum fields x^μ and p_μ of the theory describe a sigma model with target $M = T^*V$. For definiteness, we take the structure sheaf $\mathcal{O}$ of M to be functions with analytic dependence on x^μ and algebraic dependence on p_μ, but our results are actually insensitive to the regularity as functions of x^μ.

For the correct definition of the superparticle, it is necessary to exclude the states of vanishing momentum. To this end, we let M_0 be the complement in M of the zero-section. Denote by $j : M_0 \to M$ the open embedding, and by $\mathcal{O}_0 = j^*\mathcal{O}$ the structure sheaf of M_0.

The sheaf $\mathcal{A}$ is the graded commutative algebra generated over $\mathcal{O}_0$ by the variables

$$\{e, e^{-1}, c\} \cup \{\partial^l x^\mu, \partial^l p_\mu, \partial^l e, \partial^l c\}_{l>0} \cup \{\partial^l x^+_\mu, \partial^l p^{+\mu}, \partial^l e^+, \partial^l c^+\}_{l\geq 0} \cup \{\partial^l \theta_n, \partial^l \theta^+_n\}_{n\geq 0, l\geq 0}.$$

We denote its completion by $\widehat{\mathcal{A}}$.

We can now provide a concise characterization of how one extends the solution of the classical master equation for the particle to a solution for the superparticle. Introduce the composite spinor fields

$$\Psi_n = \begin{cases} (-1)^{\binom{n+1}{2}}\theta^+_{-n-1}, & n < -1 \\ \theta^+_0 + \frac{1}{2}x^+_\mu\gamma^\mu\theta_0 - 2c^+\theta_1, & n = -1 \\ \partial\theta_n + x^+_\mu\gamma^\mu\theta_{n+1} - 2c^+\theta_{n+2}, & n \geq 0. \end{cases}$$

Observe that the sheaf $\mathcal{A}$ is also generated over $\mathcal{O}_0$ by the variables

$$\{e, e^{-1}, c\} \cup \{\partial^l x^\mu, \partial^l p_\mu, \partial^l e, \partial^l c\}_{l>0} \cup \{\partial^l x^+_\mu, \partial^l p^{+\mu}, \partial^l e^+, \partial^l c^+\}_{l\geq 0} \cup \{\theta_n, \partial^l\Psi_m\}_{n\geq 0, m\in\mathbb{Z}, l\geq 0}.$$

Denote by S the solution to the classical master equation for the superparticle, and by s the associated BV differential.

Proposition 3.1.2. *The solution of the classical master equation* S *for the superparticle is characterized by the following conditions:*

(1) S *satisfies the classical master equation;*

(2) $\mathsf{S} = S + S'$ *where* $S = S_{[0]} + S_{[1]}$ *is the solution of the classical master equation for the free particle*

$$S = p_\mu\partial x^\mu - \tfrac{1}{2}e\eta^{\mu\nu}p_\mu p_\nu + (\partial e^+ - \eta^{\mu\nu}x^+_\mu p_\nu)c,$$

and S' *depends only on the fields and antifields* $\{p_\mu, \theta_n\} \cup \{x^+_\mu, e^+, c^+, \theta^+_n\}$ *and their derivatives;*

(3) for all $n \in \mathbb{Z}$*, the differential* s *acts on the composite fields* Ψ_n *as follows:*

$$\mathsf{s}\Psi_n = (-1)^{n+1}p_\mu\gamma^\mu\Psi_{n+1} - 2e^+\Psi_{n+2}.$$

3.1.3. Deriving the solution by resolving higher syzygies

We now discuss how one may start with the classical superparticle Lagrangian density and arrive at an explicit solution of the classical master equation satisfying the above conditions by successively introducing ghosts to resolve gauge symmetries. We approach this task from the physical perspective of gauge symmetries as vector fields preserving the action, as opposed to the perspective in the previous section of gauge symmetries as non-trivial BV cohomology classes.

As introduced in section 1, the classical superparticle action is given by

$$\int \left(p_\mu \left(\partial x^\mu - \tfrac{1}{2}\mathsf{T}^\mu(\theta_0, \partial\theta_0)\right) - \tfrac{1}{2} e \eta^{\mu\nu} p_\mu p_\nu\right)$$

where θ_0 is a left-handed odd Majorana-Wely spinor. This action has two gauge symmetries. First, there is the aforementioned reparameterization invariance

$$\delta_\xi x^\mu = \eta^{\mu\nu} p_\nu \xi \qquad\qquad \delta_\xi \theta_0 = 0$$

$$\delta_\xi p_\mu = 0 \qquad\qquad \delta_\xi e = \partial \xi$$

where ξ is a bosonic gauge parameter corresponding to the ghost field c above. Additionally, there is a fermionic symmetry

$$\delta_{\kappa_1} x^\mu = \tfrac{1}{2}\mathsf{T}^\mu(\theta_0, p_\nu \gamma^\nu \kappa_1) \qquad\qquad \delta_{\kappa_1}\theta_0 = -p_\mu \gamma^\mu \kappa_1$$

$$\delta_{\kappa_1} p_\mu = 0 \qquad\qquad \delta_{\kappa_1} e = 2\mathsf{T}(\partial\theta_0, \kappa_0)$$

where κ_1 is a fermionic gauge parameter transforming as a right handed Majorana–Weyl spinor and corresponds to the ghost field θ_1 above. As prescribed by the BV formalism, we introduce ghost fields c and θ_1 corresponding to these two gauge symmetries and add the following term to the action

$$\int \left(\eta^{\mu\nu} x_\mu^+ p_\nu c + \tfrac{1}{2} x_\mu^+ \mathsf{T}^\mu(\theta_0, p_\nu \gamma^\nu \theta_1) + e^+ \partial c + 2e^+ \mathsf{T}(\partial\theta, \theta_1) + p_\mu \mathsf{T}^\mu(\theta_0^+, \theta_1)\right).$$

To ensure the classical master equation is satisfied, we must incorporate the additional term

$$\int \left(-c^+ p_\mu \mathsf{T}^\mu(\theta_1, \theta_1) - e^+ x_\mu^+ \mathsf{T}^\mu(\theta_1, \theta_1)\right).$$

These additional terms introduce a new gauge symmetry

$$\begin{aligned} \delta_{\kappa_2} x^\mu &= -e^+ \mathsf{T}^\mu(\theta_0, \kappa_2) & \delta_{\kappa_2}\theta_0 &= 2e^+\kappa_2 \\ \delta_{\kappa_2} e &= 2\mathsf{T}(\theta_0^+, \kappa_2) + x_\mu^+ \mathsf{T}^\mu(\theta_0, \kappa_2) - 4c^+ \mathsf{T}(\theta_1, \kappa_2) & \delta_{\kappa_2}\theta_0^+ &= -e^+ x_\mu^+ \gamma^\mu \kappa_2 \\ \delta_{\kappa_2} c &= -4e^+ \mathsf{T}(\theta_1 \kappa_2) & \delta_{\kappa_2}\theta_1 &= p_\mu \gamma^\mu \kappa_2 \end{aligned}$$

where κ_2 is a bosonic gauge parameter transforming as a left handed Majorana–Weyl spinor. This corresponds to the additional term in the action

$$\int \left(e^+ x_\mu^+ \mathsf{T}^\mu(\theta_0, \theta_2) + 4e^+ c^+ \mathsf{T}(\theta_1, \theta_2) + p_\mu \mathsf{T}^\mu(\theta_1^+, \theta_2) + 2e^+ \mathsf{T}(\theta_0^+, \theta_2)\right)$$

where θ_2 is the ghost for ghost corresponding to κ_2. The action consisting of the sum of the terms introduced above indeed satisfies the classical master equation. One may think that this is the end of the story, but it turns out that this composite action exhibits yet another gauge symmetry, namely

$$\delta_{\kappa_3} e = 2\mathsf{T}(\theta_1^+, \kappa_3) \qquad \delta_{\kappa_3}\theta_1 = 2e^+\kappa_3 \qquad \delta_{\kappa_3}\theta_2 = -p_\mu \gamma^\mu \kappa_3,$$

where κ_3 is a fermionic, right-handed MW spinor like κ_1. This necessitates the additional term in the action

$$\int \left(p_\mu \mathsf{T}^\mu(\theta_2^+, \theta_3) + 2e^+ \mathsf{T}(\theta_1^+, \theta_1)\right)$$

where θ_3 is the ghost for ghost for ghost corresponding to κ_3. While the action built up to this point again satisfies the classical master equation, one can find another gauge symmetry

$$\delta_{\kappa_4} e = 2\mathsf{T}(\theta_2^+, \kappa_4) \qquad \delta_{\kappa_4}\theta_2 = 2e^+\kappa_4 \qquad \delta_{\kappa_4}\theta_3 = p_\mu\gamma^\mu\kappa_4$$

where κ_4 is a bosonic, left handed MW spinor like κ_2. This pattern continues *ad infinitum* and results in the presence of two infinite sums of higher ghosts in the action

$$\int \Big(p_\mu \sum_{i=0}^{\infty} \mathsf{T}^\mu(\theta_i^+, \theta_{i+1}) + 2e^+ \sum_{i=0}^{\infty} \mathsf{T}(\theta_i^+, \theta_{i+1})\Big).$$

This pattern is evident at the first levels of gauge symmetries as well, but it is accompanied by the clutter of how the bosonic fields in the theory and their ghosts transform. It is a convenient property of this superparticle action that the transformation rules for the bosonic fields and their ghosts only involve the first few θ_n's except for the einbein e, which is coupled to an infinite series of the spinor fields/anti-fields/ghosts. One reason for this is the odd manner in which the reparameterization gauge symmetry is expressed in the action. Had one expressed reparameterization invariance in the usual way using Lie derivatives, the field c would also be coupled to all of the spinor fields. This has the advantage of making the action considerably simpler and the disadvantage of the model not having manifest reparameterization invariance. We will see that proving that the superparticle is, indeed, a generally covariant model is a non-trivial task.

3.2. A vanishing theorem for BV cohomology of the superparticle

This goal of this section is to prove that the BV cohomology of the superparticle vanishes in sufficiently negative degrees. As a warm up to proving this vanishing result for the superparticle, let us consider the simpler case of the particle. Recall that the fields for the particle form a graded commutative algebra $\mathcal{A}_{\text{particle}}$ over $\mathcal{O}$ generated by the variables

$$\{e, e^{-1}, c\} \cup \{\partial^l x^\mu, \partial^l p_\mu, \partial^l e, \partial^l c\}_{l>0} \cup \{\partial^l x^+_\mu, \partial^l p^{+\mu}, \partial^l e^+, \partial^l c^+\}_{l\geq 0}$$

with completion denoted $\widehat{\mathcal{A}}_{\text{particle}}$.

In preparation for the proof, we recall a criterion of Boardman for the convergence of a spectral sequence. Let V be a complex, with differential $d : V^i \to V^{i+1}$. A decreasing filtration on V is a sequence of subcomplexes

$$\cdots \supset F^{-1}V \supset F^0 V \supset F^1 V \supset \cdots$$

The associated graded complex is

$$\operatorname{gr}^k_F V = F^k V / F^{k+1} V.$$

The filtration is **exhaustive** if for each $i \in \mathbb{Z}$,

$$\bigcup_k F^k V^i = V^i.$$

The filtration is **Hausdorff** if for each $i \in \mathbb{Z}$,

$$\bigcap_k F^k V^i = 0.$$

The filtration is **complete** if

$$V = \varprojlim_k V / F^k V.$$

The filtration $F^\bullet$ induces a filtration on the cohomology $H^*(V)$, which we denote by the same letter. The spectral sequence associated to the filtration converges if for all $(p, q) \in \mathbb{Z}^2$ the induced morphism

$$\operatorname{gr}^p_F H^{p+q}(V) \longrightarrow E^{pq}_\infty$$

is an isomorphism, and the induced filtration on $H^*(V)$ is complete, exhaustive and Hausdorff. The spectral sequence degenerates if $E_\infty = E_r$ for $r \gg 0$.

Theorem 3.2.1 (Boardman [**1**])**.** *If the spectral sequence associated to a complete, exhaustive Hausdorff filtration $(V, d, F^k V)$ degenerates, then it is convergent.*

Proof. Combine the following results from [**1**]: Theorems 8.2 and 9.2, the remark after Theorem 7.1, and Lemma 8.1. □

A filtration on a differential graded algebra A is a filtration on the underlying complex such that $F^j A \cdot F^k A \subset F^{j+k} A$. In this case, the pages (E_r, d_r) of the spectral sequence are themselves differential graded algebras, and the product on $E_{r+1} \cong H^*(E_r, d_r)$ is induced by the product on E_r.

Introduce the light-cone

$$C = \{(x^\mu, p_\mu) \in M \mid \eta^{\mu\nu} p_\mu p_\nu = 0\}.$$

Theorem 3.2.2. *The sheaf $H^i(\widehat{\mathcal{A}}_{particle}, s)$ vanishes for $i < 0$ and is concentrated on the light-cone C.*

Let $\widetilde{\mathcal{A}}_{particle}$ be the quotient of $\widehat{\mathcal{A}}_{particle}$ by constant multiples of the identity. The sheaf $H^i(\widetilde{\mathcal{A}}_{particle}, s)$ also vanishes for $i < 0$ and is concentrated on the light-cone.

Here, s is the BV differential associated to the solution S of the classical master equation for the free particle.

Proof. Introduce an auxiliary grading on the sheaf of algebras $\mathcal{A}_{\text{particle}}$. The structure sheaf $\mathcal{O}$ of the manifold M is placed in degree 0, and the generators of $\mathcal{A}_{\text{particle}}$ over $\mathcal{O}$ are assigned the degrees in the following table:

φ	$\partial^\ell x^\mu$	$\partial^\ell p_\mu$	$\partial^\ell e$	e^{-1}	$\partial^\ell c$	$\partial^\ell x^+_\mu$	$\partial^\ell p^{+\mu}$	$\partial^\ell e^+$	$\partial^\ell c^+$
$\deg(\varphi)$	0	0	0	0	3	0	0	-1	-1

Write $\mathrm{gh}(f) = \mathrm{gh}_+(f) - \mathrm{gh}_-(f)$, where $\mathrm{gh}_+(f)$ and $\mathrm{gh}_-(f)$ are the contributions of the fields, respectively antifields, to the ghost number. Rearranging, we see that

$$\mathrm{gh}_-(f) = \mathrm{gh}_+(f) - \mathrm{gh}(f).$$

Since

$$\mathrm{gh}_+(f) + 2\,\mathrm{gh}(f) \leq \deg(f) \leq 3\,\mathrm{gh}_+(f),$$

we see that

$$\tfrac{1}{3}\big(\deg(f) - 3\,\mathrm{gh}(f)\big) \leq \mathrm{gh}_-(f) \leq \deg(f) - 3\,\mathrm{gh}(f). \tag{3.1}$$

From this grading, we construct an exhaustive and Hausdorff descending filtration on $\mathcal{A}_{\mathrm{particle}}$: $G^k\mathcal{A}^i_{\mathrm{particle}}$ is the span of elements $f \in \mathcal{A}^i_{\mathrm{particle}}$ such that $\deg(f) \geq k$. By (3.1), the completion of this filtration is isomorphic to $\widehat{\mathcal{A}}_{\mathrm{particle}}$.

We now consider the spectral sequence for the filtration induced by G on $\widehat{\mathcal{A}}_{\mathrm{particle}}$. We will show that $E^{pq}_\infty = E^{pq}_2$, that the sheaf E^{pq}_2 vanishes if $p + q < 0$, and that it is supported on the light-cone C. This establishes the theorem.

The differential s_0 of the zeroth page E^{pq}_0 of the spectral sequence equals

$$s_0 = \mathrm{ev}\left(\big(\partial x^\mu - \eta^{\mu\nu} e p_\nu\big) \frac{\partial}{\partial p^{+\mu}} - \partial p_\mu \frac{\partial}{\partial x^+_\mu} + \partial e^+ \frac{\partial}{\partial c^+} \right).$$

This is a Koszul differential and its cohomology E_1 is the graded commutative algebra generated over $\mathcal{O}$ by the variables $\{\partial^\ell e, e^{-1}, \partial^\ell c, e^+\}$.

The differential s_1 of the first page E_1 of the spectral sequence equals

$$s_1 = \mathrm{ev}\left(-\tfrac{1}{2}\eta^{\mu\nu}p_\mu p_\nu\,\frac{\partial}{\partial e^+}\right).$$

The element $\eta^{\mu\nu}p_\mu p_\nu \in E_1^{00}$ is not a zero divisor in E_1. We conclude that E_2 vanishes in negative degrees and is concentrated on the zero-locus of $\eta^{\mu\nu}p_\mu p_\nu$ in M, namely the light-cone C.

We see that the second page E_2 of the spectral sequence is a graded commutative algebra, generated over $\mathcal{O}_N$ by the variables $\{\partial^\ell e, e^{-1}\}$ and $\{\partial^\ell c\}$. Thus E_2^{pq} vanishes unless $p \geq 0$ and $p + 2q = 0$, hence $s_r = 0$ for $r > 2$ and the spectral sequence is seen to degenerate at E_2.

Turning to the case of the sheaf $\widetilde{\mathcal{A}}_{\text{particle}}$, we have a long exact sequence for cohomology sheaves

$$0 \longrightarrow H^{-1}(\widetilde{\mathcal{A}}_{\text{particle}}, s) \longrightarrow \mathbb{R} \longrightarrow H^0(\widehat{\mathcal{A}}_{\text{particle}}, s) \longrightarrow \cdots$$

But the above proof shows that the morphism $\mathbb{R} \to E_\infty^{00}$ is an injection, and hence that
$H^{-1}(\widetilde{\mathcal{A}}_{\text{particle}}, s) = 0$. □

Corollary 3.2.3. *Let $\mathcal{F}_{particle} = \widehat{\mathcal{A}}_{particle}/\widetilde{\mathcal{A}}_{particle}$. The cohomology sheaf $H^i(\mathcal{F}_{particle}, s)$ vanishes for $i < -1$ and is concentrated on the light-cone C.*

Proof. The sheaf $\mathcal{F}_{\text{particle}}$ has a resolution

$$0 \longrightarrow \widetilde{\mathcal{A}}_{\text{particle}} \xrightarrow{\partial} \widehat{\mathcal{A}}_{\text{particle}} \longrightarrow \mathcal{F}_{\text{particle}} \longrightarrow 0.$$

The associated long exact sequence implies that $H^i(\mathcal{F}_{\text{particle}}, s) = 0$ for $i < -1$. □

We now prove the analogous result for the superparticle. Let

$$C_0 = C \cap M_0.$$

Theorem 3.2.4. *The sheaf $H^i(\widehat{\mathcal{A}}, \mathsf{s})$ vanishes for $i < 0$ and is concentrated on the light-cone C_0.*

Let $\widetilde{\mathcal{A}}$ be the quotient of $\widehat{\mathcal{A}}$ by constant multiples of the identity. The sheaf $H^i(\widetilde{\mathcal{A}}, \mathsf{s})$ also vanishes for $i < 0$ and is concentrated on the light-cone C_0.

In the proof of Theorem 3.2.4, we need the formula for the differential $\mathsf{s} = s + s'$ on fields and antifields of the theory, where s is the differential of the particle, and s' is the contribution to the differential from S'. We see that s' vanishes on the fields and antifields $\{p_\mu\} \cup \{x^+_\mu, e^+, c^+\}$, and

$$\begin{aligned}
s'\theta_n &= (-1)^{n+1} p_\mu \gamma^\mu \theta_{n+1} - 2e^+\theta_{n+2} \\
s'p^{+\mu} &= -\tfrac{1}{2}\mathsf{T}^\mu(\theta_0, \partial\theta_0) - c^+\mathsf{T}^\mu(\theta_1,\theta_1) + \sum_{n=1}^{\infty} (-1)^{\binom{n+1}{2}} \mathsf{T}^\mu(\Psi_{-n}, \theta_n) \\
s'x^\mu &= -\tfrac{1}{2} p_\nu \mathsf{T}^\nu(\gamma^\mu\theta_0, \theta_1) + e^+\big(\mathsf{T}^\mu(\theta_1,\theta_1) + \mathsf{T}^\mu(\theta_0,\theta_2)\big) \\
s'e &= x^+_\mu \mathsf{T}^\mu(\theta_1,\theta_1) - 4c^+\mathsf{T}(\theta_1,\theta_2) - 2\sum_{n=0}^{\infty} (-1)^{\binom{n}{2}} \mathsf{T}(\Psi_{-n}, \theta_{n+1}) \\
s'c &= -p_\mu \mathsf{T}^\mu(\theta_1,\theta_1) - 4e^+\mathsf{T}(\theta_1,\theta_2).
\end{aligned}$$

The infinite sums in the formulas for $s'p^{+\mu}$ and $s'e$ make sense by the completeness property of $\widehat{\mathcal{A}}$.

Proof of Theorem 3.2.4. We define an auxiliary grading on $\mathcal{A}$ extending the grading on $j^*\mathcal{A}_{\text{particle}}$ used in the proof of Theorem 3.2.2: the generators of $\mathcal{A}$ over $j^*\mathcal{A}_{\text{particle}}$ are assigned the degrees in the following table:

Φ	θ_n	$\partial^\ell\Psi_n$	$\partial^\ell\Psi_{-n}$
$\deg(\Phi)$	$3n+1$	$3n$	$-2n$

Observe that

$$\deg(f) \leq 4\,\mathrm{gh}_+(f) + 16;$$

the factor 4 accounts for the field θ_1, which has ghost number 1 and degree 4, while the constant 16 accounts for the 16 modes of the fermionic field θ_0, which have ghost number 0 and degree 1. In the other direction, we have

$$\mathrm{gh}_+(f) + 2\,\mathrm{gh}(f) \leq \deg(f).$$

Combining these two inequalities, we see that

$$\tfrac{1}{4}\deg(f) - \mathrm{gh}(f) - 4 \leq \mathrm{gh}_-(f) \leq \deg(f) - 3\,\mathrm{gh}(f). \tag{3.2}$$

From this grading, we construct an exhaustive and Hausdorff descending filtration on $\mathcal{A}$: $G^k\mathcal{A}^i$ is the span of elements $f \in \mathcal{A}^i$ such that $\deg(f) \geq k$. By (3.2), the completion of this filtration is isomorphic to $\widehat{\mathcal{A}}$.

The differential s_0 on the zeroth page of the spectral sequence E_0^{pq} equals

$$\mathsf{s}_0 = \mathrm{ev}\left(\partial x^\mu \frac{\partial}{\partial p^{+\mu}} - \partial p_\mu \frac{\partial}{\partial x^+_\mu} - \partial e^+ \frac{\partial}{\partial c^+}\right).$$

This is a Koszul differential and its cohomology E_1 is the graded commutative algebra freely generated over $\mathcal{O}_0$ by the variables

$$\{\partial^\ell e, e^{-1}, \partial^\ell c, e^+\} \cup \{\theta_n \mid n \geq 0\} \cup \{\partial^\ell \Psi_n \mid n \in \mathbb{Z}, \ell \geq 0\}.$$

The differential s_1 on the first page E_1 of the spectral sequence is given by the formula

$$\mathsf{s}_1 = \mathrm{ev}\left(-\tfrac{1}{2}\eta^{\mu\nu} p_\mu p_\nu \frac{\partial}{\partial e^+}\right).$$

The element $\eta^{\mu\nu}p_\mu p_\nu$ is not a zero divisor in E_1: its zero-locus in M_0 is the light-cone $i : C_0$, with structure sheaf $\mathcal{O}_{C_0}$.

We see that the second page E_2 of the spectral sequence is a graded commutative algebra generated over $\mathcal{O}_{C_0}$ by the variables

$$\{\partial^\ell e, e^{-1}, \partial^\ell c\} \cup \{\theta_n \mid n \geq 0\} \cup \{\partial^\ell \Psi_n \mid n \in \mathbb{Z}, \ell \geq 0\}.$$

The differential s_2 on the second page E_2 of the spectral sequence is given by the formula

$$\mathsf{s}_2 = \sum_{n=1}^{\infty} \mathrm{ev}\left(p_\mu \mathsf{T}^\mu \left(\Psi_{1-n}, \frac{\partial}{\partial \Psi_{-n}}\right)\right).$$

On the light-cone C_0, the operator

$$p_\mu \gamma^\mu : \mathbb{S}_\pm \longrightarrow \mathbb{S}_\mp$$

has square zero, since $(p_\mu \gamma^\mu)^2 = \eta^{\mu\nu} p_\mu p_\nu = 0$. The cohomology of this operator vanishes, in the sense that

$$\ker(p_\mu \gamma^\mu) = \mathrm{im}(p_\mu \gamma^\mu).$$

To see this, choose a vector q_μ such that $\eta^{\mu\nu} p_\mu q_\nu > 0$: then $q_\mu \gamma^\mu$ yields a contracting homotopy for the differential $p_\mu \gamma^\mu$. (This is where in the proof we need to have localized away from the zero section of M.)

The third page E_3^{p+q} is generated over E_3^{00} by the variables $\{\partial^\ell c\}$ in $E_3^{3,-2}$, $\{\theta_n\}$ in $E_3^{3n+1,-2n-1}$, and $\{\partial^\ell \Psi_n \mid n \geq 0\}$ in $E_3^{3n,-2n}$. Thus, E_r^{pq} vanishes unless $p \geq 0$, $p + q \geq 0$, and $3p + 4q \geq -16$; this last inequality is saturated by the product of the 16 modes of the field θ_0, located in $E_0^{16,-16}$, with monomials in the variables $\{\partial^\ell \theta_1\}$. The differential s_r of the rth page of the spectral sequence vanishes for $r > 20$, and hence the spectral sequence degenerates, proving the first part of the theorem.

The proof of the vanishing of the cohomology sheaves $H^i(\widetilde{\mathcal{A}}, \mathsf{s})$ follows the same lines as the proof of the analogous result for the particle. □

Corollary 3.2.5. *Let $\mathcal{F} = \widehat{\mathcal{A}}/\partial\widehat{\mathcal{A}}$. The sheaf $H^i(\mathcal{F}, \mathsf{s})$ vanishes for $i < -1$, and is concentrated on the light-cone C_0.*

3.3. General covariance, supersymmetry, and Lorentz covariance

3.3.1. The superparticle as a covariant field theory

In this section, using the Thom–Whitney formalism, we will show that the superparticle is a global covariant field theory, in the terminology of [**5**].

Let $\mathsf{D} \in \Gamma(M_0, \widehat{\mathcal{A}}^{-1})$ be the element

$$\mathsf{D} = x^+_\mu \partial x^\mu + p^{+\mu}\partial p_\mu - e\partial e^+ + c^+\partial c + \sum_{n=0}^{\infty} \mathsf{T}(\theta^+_n, \partial\theta_n)$$

and recall the following definition.

Definition 3.3.2. A global covariant field theory is a solution of the curved Maurer–Cartan equation in $\|\mathcal{F}(N_\bullet\mathcal{U})[u]\|$, where $\mathcal{U}$ is a cover of M_0:

$$\delta\textstyle\int\mathsf{S}_u + \frac{1}{2}\left(\int\mathsf{S}_u, \int\mathsf{S}_u\right) = -u\int\mathsf{D}. \tag{3.3}$$

If S_u is a covariant field theory with respect to a cover $\mathcal{U}$ of M_0 and $(\mathcal{V}, \varphi)$ is a refinement of $\mathcal{U}$, $\Phi^*\mathsf{S}_u$ is again a global covariant field theory with respect to the refined cover.

Theorem 3.3.3. *There is a global covariant field theory*

$$\mathsf{S}_u = \mathsf{S} + \sum_{n=0}^{\infty} u^{n+1}\mathsf{G}_n$$

such that S is the solution of the classical master equation for the superparticle.

Proof. Consider the open affine cover $\mathcal{U} = \{U_\mu\}_{0\le\mu\le 9}$ of M_0, where

$$U_\mu = \{p_\mu \neq 0\}.$$

We must construct a series of cochains

$$\textstyle\int \mathsf{G}_n \in \|\mathcal{F}(N_\bullet\mathcal{U})\|^{-2n-2},$$

in the Thom–Whitney totalization $\|\mathcal{F}(N_\bullet\mathcal{U})\|$ of the cosimplicial graded Lie superalgebra $\mathcal{F}(N_\bullet\mathcal{U})$, satisfying the curved Maurer–Cartan equation

$$\delta\left(\textstyle\int \mathsf{S}_u\right) + \tfrac{1}{2}\left(\textstyle\int \mathsf{S}_u, \int \mathsf{S}_u\right) = -u \textstyle\int \mathsf{D}.$$

Equivalently, we must find a solution G_0 of the equation

$$(\delta + \mathsf{s}) \textstyle\int \mathsf{G}_0 = -\int \mathsf{D}, \tag{3.4}$$

and for $n > 0$, solutions of the equations

$$(\delta + \mathsf{s})\left(\textstyle\int \mathsf{G}_n\right) = -\frac{1}{2} \sum_{j+k=n-1} \left(\textstyle\int \mathsf{G}_j, \int \mathsf{G}_k\right). \tag{3.5}$$

Assuming that we have solved these equations for $(\mathsf{G}_0, \dots, \mathsf{G}_{n-1})$, we see that

$$\frac{1}{2} \sum_{j+k=n-1} (\delta + \mathsf{s}) \left(\textstyle\int \mathsf{G}_j, \int \mathsf{G}_k\right) = -\left(\textstyle\int \mathsf{D}, \int \mathsf{G}_{n-1}\right) - \sum_{i+j+k=n-2} \left(\left(\textstyle\int \mathsf{G}_i, \int \mathsf{G}_j\right), \int \mathsf{G}_k\right).$$

The first term vanishes since $\int \mathsf{D}$ lies in the center of $\mathcal{F}$, while the second term vanishes by the Jacobi relation for graded Lie superalgebras. Thus, the right-hand side of (3.5) is a cocycle. Since the cohomology of the complex $\|\mathcal{F}(N_\bullet\mathcal{U})\|$ vanishes below degree -1 by Theorem 3.2.4, we may solve the equation for G_n.

Rewrite the formula for D, using the definition (3.11) of Ψ_n and the formula for the action of s:

$$\mathsf{D} = -\mathsf{s}(x^+_\mu p^{+\mu} + ec^+) + \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}}\, \mathsf{T}(\Psi_{-n}, \Psi_{n-1}). \tag{3.6}$$

Introduce the vector

$$q_\mu = \frac{t_\mu}{2\eta^{\mu\nu} p_\nu},$$

and its de Rham differential

$$\delta q_\mu = \frac{\delta t_\mu}{2\eta^{\mu\nu} p_\nu}.$$

We will show that the expression

$$\mathsf{G}_0 = x^+_\mu p^{+\mu} + ec^+ + \frac{1}{2}\sum_{k\geq 0}\sum_{\nu_0\ldots\nu_k} (-1)^k\, q_{\nu_0}\delta q_{\nu_1}\ldots\delta q_{\nu_k} \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}}\, \mathsf{T}^{\nu_0\ldots\nu_k}(\Psi_{-n},\Psi_{n-k-2}) \tag{3.7}$$

in $\|\mathcal{A}(N_\bullet\mathcal{U})\|^{-2}$ gives a solution of the equation

$$(\delta + \mathsf{s})\mathsf{G}_0 = -\mathsf{D}, \tag{3.8}$$

yielding (3.4). By (3.6), it suffices to show that

$$\mathsf{s}\sum_{\nu_0\ldots\nu_k} q_{\nu_0}\delta q_{\nu_1}\ldots\delta q_{\nu_k} \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}}\, \mathsf{T}^{\nu_0\ldots\nu_k}(\Psi_{-n},\Psi_{n-k-2}) \tag{3.9}$$

$$= \begin{cases} \displaystyle\sum_{\nu_0\ldots\nu_{k-1}} \delta q_{\nu_0}\ldots\delta q_{\nu_{k-1}} \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}}\, \mathsf{T}^{\nu_0\ldots\nu_{k-1}}(\Psi_{-n},\Psi_{n-k-1}), & k>0, \\ \displaystyle -\sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}}\, \mathsf{T}(\Psi_{-n},\Psi_{n-1}), & k=0. \end{cases}$$

We have

$$\begin{aligned} \mathsf{s}\sum_{n=-\infty}^{\infty} & (-1)^{\binom{n}{2}}\, \mathsf{T}^{\nu_0\ldots\nu_k}(\Psi_{-n},\Psi_{n-k-2}) \\ &= \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}}\, \mathsf{T}^{\nu_0\ldots\nu_k}((-1)^{n+1} p_\mu\gamma^\mu\Psi_{-n+1} - 2e^+\Psi_{-n+2},\Psi_{n-k-2}) \\ &+ \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}+n+1}\, \mathsf{T}^{\nu_0\ldots\nu_k}(\Psi_{-n},(-1)^{n+k+1} p_\mu\gamma^\mu\Psi_{n-k-1} - 2e^+\Psi_{n-k}) \\ &= p_\mu \sum_{n=-\infty}^{\infty} (-1)^{\binom{n}{2}} \Big(\mathsf{T}^{\nu_0\ldots\nu_k}(\gamma^\mu\Psi_{-n},\Psi_{n-k-1}) + (-1)^k\, \mathsf{T}^{\nu_0\ldots\nu_k}(\Psi_{-n},\gamma^\mu\Psi_{n-k-1})\Big) \\ &- 2e^+ \sum_{n=-\infty}^{\infty} \Big((-1)^{\binom{n+2}{2}} + (-1)^{\binom{n}{2}}\Big)\mathsf{T}^{\nu_0\ldots\nu_k}(\Psi_{-n},\Psi_{n-k}). \end{aligned}$$

The sum on the last line vanishes, since $(-1)^{\binom{n+2}{2}} = -(-1)^{\binom{n}{2}}$. Using the identity

$$\mathsf{T}^{\mu_1\dots\mu_k}(\gamma^\mu\alpha,\beta) - (-1)^k\,\mathsf{T}^{\mu_1\dots\mu_k}(\alpha,\gamma^\mu\beta) = 2\sum_{j=1}^{k}(-1)^{k-j}\eta^{\mu\mu_j}\mathsf{T}^{\mu_1\dots\widehat{\mu}_j\dots\mu_k}(\alpha,\beta),$$

we conclude that

$$\mathsf{s}\sum_{n=-\infty}^{\infty}(-1)^{\binom{n}{2}}\,\mathsf{T}^{\nu_0\dots\nu_k}(\Psi_{-n},\Psi_{n-k-2}) = 2\sum_{j=0}^{k}(-1)^{k-j}p^{\nu_j}\sum_{n=-\infty}^{\infty}(-1)^{\binom{n}{2}}\mathsf{T}^{\nu_0\dots\widehat{\nu}_j\dots\nu_k}(\Psi_{-n},\Psi_{n-k-1}),$$

from which (3.9) follows. □

Corollary 3.3.4. *The long exact sequence*

$$\begin{tikzcd} \cdots \arrow[r] & H^{-1}(\widetilde{\mathcal{A}},\mathsf{s}) \arrow[r, "\partial"] & H^{-1}(\widehat{\mathcal{A}},\mathsf{s}) \arrow[r] & H^{-1}(\mathcal{F},\mathsf{s}) \arrow[dll, out=-10, in=170] & \\ & H^{0}(\widetilde{\mathcal{A}},\mathsf{s}) \arrow[r, "\partial"] & H^{0}(\widehat{\mathcal{A}},\mathsf{s}) \arrow[r] & H^{0}(\mathcal{F},\mathsf{s}) \arrow[r] & \cdots \end{tikzcd}$$

splits, in the sense that the morphisms ∂ *vanish.*

By an extension of this method, we may show that the space of solutions of (2.2) is a contractible simplicial set. This amounts to showing that for each $n > 0$, any solution of (2.2) in $\Omega(\partial\Delta^n)\otimes\|\mathcal{F}(N_\bullet\mathcal{U})[[u]]\|$ may be extended to a solution of (2.2) in

$$\Omega(\Delta^n)\otimes\|\mathcal{F}(N_\bullet\mathcal{U})[[u]]\| = \Omega_n\otimes\|\mathcal{F}(N_\bullet\mathcal{U})[[u]]\|.$$

In particular, the case $n = 1$ shows that there is a solution of (2.2) in $\Omega_1\otimes\|\mathcal{F}(N_\bullet\mathcal{U})[[u]]\|$ interpolating between any pair of solutions of (2.2) in $\|\mathcal{F}(N_\bullet\mathcal{U})[[u]]\|$.

3.3.5. Supersymmetry and Lorentz invariance of the solution

The reason for the interest of the superparticle, and of the Green–Schwarz superstring for which it is a toy model, is that it is manifestly supersymmetric. The supersymmetry is generated by the functional $\int Q$, where

$$Q = \theta_0^+ - \tfrac{1}{2} x_\mu^+ \gamma^\mu \theta_0 \in \mathbb{S}_- \otimes \mathcal{A}^{-1}.$$

The formula $\mathsf{s}Q = \partial\big(p_\mu \gamma^\mu \theta_0 - 2e^+\theta_1\big)$ implies the vanishing of the Batalin–Vilkovisky antibracket

$$(\textstyle\int Q, \int \mathsf{S}) = 0. \tag{3.10}$$

There is an interesting, if not completely rigorous, explanation for this which hints at some fundamental structure of the superparticle action S. Consider the two parameter family of composite fields $\Psi_m^{(k)}$ defined recursively by

$$\Psi_n^{(k)} = \begin{cases} \theta_n, & k = 0, n \geq 0 \\ \Psi_n, & k = 1, n \in \mathbb{Z} \\ \partial \Psi_n^{(k-1)} + x_\mu^+ \gamma^\mu \Psi_{n+1}^{(k-1)} - 2c^+ \Psi_{n+2}^{(k-1)}, & k > 1, n \in \mathbb{Z}. \end{cases} \tag{3.11}$$

As with Ψ_n, the formula for the action of the BV differntial on these composite fields is elegant:

$$\mathsf{s}\Psi_n^{(k)} = (-1)^{n+1} p_\mu \gamma^\mu \Psi_{n+1}^{(k)} - 2e^+ \Psi_{n+2}^{(k)}.$$

Note that $\Psi_n^{(k)}$ is only defined for either $k > 0$ or $k = 0$ and $n \geq 0$. In particular, the composite field $\Psi_{-1}^{(0)}$ is not defined (it would contain anti-derivatives of the anti-fields if it was!). However, the formulas above do give us a way to express its derivative:

$$\partial \Psi_{-1}^{(0)} = \Psi_{-1}^{(1)} - x_\mu^+ \gamma^\mu \Psi_0^{(0)} + 2c^+ \Psi_1^{(0)} = Q.$$

Then, since s is evolutionary, we see that

$$\mathsf{s}Q = \mathsf{s}\left(\partial \Psi_{-1}^{(0)}\right) = \partial\left(\mathsf{s}\Psi_{-1}^{(0)}\right) = \partial\left(p_\mu \gamma^\mu \Psi_0^{(0)} - 2e^+ \Psi_1^{(0)}\right)$$

where, though $\Psi_{-1}^{(0)}$ is not defined, both $\partial \Psi_{-1}^{(0)}$ and $\mathsf{s}\Psi_{-1}^{(0)}$ are.

Let $\mathcal{A}_\star$ be the subalgebra of the sheaf $\mathcal{A}$ generated by the fields

$$\{\partial^\ell p_\mu, \partial^\ell x^+_\mu, \partial^\ell e^+, \partial^\ell c^+\}_{\ell\geq 0} \cup \{\partial^\ell \Psi_n\}_{n\in\mathbb{Z}, \ell\geq 0}.$$

Let $\widehat{\mathcal{A}}_\star \subset \widehat{\mathcal{A}}$ be its associated completion, with respect to the fields of negative degree

$$\{\partial^\ell x^+_\mu, \partial^\ell e^+, \partial^\ell c^+\}_{\ell\geq 0} \cup \{\partial^\ell \Psi_n\}_{n<0, \ell\geq 0}.$$

Both $\mathcal{A}_\star$ and $\widehat{\mathcal{A}}_\star$ may be viewed as sheaves over the momentum space $V_0^\vee = V^\vee \setminus \{0\}$, which is the fibre of M_0 over $0 \in V$.

Lemma 3.3.6. *The subsheaf $\widehat{\mathcal{A}}_\star \subset \widehat{\mathcal{A}}$ satisfies $\partial[\widehat{\mathcal{A}}_\star] \subset \widehat{\mathcal{A}}_\star$ and is closed under the Soloviev bracket.*

Proof. It follows directly from its definition that $\widehat{\mathcal{A}}_\star$ is preserved by the action of ∂. In order for $\widehat{\mathcal{A}}_\star$ to be closed under the Soloviev bracket, it suffices to observe that for all fields Φ that generate $\mathcal{A}_\star$, we have

$$\frac{\partial\Phi}{\partial(\partial^k x^\mu)} = \frac{\partial\Phi}{\partial(\partial^k p^{+\mu})} = \frac{\partial\Phi}{\partial(\partial^k e)} = \frac{\partial\Phi}{\partial(\partial^k c)} = 0.$$

This implies that

$$(\!(f,g)\!) = \sum_{n=0}^{\infty}(-1)^{(n+1)(p(f)+1)} \sum_{k,\ell=0}^{\infty}\left(\partial^\ell\left(\frac{\partial f}{\partial(\partial^k\theta_n)}\right)\partial^k\left(\frac{\partial g}{\partial(\partial^\ell\theta^+_n)}\right) + (-1)^{p(f)}\partial^\ell\left(\frac{\partial f}{\partial(\partial^k\theta^+_n)}\right)\partial^k\left(\frac{\partial g}{\partial(\partial^\ell\theta_n)}\right)\right).$$

It only remains to observe that for all $n \geq 0$ and all $k, \ell \geq 0$, the terms in the above formula are in $\widehat{\mathcal{A}}_\star$ for any f and g in $\widehat{\mathcal{A}}_\star$. □

We now have the following analogue of Theorem 3.2.5. The proof follows the same lines, but is actually somewhat simpler.

Lemma 3.3.7. *Let* $\mathcal{F}_\star = \widehat{\mathcal{A}}_\star / \partial \widehat{\mathcal{A}}_\star$. *The cohomology sheaf* $H^i(\mathcal{F}_\star, \mathsf{s})$ *vanishes unless* $i \in \{-1, 0\}$.

Proof. The sheaf $\widehat{\mathcal{A}}_\star$ is an algebra over the momentum space $M_\star$, whose structure sheaf is the algebra of rational functions in the variables $\{p_\mu\}$. The filtration of $\widehat{\mathcal{A}}$ induces a filtration of $\widehat{\mathcal{A}}_\star$, and the differential s_0 on the zeroth page of the associated spectral sequence E_0^{pq} equals

$$\mathsf{s}_0 = -\operatorname{ev}\left(\partial p_\mu \frac{\partial}{\partial x^+_\mu} + \partial e^+ \frac{\partial}{\partial c^+}\right).$$

This is a Koszul differential and its cohomology E_1 is the graded commutative algebra freely generated over the structure sheaf $\mathcal{O}_{M_\star}$ by the variables

$$\{e^+\} \cup \{\partial^\ell \Psi_n \mid n \in \mathbb{Z}, \ell \geq 0\}.$$

The differential s_1 on the first page E_1 of the spectral sequence is given by the formula

$$\mathsf{s}_1 = \operatorname{ev}\left(-\tfrac{1}{2}\eta^{\mu\nu} p_\mu p_\nu \frac{\partial}{\partial e^+}\right).$$

The element $\eta^{\mu\nu} p_\mu p_\nu$ is not a zero divisor in E_1: its zero-locus is the light-cone

$$\{p_\mu \neq 0 \mid \eta^{\mu\nu} p_\mu p_\nu = 0\}.$$

We see that the second page E_2 of the spectral sequence is a sheaf of graded commutative algebras generated over $\mathcal{O}_{C_\star}$ by the variables

$$\{\partial^\ell \Psi_n \mid n \in \mathbb{Z}, \ell \geq 0\}.$$

The differential s_2 on the second page E_2 of the spectral sequence is given by the formula

$$\mathsf{s}_2 = \sum_{n=1}^{\infty} \operatorname{ev}\left(p_\mu T^\mu\left(\Psi_{1-n}, \frac{\partial}{\partial \Psi_{-n}}\right)\right).$$

On the light-cone, the operator

$$p_\mu \gamma^\mu : \mathbb{S}_\pm \longrightarrow \mathbb{S}_\mp$$

has vanishing cohomology. We see that the third page E_3^{p+q} is generated over E_3^{00} by the variables $\{\partial^\ell \Psi_n \mid n \geq 0\}$ in $E_3^{3n,-2n}$, and hence the differential s_r of the rth page of the spectral sequence vanishes for $r > 3$.

The remainder of the proof follows the proof of Theorem 3.2.5. $\square$

Theorem 3.3.8. *There is a choice of the solution* S_u *to the equation* (2.2) *such that*

$$\left(\int Q, \int \mathsf{S}_u\right) = 0.$$

Proof. Let q be the Hamiltonian vector field associated to $\int Q$. It is easily seen that $\mathsf{q}\Psi_n = 0$, and hence that q annihilates $\widehat{\mathcal{A}}_\star$. It is easily seen that $\mathsf{q}\mathsf{G}_0 = 0$. We prove the theorem by showing that for all $n > 0$, G_n may be chosen in $\|\mathcal{F}_\star(N_\bullet\mathcal{U})\|^{-2n-2}$. In view of Lemma 3.3.7, it suffices to show that the cocycle

$$-\frac{1}{2} \sum_{j+k=n-1} \left(\int \mathsf{G}_j, \int \mathsf{G}_k\right)$$

lies in $\|\mathcal{F}_\star(N_\bullet\mathcal{U})\|^{-2n-1}$ for $n > 0$. By induction, we may assume that this holds for all of the terms of this sum with $j, k > 0$. It remains to check that

$$\left(\int \mathsf{G}_0, \int \mathsf{G}_{n-1}\right) \in \|\mathcal{F}_\star(N_\bullet\mathcal{U})\|^{-2n-1}.$$

But modulo $\widehat{\mathcal{A}}_\star$, $\mathsf{G}_0 = x_\mu^+ p^{+\mu} + ec^+$, and it is easily seen that

$$(\!(x_\mu^+ p^{+\mu} + ec^+, \widehat{\mathcal{A}}_\star)\!) \subset \widehat{\mathcal{A}}_\star.$$

Indeed, on restriction to $\widehat{\mathcal{A}}_\star$, the Soloviev bracket $\operatorname{ad}(x_\mu^+ p^{+\mu} + ec^+)$ is given by the evolutionary vector field

$$\operatorname{ev}\left(-x_\mu^+ \frac{\partial}{\partial p_\mu} + c^+ \frac{\partial}{\partial e^+}\right),$$

which preserves $\widehat{\mathcal{A}}_\star$. $\square$

We now turn to the question of Lorentz invariance of the solution S_u to (2.2) that we have obtained. Let $C^*(\mathfrak{so}(9,1))$ be the graded commutative algebra of the Lie algebra $\mathfrak{so}(9,1)$ of the Lorentz group (the exterior algebra generated by $\mathfrak{so}(9,1)^\vee$), with differential $\mathbf{d}$. The action of the Lie algebra $\mathfrak{so}(9,1)$ on the space of fields of the superparticle is generated by the Batalin–Vilkovisky currents

$$M^{\mu\nu} = \eta^{\lambda[\mu}x^{\nu]}x^+_\lambda - \eta^{\lambda[\mu}p^{+\nu]}p_\lambda - \sum_{n=0}^\infty \mathsf{T}^{\mu\nu}(\theta^+_n,\theta_n). \tag{3.12}$$

Let $\mathsf{S}(\epsilon) = \mathsf{S} + M^{\mu\nu}\epsilon_{\mu\nu}$, where $\epsilon_{\mu\nu}$ is the dual basis of $\mathfrak{so}(9,1)^\vee$; this is an element of total degree 0 in the tensor product of $C^*(\mathfrak{so}(9,1))$ and the Batalin–Vilkovisky graded Lie algebra. The Lorentz invariance of the action S may be expressed by the following extension of the classical master equation:

$$\mathbf{d}\mathsf{S}(\epsilon) + \tfrac{1}{2}\left(\textstyle\int\mathsf{S}(\epsilon),\int\mathsf{S}(\epsilon)\right) = 0.$$

Indeed, the coefficient of $\epsilon_{\mu\nu}$ in the above equation says that

$$\left(\textstyle\int\mathsf{S},\int M^{\mu\nu}\right) = 0.$$

The Lorentz group does not act on Thom–Whitney complex, because the open cover itself is not Lorentz invariant; in particular, G_0 is not invariant under the action of $\mathfrak{so}(9,1)$. Nevertheless, it may be proved that S_u has an enhancement

$$\mathsf{S}_u(\epsilon) = \mathsf{S}(\epsilon) + \sum_{n=0}^\infty u^{n+1}\mathsf{G}_n(\epsilon), \tag{3.13}$$

where $\mathsf{G}_n(\epsilon)$ is an element of total degree $-2n-2$ in the tensor product of $C^*(\mathfrak{so}(9,1))$ and the Thom–Whitney extension of the Batalin–Vilkovisky graded Lie algebra, such that the following extension of (2.2) holds:

$$(\mathbf{d}+\delta)\mathsf{S}_u(\epsilon) + \tfrac{1}{2}\left(\textstyle\int\mathsf{S}_u(\epsilon),\int\mathsf{S}_u(\epsilon)\right) = -u\textstyle\int\mathsf{D}. \tag{3.14}$$

Moreover, we can choose the $\mathsf{G}_n(\epsilon)$ to be supersymmetric, that is, $\mathsf{q}\mathsf{G}_n(\epsilon) = 0$. In mathematical terms, this equation, which is nothing but the BRST formalism for the global symmetry Lie algebra $\mathfrak{so}(9,1)$, expresses that the covariant field theory is invariant under supersymmetry and Lorentz invariant up to homotopy.

We solve (3.14) inductively, by an extension of the method used to prove Theorem 3.3.8. Write

$$\mathsf{G}_n(\epsilon) = \sum_{k=0}^{10} \mathsf{G}_{n,k},$$

where $\mathsf{G}_{0,0}$ equals the explicit solution $\mathsf{G}_0 \in \|\mathcal{F}(N_\bullet\mathcal{U})\|^{-2}$ of (3.7), and $\mathsf{G}_{n,k} \in C^k(\mathfrak{so}(9,1)) \otimes \|\mathcal{F}_\star(N_\bullet\mathcal{U})\|^{-2n-k-2}$ for $n > 0$ or $k > 0$. Assuming we have found $\mathsf{G}_{m,\ell}$ for $m < n$ or $m = n$ and $\ell < k$, we must solve the equation

$$(\delta + \mathsf{s})\mathsf{G}_{n,k} = -\mathbf{d}\mathsf{G}_{n,k-1} - (M^{\mu\nu}\epsilon_{\mu\nu}, \mathsf{G}_{n,k-1}) - \frac{1}{2}\sum_{m=0}^{n-1}\sum_{\ell=0}^{k}(\mathsf{G}_{m,\ell}, \mathsf{G}_{n-m-1,k-\ell}) \in C^k(\mathfrak{so}(9,1)) \otimes \|\mathcal{F}_\star(N_\bullet\mathcal{U})\|^{-2n-k-1}. \tag{3.15}$$

By the Lorentz invariance of S, $\mathsf{s}M^{\mu\nu} = 0$. For $n > 0$ or $k > 0$, this is sufficient to imply that the right-hand side of (3.15) is a cocycle. In the case $n = 0$ and $k = 1$, we need in addition the formula

$$(M^{\mu\nu}, \mathsf{D}) = 0.$$

By Lemma 3.3.7, there is a solution

$$\mathsf{G}_{n,k} \in C^k(\mathfrak{so}(9,1)) \otimes \|\mathcal{F}_\star(N_\bullet\mathcal{U})\|^{-2n-k-2}.$$

Thus there exists a supersymmetric solution to the equation (3.14).

CHAPTER 4

The quantum superparticle

We now discuss first steps towards quantizing the superparticle. Previous attempts have approached this task at the expense of breaking Lorentz symmetry. One method involves the choice of a light-cone gauge in which we single out a pair of light-like velocity vectors $q_\uparrow$ and $q_\downarrow$ which satisfy

$$\langle q_\uparrow, q_\downarrow \rangle = 1.$$

In the following section, we will explain how such a choice allows us to choose a suitable Lagrangian gauge condition; for now we simply observe that such a choice is not fixed under Lorentz transformations. While attempts at choosing a Lorentz covariant gauge fixing have been made [**10**, **11**], these approaches introduce new infinite families of auxiliary fields which lead to subtleties and discrepancies among the various prescriptions. We instead choose to study families of homotopies relating different choices of the light-cone gauge to one another.

4.1. Light-cone gauge

We first review gauge fixing for the superparticle in the light-cone gauge. As mentioned above, this gauge fixing depends on a choice of velocity vectors $q_\uparrow$ and $q_\downarrow$ which satisfy

$$\langle q_\uparrow, q_\uparrow \rangle = \langle q_\downarrow, q_\downarrow \rangle = 0 \qquad\qquad \langle q_\uparrow, q_\downarrow \rangle = 1.$$

We call the pair $(q_\uparrow, q_\downarrow)$ a light-cone frame. This choice gives an orthogonal decomposition of ten-dimensional Minkowski space

$$\mathbb{R}^{9,1} = \mathrm{span}\{q_\uparrow, q_\downarrow\} \oplus \{q_\uparrow, q_\downarrow\}^\perp,$$

where $\{q_\uparrow, q_\downarrow\}^\perp$ is an eight-dimensional Euclidean vector space. Let $\{\mathbf{e}_a\}_{a=1}^8$ be a basis for $\{q_\uparrow, q_\downarrow\}^\perp$ and $\{\mathbf{f}^a\}_{a=1}^8$ be its dual basis. Define $\gamma^{\uparrow\downarrow} = \eta^{\mu\nu} q^\mu_{\downarrow\uparrow}\gamma^\nu$ and $p_{\uparrow\downarrow} = q^\mu_{\uparrow\downarrow} p_\mu$. Clifford multiplication by p is given by the formula

$$p_\mu \gamma^\mu = p_\uparrow \gamma^\uparrow + p_\downarrow \gamma^\downarrow + \sum_{a=1}^{8} (p_\mu \mathbf{e}^\mu_a)(\gamma^\mu \mathbf{f}^a_\mu).$$

Define the projection operators $P_{\uparrow\downarrow} : \mathbb{S} \to \ker(\gamma^{\downarrow\uparrow})$ by

$$P_{\uparrow\downarrow} = \tfrac{1}{2}\gamma^{\downarrow\uparrow}\gamma^{\uparrow\downarrow}.$$

Since $P_{\uparrow\downarrow}P_{\downarrow\uparrow} = 0$ and $[P_{\uparrow\downarrow}, \gamma^\mu \mathbf{f}^a_\mu] = 0$ for $1 \leq a \leq 8$, we see that $\mathbb{S}$ has a direct sum decomposition into representations of Spin(8)

$$\mathbb{S} \cong \ker(\gamma^{\uparrow\downarrow}) \oplus \ker(\gamma^{\downarrow\uparrow}). \tag{4.1}$$

On the open subspace defined by $p_\uparrow > 0$ we consider the Lagrangian subsupermanifold defined by the equations $x^+_\mu = p^{+\mu} = c^+ = 0$, $e = 1$, and

$$\gamma^\downarrow \theta_i = \gamma^\downarrow \theta^+_i = 0 \tag{4.2}$$

for $i \geq 0$. We use the notation

$$u_i = P_\uparrow \theta_i, \qquad u^+_i = P_\uparrow \theta^+_i$$

so that, when restricting to the gauge fixing above, $\theta_i = u_i$ and $\theta^+_i = u^+_i$. On this subspace, we have the relation

$$p_\mu \mathsf{T}^\mu(-,-) = p_\uparrow \mathsf{T}^\uparrow(-,-) : \ker(\gamma^\downarrow) \otimes \ker(\gamma^\downarrow) \to \mathbb{R}$$

for any choice of momentum p. Moreover, the pairing $\mathsf{T}(-,-)$ vanishes when restricted to $\ker(\gamma^{\downarrow})$ since

$$\mathsf{T}(P_{\uparrow}\alpha, P_{\uparrow}\beta) = \mathsf{T}(\tfrac{1}{2}\gamma^{\downarrow}\gamma^{\uparrow}\alpha, \tfrac{1}{2}\gamma^{\downarrow}\gamma^{\uparrow}\beta) = \tfrac{1}{4}\mathsf{T}(\gamma^{\uparrow}\gamma^{\downarrow}\gamma^{\downarrow}\gamma^{\uparrow}\alpha, \beta)$$

and $\gamma^{\downarrow}\gamma^{\downarrow} = 0$. Using these facts, we see that the restriction of the superparticle action S to this Lagrangian is given by

$$\mathsf{S}_{gf} = p_{\mu}\partial x^{\mu} - \tfrac{1}{2}\eta^{\mu\nu}p_{\mu}p_{\nu} + e^{+}\partial c + \tfrac{1}{2}p_{\uparrow}\mathsf{T}^{\uparrow}(\partial u_0, u_0) + \sum_{i=0}^{\infty} p_{\uparrow}\mathsf{T}^{\uparrow}(u_i^{+}, u_{i+1}). \tag{4.3}$$

The condition $p_{\uparrow} > 0$ ensures that the pairing $p_{\uparrow}\mathsf{T}^{\uparrow}(-,-)$ is non-degenerate, allowing us to perform the path integral over the fermionic variables u_0. Making the change of variables

$$u_0 \mapsto \widetilde{u}_0 = p_{\uparrow}^{1/2}u_0$$

the term in the gauge-fixed action involving $\widetilde{u}_0$ becomes $\mathsf{T}^{\uparrow}(\partial\widetilde{u}_0, \widetilde{u}_0)$. This now looks like the typical kinetic term for fermions and can be handled by the usual path integral methods from quantum field theory. Observe that we could have instead restricted to the subset $p_{\uparrow} < 0$, in which case a similar change of variables,

$$u_0 \mapsto \widetilde{u}_0 = (-p_{\uparrow})^{1/2}u_0,$$

could be performed. In this case, the kinetic term in the gauge fixed action for $\widetilde{u}_0$ would be identical up to change of sign.

4.2. Homotopies between local light-cone gauge choices

The procedure described above is only valid on the subsets $\{p_{\uparrow} > 0\}$ and $\{p_{\uparrow} < 0\}$. We now construct a global Lagrangian in the sense of [**7**] using a cover of momentum space by charts of this form.

Fix a light-like vector v satisfying $v^0 < 0$. We will consider the space of vectors q which satisfy $\langle q, q\rangle = 0$ and $\langle q, v\rangle = 1$ so that the pair (q, v) forms a light-cone frame. Denote this space by C_q.

Lemma 4.2.1. *For any momentum vector $p \neq 0$, there is a choice of $q \in C_q$ for which $p_\mu q^\mu \neq 0$.*

Proof. Choose a Lorentz frame so that $v^0 = -1$, $v^1 = 1$, and $v^a = 0$ for $2 \leq a \leq 9$. Then $q \in C_q$ satisfies the following equations

$$q^0 + q^1 = 1, \qquad q^0 - q^1 = (q^2)^2 + \cdots + (q^9)^2.$$

The lemma follows since both C_q and the subspace

$$\{q \in \mathbb{R}^{9,1} \mid p_\mu q^\mu = 0\}$$

are eight-dimensional and the latter is a linear subspace while the former is not. □

Corollary 4.2.2. *The open sets $\mathcal{U} = \{U_q^\pm\}_{q\in C_q}$ defined by*

$$U_q^\pm = \{\pm p_\mu q^\mu > 0\}$$

form a cover of M_0.

On $U_q^\pm$ we may define the Lagrangian L_q by the equations (4.2) with light-cone frame $q_\uparrow = q$ and $q_\downarrow = v$.

Proposition 4.2.3. *Let $q_0, \cdots, q_k$ be points in C_q. There is a map*

$$q : \Delta^k \to C_q$$

which attains the value q_l at the lth vertex of the simplex Δ^k.

Proof. Choose a basepoint $q_* \in C_q$. There is a diffeomorphism $f : \{q_*, v\}^\perp \to C_q$ defined by

$$f(w) = q_* + w - \tfrac{1}{2}v\langle w, w\rangle.$$

Indeed,

$$\langle f(w), v\rangle = \langle q_*, v\rangle + \langle w, v\rangle - \tfrac{1}{2}\langle v, v\rangle\langle w, w\rangle = \langle q_*, v\rangle = 1$$

since $\langle w, v\rangle = 0$ and v is light-like. Additionally,

$$\begin{aligned}\langle f(w), f(w)\rangle &= \langle q_*, q_*\rangle + 2\langle q_*, w\rangle - \langle q_*, v\rangle\langle w, w\rangle + \langle w, w\rangle - \langle w, v\rangle\langle w, w\rangle + \tfrac{1}{4}\langle v, v\rangle\langle w, w\rangle\langle w, w\rangle \\ &= -\langle q_*, v\rangle\langle w, w\rangle + \langle w, w\rangle = 0\end{aligned}$$

since $\langle q_*, v\rangle = 1$, $\langle q_*, w\rangle = \langle w, v\rangle = 0$, and q_* and v are light-like. Using this, we may define the map q by

$$\begin{aligned}q(t_0, \dots, t_k) &= f\left(t_0 f^{-1}(q_0) + \cdots t_k f^{-1}(q_k)\right) \\ &= q_* + \sum_{i=0}^{k} t_i w_i - \tfrac{1}{2} v \sum_{i,j=0}^{k} t_i t_j \langle w_i, w_j\rangle\end{aligned}$$

where $w_i = f^{-1}(q_i)$. $\square$

Define

$$U_{q_0 \cdots q_k}^{\epsilon_0 \cdots \epsilon_k} = U_{q_0}^{\epsilon_0} \cap \cdots \cap U_{q_k}^{\epsilon_k}$$

where $\epsilon_l \in \{+, -\}$ for $0 \leq l \leq k$. A family of Lagrangians in $U_{q_0 \cdots q_k}^{\epsilon_0 \cdots \epsilon_k}$ is described by a map

$$\iota : L \times \Delta^k \longrightarrow U_{q_0 \cdots q_k}^{\epsilon_0 \cdots \epsilon_k}$$

where, for each $t \in \Delta^k$, the image of the restriction $\iota|_{L \times \{t\}}$ is a Lagrangian subsupermanifold. A Lagrangian subsupermanifold of $N_\bullet \mathcal{U}$ is a supermanifold L with cover $\mathcal{L} = \{L_q^{\pm}\}$ and a collection of Lagrangian families

$$\iota_{q_0 \cdots q_k}^{\epsilon_0 \cdots \epsilon_k} : L_{q_0 \cdots q_k}^{\epsilon_0 \cdots \epsilon_k} \times \Delta^k \longrightarrow U_{q_0 \cdots q_k}^{\epsilon_0 \cdots \epsilon_k}$$

such that, for each morphism $f : [k] \to [\ell]$ in the simplicial category, the following diagram commutes

$$\begin{tikzcd} N_\ell \mathcal{L} \times \Delta^k \arrow[r, "\mathrm{id}\times f_*"] \arrow[dd, "f^*\times\Delta^k"'] & N_\ell \mathcal{L} \times \Delta^\ell \arrow[d, "\iota_\ell"] \\ & N_\ell \mathcal{U} \arrow[d, "f^*"] \\ N_k \mathcal{L} \times \Delta^k \arrow[r, "\iota_k"'] & N_k \mathcal{U}. \end{tikzcd}$$

We define the supermanifold L with coordinates $(x^\mu, p_\mu, e^+, c, \{u_i\}_{i\geq 0}, \{u_i^+\}_{i\geq 0})$ where $p \neq 0$ and u_i and u_i^+ have analogous partity to θ_i and θ_i^+, but take values in the eight-dimensional Majorana–Weyl representations $\ker(\gamma^\downarrow)$ of Spin(8) (for any choice of $q_\uparrow$ these representations are isomorphic). Let $\mathcal{L}$ be the cover of L defined by $L_q^\pm = \{\pm p_\mu q^\mu > 0\}$. Define $\gamma^\downarrow(t) = \eta^{\mu\nu} q^\mu(t)\gamma^\nu$, where $q(t)$ is the map constructed in Proposition 4.2.3, and $\gamma^\uparrow = \eta^{\mu\nu} v^\mu \gamma^\nu$. We define the families of Lagrangians $\iota = \iota_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k} : L_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k} \times \Delta^k \to U_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}$ by

$$\begin{aligned} &\iota^* x^\mu = x^\mu, && \iota^* p_\mu = p_\mu, && \iota^* e = 1, && \iota^* c = c, && \iota^*\theta_i = P_\uparrow(t)\theta_i, \\ &\iota^* x_\mu^+ = 0, && \iota^* p^{+\mu} = 0, && \iota^* e^+ = e^+, && \iota^* c^+ = 0, && \iota^*\theta_i^+ = P_\uparrow(t)\theta_i^+. \end{aligned}$$

where $P_\uparrow(t) = \frac{1}{2}\gamma^\downarrow(t)\gamma^\uparrow$. Explicitly, the formula for $P_\uparrow(t)$ can be written as

$$\begin{aligned} P_\uparrow(t) &= \tfrac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma} q^\mu(t) v^\rho \gamma^\nu\gamma^\sigma \\ &= \tfrac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\left(q_*^\mu + \sum_{i=0}^k t_i w_i^\mu - \tfrac{1}{2}\sum_{i,j=0}^k t_i t_j v^\mu \langle w_i, w_j\rangle\right) v^\rho\gamma^\nu\gamma^\sigma \\ &= \tfrac{1}{2}\eta^{\mu\nu}\eta^{\rho\sigma}\left(\sum_{i=0}^\infty t_i\left(q_*^\mu + w_i^\mu - \tfrac{1}{2}v^\mu\langle w_i, w_i\rangle\right)\right) v^\rho\gamma^\nu\gamma^\sigma \\ &= \tfrac{1}{2}\sum_{i=0}^k t_i \eta^{\mu\nu}\eta^{\rho\sigma} q_i^\mu v^\rho \gamma^\nu\gamma^\sigma \end{aligned}$$

where we have used $\gamma^{\uparrow}\gamma^{\uparrow} = 0$ and $\sum_{i=0}^{k} t_i = 1$. The gauge-fixed action (4.3) is the pull-back of the solution to the classical master equation S along $\iota_q^{\pm}$. For a general k-simplex, the pull-back of S along $\iota_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}$ will involve additional terms expressing the dependence on $t \in \Delta^k$. We describe these generalizations of the gauge-fixed action in [**7**].

Let X be the one-form on Δ^k taking values in vector fields which describes the flow of the family $\iota_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}$,

$$X = \frac{\partial}{\partial t^l}\iota_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}\delta^l .$$

Contracting the Batalin–Vilkovisky symplectic form ω with X and pulling back by $\iota_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}$ produces a differential form

$$\big(\iota_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}\big)^*(X \lrcorner \omega) \in \Omega^1\big(L_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}\big)\widehat{\otimes}\Omega_k . \tag{4.4}$$

The condition that $\iota_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}$ defines a family of Lagrangians is equivalent to the condition that this differential form is closed under the exterior differential d on L,

$$d\big(\iota_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}\big)^*(X \lrcorner \omega) = 0.$$

Since the differential form (4.4) has ghost-number -1 and the de Rham cohomology of L vanishes outside of ghost-number zero, we may find a differential form $\tau \in \mathcal{O}\big(L_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}\big)\widehat{\otimes}\Omega_k$ such that

$$d\tau_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k} = \big(\iota_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}\big)^*(X \lrcorner \omega)$$

and $\delta\tau_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k} = 0$. In the present case,

$$X = \sum_{i=0}^{\infty} \mathsf{T}\left(\delta P_{\uparrow}(t)\theta_i, \frac{\partial}{\partial \theta_i}\right) + \sum_{i=0}^{\infty} \mathsf{T}\left(\delta P_{\uparrow}(t)\theta_i^{+}, \frac{\partial}{\partial \theta_i^{+}}\right).$$

The contraction of this vector field with ω can then be computed to be

$$X \lrcorner \omega = \sum_{i=0}^{\infty} \mathsf{T}(\delta P_{\uparrow}(t)\theta_i, d\theta_i^{+}) + \sum_{i=0}^{\infty} \mathsf{T}(\delta P_{\uparrow}(t)\theta_i^{+}, d\theta_i).$$

The identity $\langle q(t), v\rangle = 1$ implies that $\frac{\partial}{\partial t^l}\langle q(t), v\rangle = 0$, hence $\delta\gamma^{\downarrow}(t)\gamma^{\uparrow} = -\gamma^{\uparrow}\delta\gamma^{\downarrow}(t)$. Using this, we can rewrite $X \lrcorner \omega$ as

$$\begin{aligned}
X \lrcorner \omega &= \tfrac{1}{2}\sum_{i=0}^{\infty} \mathsf{T}\big(\delta\gamma^{\downarrow}(t)\gamma^{\uparrow}\theta_i, d\theta_i^+\big) + \tfrac{1}{2}\sum_{i=0}^{\infty} \mathsf{T}\big(\delta\gamma^{\downarrow}(t)\gamma^{\uparrow}\theta_i^+, d\theta_i\big) \\
&= \tfrac{1}{2}\sum_{i=0}^{\infty} \mathsf{T}\big(\delta\gamma^{\downarrow}(t)\gamma^{\uparrow}\theta_i, d\theta_i^+\big) + \tfrac{1}{2}\sum_{i=0}^{\infty}(-1)^i \mathsf{T}\big(\theta_i^+, \gamma^{\uparrow}\delta\gamma^{\downarrow}(t)d\theta_i\big) \\
&= \tfrac{1}{2}\sum_{i=0}^{\infty} \mathsf{T}\big(d\theta_i^+, \delta\gamma^{\downarrow}(t)\gamma^{\uparrow}\theta_i\big) + \tfrac{1}{2}\sum_{i=0}^{\infty}(-1)^{i+1} \mathsf{T}\big(\theta_i^+, \delta\gamma^{\downarrow}(t)\gamma^{\uparrow}d\theta_i\big) \\
&= \sum_{i=0}^{\infty}\Big(\mathsf{T}(d\theta_i^+, \delta P_{\uparrow}(t)\theta_i) + (-1)^{i+1}\mathsf{T}(\theta_i^+, \delta P_{\uparrow}(t)d\theta_i)\Big)
\end{aligned}$$

so we see that $\tau_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}$ is given by the formula

$$\tau_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k} = \sum_{i=0}^{\infty} \mathsf{T}(u_i^+, \delta P(t)u_i).$$

Let $\{\eta_q^{\pm}\}_{q\in C_q}$ be a partition of unity subordinate to the cover $\mathcal{U}$. In [**7**] we describe how to construct a linear form

$$\mathsf{Z} : ||\Omega^{1/2}(N_{\bullet}\mathcal{U})|| \to \mathbb{C}$$

from the data of the cover $\mathcal{U}$, the partition of unity $\{\eta_q^{\pm}\}_{q\in C_q}$, the families of Lagrangian subsupermanifolds $\iota_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}$, and the generating one-forms $\tau_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k}$. Define the first order differential operator on half-forms

$$\mathbf{H}_q^{\pm} = [\Delta, \eta_q^{\pm}]$$

where Δ is the Batalin-Vilkovisky Laplacian described in Proposition 2.2.3. We also make use of the notation

$$\eta_{q_0\cdots q_k}^{\epsilon_0\cdots\epsilon_k} = \frac{\hbar^k}{k+1}\sum_{i=0}^{k}(-1)^i \mathbf{H}_{q_0}^{\epsilon_0}\cdots\mathbf{H}_{q_{i-1}}^{\epsilon_{i-1}}\eta_{q_i}^{\epsilon_i}\mathbf{H}_{q_{i+1}}^{\epsilon_{i+1}}\cdots\mathbf{H}_{q_k}^{\epsilon_k}.$$

The integral of a Thom-Whitney cochain $\sigma_\bullet \in ||\Omega^{1/2}(N_\bullet\mathcal{U})||$ is then defined as

$$\mathsf{Z}(\sigma_\bullet) := \sum_{k=0}^{\infty}(-1)^k \sum_{(\epsilon_0,q_0)\cdots(\epsilon_k,q_k)} \int_{\Delta^k}\int_{L^{\epsilon_0\cdots\epsilon_k}_{q_0\cdots q_k}} e^{-\tau^{\epsilon_0\cdots\epsilon_k}_{q_0\cdots q_k}/\hbar}\big(\iota^{\epsilon_0\cdots\epsilon_k}_{q_0\cdots q_k}\big)^*\big(\eta^{\epsilon_0\cdots\epsilon_k}_{q_0\cdots q_k}\sigma_\bullet\big).$$

In [**7**] we establish the following properties of this linear form.

Proposition 4.2.4 (Stokes' theorem)**.** *Let* $\sigma_\bullet \in ||\Omega^{1/2}(N_\bullet\mathcal{U})||$. *Then*

$$\mathsf{Z}\big((\delta + \hbar\Delta)\sigma_\bullet\big) = 0.$$

Proposition 4.2.5 (Lorentz-covariance)**.** *Let* $\sigma_\bullet \in C^*(\mathfrak{so}(9,1)) \otimes ||\Omega^{1/2}(N_\bullet\mathcal{U})||$ *be a Thom-Whitney cochain with values in the Chevalley-Eilenberg complex of the Lorentz Lie algebra. Then*

$$\mathsf{Z}\big((\mathbf{d} + \delta + \hbar\Delta)\sigma_\bullet\big) = \mathbf{d}\mathsf{Z}(\sigma_\bullet)$$

where **d** *is the Chevalley-Eilenberg differential.*

Together, these results imply that if $\mathbf{d}\sigma_\bullet = 0$, then $\mathbf{d}\mathsf{Z}(\sigma_\bullet) = 0$. In other words, the integral of a Lorentz invariant cochain is, itself, Lorentz invariant. In particular, the integral of a Lorentz invariant zero-cocycle σ of the form

$$\sigma = e^{i\mathsf{S}(\hbar)/\hbar}$$

is Lorentz invariant. This zero-cochain is a zero-cocycle when $\mathsf{S}(\hbar)$ is a global solution of the quantum master equation. In this case, $\mathsf{Z}(\sigma)$ defines the Lorentz invariant partition function for the superparticle.

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