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A Study of the Equivariant Gromov-Witten Theory of the Projective Line and Eynard-Orantin Recursion

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Abstract

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Using Eynard-Orantin topological recursion, we prove here a result concerning the equivariant Gromov-Witten invariants for the projective line equipped with the standard action of the 2-torus. Our result is that the genus g, n point Gromov-Witten potential with arbitrary primary insertions may be written as a sum over certain genus 0, 2 point Gromov-Witten potentials.

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CHAPTER 1

Introduction

1.1. Motivations

It was recently discovered [**15**] that the equivariant Gromov-Witten theory [**18**] of $\mathbb{P}^1$ equipped with the standard action of $T = (\mathbf{C}^\times)^2$ was equivalent to spectral curve data in the sense of Eynard and Orantin [**6**, **8**]. Motivated by [**4**], in [**15**] it was shown that through a particular change of coordinates, the genus g, n-point Gromov-Witten potential with arbitrary primary insertions could be identified with the genus g, n-point correlation function arising from the data of a spectral curve Σ and two analytic functions defined on it.

Eynard-Orantin theory has been linked to many counting problems in enumerative geometry (see [**13**] for a review). Given a counting problem, however, there is no general recipe for constructing the spectral curve data (although there has been progress on this front – see [**4**] for instance). It appears to be a case of mirror symmetry: given a counting problem A, computation of the spectral curve B amounts to finding the mirror dual of A. Moreover, it is conjectured [**3**] that the spectral curve data is a Laplace transform of the counting problem:

Conjecture 1.1 (The Laplace transform conjecture [**3**]). *Given the solutions to the counting problem for the unstable cases* $(g, n) = (0, 1)$ *and* $(0, 2)$ *on the A-model side, their Laplace transforms determine the spectral curve and the recursion kernel of the Eynard-Orantin formalism on the mirror B-model side.*

A hope is that our study of one proven instance of the above might shed new light on the role of Eynard-Orantin theory in mirror symmetry.

1.2. Gromov-Witten Theory

Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ denote the moduli space of stable maps f of degree d from a curve Σ with marked points $(x_1, \ldots, x_n)$ to $\mathbb{P}^1$. Let $\mathrm{ev}_j : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \to \mathbb{P}^1$ denote the evaluation map obtained by applying f to the j^{th} marked point. Suppose that T acts on $\mathbb{P}^1$ as $(\theta_1, \theta_2) \cdot [z_1; z_2] = [\theta_1 z_1; \theta_2 z_2]$. This induces an action of T on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$, and so we have induced maps on equivariant cohomology

$$\mathrm{ev}_j^* : H_T^*(\mathbb{P}^1, \mathbf{Q}) \to H_T^*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d), \mathbf{Q}).$$

For $j = 1, \ldots, n$, let ψ_j denote the first equivariant Chern class of the line bundle $\mathcal{L}_i$ on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ whose fibre over a point $(\Sigma, (x_1, \ldots, x_n), f)$ is given by $T^*_{x_j}\Sigma$.

Given $2g - 2 + n > 0$ and classes $\alpha_1, \ldots, \alpha_n \in H_T^*(\mathbb{P}^1, \mathbf{Q})$, define the generating function

$$\left\langle\!\!\left\langle \frac{\alpha_1}{1 - z_1\psi_1}, \cdots, \frac{\alpha_n}{1 - z_n\psi_n} \right\rangle\!\!\right\rangle_{g,n} = \sum_{\substack{d,l \geq 0 \\ a_1 \ldots a_n \geq 0}} \frac{Q^d}{l!} \langle \tau_{a_1}(\alpha_1) \ldots \tau_{a_n}(\alpha_n) \underbrace{\tau_0(\mathbf{t}) \ldots \tau_0(\mathbf{t})}_{l \text{ times}} \rangle_{g,n+l,d} \prod_{i=1}^{n} z_i^{a_i}$$

where $\mathbf{t} = t_0 e + t_1 H$ and

$$\langle \tau_{a_1}(\alpha_1) \ldots \tau_{a_n}(\alpha_n) \rangle_{g,n} = \int_{\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)\right]^{vir}} \prod_{1 \leq j \leq n} \psi_j^{a_j} \, \mathrm{ev}_j^*(\alpha_j) \ \in \ \mathbf{Q}[s_1, s_2]$$

with $\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)\right]^{vir}$ the virtual fundamental cycle, and $\mathbf{Q}[s_1, s_2] = H_T^*(\{*\}, \mathbf{Q})$. In Chapter 3 we give a definition of this generating function in the case $(g, n) = (0, 2)$

1.3. Eynard-Orantin Theory

Consider the Eynard-Orantin spectral curve Σ in $\mathbf{C}^2$ identified as the locus of points

$$x = t_0 + Y + \frac{Q\,\mathrm{e}^{t_1}}{Y} + s_1 \log(Y) + s_2 \log\left(\frac{Q\,\mathrm{e}^{t_1}}{Y}\right)$$

$$y = \log(Y)$$

This is a cover $\Sigma \to \mathbf{C}^\times$ under $(x, y) \mapsto Y$. Let $Y = P_1, P_2$ denote the two branch points of the map $x \to Y$. These branch points are of order two, and so in a neighbourhood of a branch point P_i we have a unique involution $Y \mapsto \hat{Y}$ defined by $x(Y) = x(\hat{Y})$ with $Y \neq \hat{Y}$ and $\hat{Y} \to P_i$ as $Y \to P_i$.

Given indices $K = (i_1, \ldots, i_k)$, write $Y_K = (Y_{i_1}, \ldots, Y_{i_k})$.

Let

$$B(Y, Y') = \frac{dY dY'}{(Y - Y')^2}$$

be a bilinear form defined on $\Sigma \times \Sigma$. For integers g, n, $n > 0$, the correlation functions $\omega_{g,n}$ are n-linear differential forms on $\Sigma^{\times n}$ defined recursively as follows:

$$\omega_{g,n}(Y_1, \ldots, Y_N) = 0 \qquad\qquad g < 0$$

$$\omega_{0,1}(Y) = 0$$

$$\omega_{0,2}(Y_1, Y_2) = B(Y_1, Y_2)$$

and, for $K = (1, \ldots, n)$

$$\omega_{g,n+1}(Y, Y_K) = \sum_{i=1,2} \operatorname*{Res}_{Y \to P_i} \frac{\int_{\xi=Y}^{\hat{Y}} B(Y, \xi)}{\left(y(Y) - y(\hat{Y})\right) dx(Y)} W_{g,n}(Y, \hat{Y}, Y_K)$$

where

$$W_{g,n}(Y, \hat{Y}, Y_K) = \omega_{g-1,n+2}(Y, \hat{Y}, Y_K) + \sum_{\substack{g_1+g_2=g \\ J \coprod J'=K}} \omega_{g_1,|J|+1}(Y, Y_J)\omega_{g_2,|J'|+1}(\hat{Y}, Y_{J'}).$$

1.4. The Results of this Work

One of the main results in [**15**] is as follows. Given integers g, n with $n > 0$ and $2g-2+n > 0$, we have

$$\left\langle\!\!\left\langle \frac{z_1\kappa(\mathcal{L}_1, z_1)}{1 - z_1\psi_1}, \cdots, \frac{z_n\kappa(\mathcal{L}_n, z_n)}{1 - z_n\psi_n} \right\rangle\!\!\right\rangle_{g,n} = \int_{\gamma(\mathcal{L}_1)} \cdots \int_{\gamma(\mathcal{L}_n)} \exp\left(\sum_{i=1}^{n} z_i x(Y_i)\right) \omega_{g,n} \tag{1.1}$$

where $\kappa(\mathcal{L}, z)$ is a certain expression in the equivariant Chern classes of the equivariant line bundle $\mathcal{L}$ and the equivariant tangent bundle of $\mathbb{P}^1$, and $\gamma(\mathcal{L}_i)$ is a certain contour on Σ.

The main result of this thesis is a consequence of aforementioned theorem in [**15**]: given integers g, n, $n > 0$, $2g - 2 + n > 0$, and the unit e and hyperplane class $H = c_1^T(\mathcal{O}(1))$ in $H_T^*(\mathbb{P}^1, \mathbf{Q})$ we may write $\left\langle\!\!\left\langle \frac{\alpha_1}{1 - z_1\psi_1}, \cdots, \frac{\alpha_n}{1 - z_n\psi_n} \right\rangle\!\!\right\rangle_{g,n}$ as a finite sum of terms involving genus zero, two point generating functions, of the form

$$\prod_{j=1}^{n} \left\langle\!\!\left\langle \beta_{i_j}, \frac{\alpha_j}{1 - z_j\psi_j} \right\rangle\!\!\right\rangle_{0,2}$$

where β_{i_j} is either e or H, with coefficients in $\mathbf{Q}\left[s, \mathrm{e}^{-t_1}, (s^2 + 4\,\mathrm{e}^{t_1})^{\pm\frac{1}{2}}, z_1, \ldots, z_n\right]$. The proof amounts to an analysis of (1.1).

First, we observe that the generating functions $\kappa(\mathcal{L})$ generate $H_T^*(\mathbb{P}^1, \mathbf{Q})$. Second, we show that we can find an n-linear differential $\Omega_{g,n}$ such that for any choice of line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_n$,

$$\int_{\gamma(\mathcal{L}_1)} \cdots \int_{\gamma(\mathcal{L}_n)} \exp\left(\sum_{i=1}^{n} z_i x(Y_i)\right) \omega_{g,n} = \int_{\gamma(\mathcal{L}_1)} \cdots \int_{\gamma(\mathcal{L}_n)} \exp\left(\sum_{i=1}^{n} z_i x(Y_i)\right) \Omega_{g,n} \tag{1.2}$$

but where the evaluation of the right hand side of (1.2) is straightforward.

Last, we show that the form of $\Omega_{g,n}$ implies the main result holds.

1.5. Organization

The organization of this thesis is as follows. In chapter 2 we review Eynard-Orantin recursion, and note some of the earlier results in the theory. In chapter 3, we give a brief overview of the relevant elements of Gromov-Witten. In chapter 4 we recall the details of [**15**] that we need. We pause to study the non-standard nature of the spectral curve data Σ, before presenting our main result, and proving it.

CHAPTER 2

Eynard-Orantin Topological Recursion

In this chapter we explain the Eynard-Orantin data and define the correlation functions $\omega_{g,n}$ associated to this data. The correlation functions are recursively defined in a manner similar to the way in which a genus g Riemann surface with n marked points may degenerate. Eynard and Orantin refer to this as topological recursion, while other authors pay homage to its inventors. The recursion was originally discovered as a Ward identity in the context of matrix models [**5**], of which Witten's model [**26**] is a special case. The low temperature expansion of such a matrix model could be expressed as a sum over weighted ribbon graphs. Eynard observed that in the one-matrix model, the one and two loop insertion expectation values — corresponding to the genus 0, 1 point and genus 0, 2 point contributions to the sum over ribbon graphs, could be used to define a hyperelliptic curve. Moreover, the algebraic geometry of this hyperelliptic curve alone could be used to recursively determine the entire statistical theory of the model, that is, the contributions the ribbon graphs for all g, n to the partition function of the one-matrix model in question.

The recursion is somewhat similar to a Feynmann diagram expansion[1]. Studying the recursion, Eynard and Orantin discovered [**8**] that the recursion was entirely geometric in origin, and, with some mild limitations, could be seen to arise from any Riemann surface with two meromorphic functions x and y defined on it. While established for compact Riemann surfaces, Eynard-Orantin recursion can be made to work for many cases in which the Riemann is not compact.

[1]However, it is not a Feynmann diagram expansion as there are non-local constraints on graphs

2.1. Eynard-Orantin Recursion

Definition 2.1 (Eynard-Orantin spectral curve data)**.** *The data for Eynard-Orantin recursion is a compact genus* $\mathfrak{g}$ *Riemann surface* Σ *known as the spectral curve, equipped with two meromorphic functions* x*, and* y *and assume that the surface comes equipped with a marking, that is, a basis of 1-cycles* $\{A\}_{1\leq i\leq \mathfrak{g}}$ *and* $\{B\}_{1\leq i\leq \mathfrak{g}}$ *so that*

$$A_i \cdot A_j = B_i \cdot B_j = 0 \qquad\qquad A_i \cdot B_j = \delta_{ij}.$$

We require that all ramification of the map $(x, y) \mapsto x$ *are simple, .*

About a branch point a of the map $(x, y) \mapsto x$, with $q \in \Sigma$ sufficiently near a we define a conjugation $q \to \hat{q}$ as follows. We have $y(q) \sim y(a) + A\sqrt{x(q) - x(a)}$ as $q \to a$. Define $\hat{q}$ to be the point such that

$$x(q) = x(\hat{q})$$
$$y(\hat{q}) \sim y(a) - A\sqrt{x(q) - x(a)}$$

The data of the A cycles determines the so-called fundamental normalized differential of the second kind [**16**], sometimes known as the Bergmann kernel. This is the unique meromorphic section $B(p, q)$ of $T^*\Sigma \otimes T^*\Sigma \to \Sigma \times \Sigma$ that is symmetric, analytic away from the diagonal $\Sigma \mapsto \Sigma \times \Sigma$, and for p, q nearby points on Σ with z a local coordinate on Σ near p, q

$$B(p, q) \sim_{p\to q} \frac{dz(p)dz(q)}{(z(p) - z(q))^2} + b(z(p), z(q))$$
$$\oint_{p\in\mathcal{A}_i} B(p, q) = 0$$

where $b(z(p), z(q))$ is analytic.

In the case we are interested in, $\mathfrak{g} = 0$, $\Sigma = \mathbb{P}^1$, with coordinate z, we have

$$B(z(p), z(q)) = \frac{dz(p)dz(q)}{(z(p) - z(q))^2}.$$

which is uniquely defined since $\mathfrak{g} = 0$.

2.2. Correlation Functions and Eynard-Orantin Recursion

The correlation functions are our main objects of study. In practice, they arise as generating functions for counting problems in enumerative geometry.

Definition 2.2 (Correlation functions)**.** *Suppose we have Eynard-Orantin spectral curve data* (Σ, x, y). *The correlation functions* $\omega_{g,k}$ *are sections of* $(T^*\Sigma)^{\otimes k} \to \Sigma^k$, *defined by the following recursion.*

Denote by $\mathbf{a}$ *the set of branch points of the map* $(x, y) \mapsto x$. *For* $g, n \in \mathbf{Z}$ *with* $n > 0$, *define*

$$\omega_{g,k}(p_1, \ldots, p_k) = 0 \qquad g < 0$$

$$\omega_{0,1}(p) = 0$$

$$\omega_{0,2}(p_1, p_2) = B(p_1, p_2)$$

and, for $K = \{1, \ldots, k\}$

$$\omega_{g,k+1}(p, p_K) = \sum_{\mathbf{a}} \operatorname*{Res}_{q \to \mathbf{a}} \frac{\int_{\xi=q}^{\hat{q}} B(p, \xi)}{(y(q) - y(\hat{q}))\, dx(q))} W_{g,n}(q, \hat{q}, p_K) \tag{2.1}$$

where

$$W_{g,n}(q, \hat{q}, p_K) = \omega_{g-1,k+2}(q, \hat{q}, p_K) + \sum_{\substack{g_1+g_2=g \\ J \coprod J'=K}} \omega_{g_1,|J|+1}(q, p_J)\omega_{g_2,|J'|+1}(\hat{q}, p_{J'}).$$

This recursion is effective: all $\omega_{h,m}$ appearing in non-zero terms on the right hand side of (2.1) have $2h-2+m<2g-2+k$. We note that there is a graphical representation of the correlation functions and the recursion. We refer the reader to [**8**] for details.

One important result we need regarding the correlation functions is the following

Theorem 2.3 (Theorem 4.6:[**8**]). *$\omega_{g,n}(p_1,\ldots,p_n)$ is symmetric in its arguments.*

We not that the proof is a computation using only local properties of the curve about the branch points. It can be found in [**8**].

2.3. Example: The Airy Curve and Intersection Theory on the Moduli Space of Curves.

In [**20**], it was proved that a certain generating function of intersection numbers on the moduli space of curves could be expressed as a matrix integral. Given the origins of Eynard-Orantin theory, we should expect to be able to find a spectral curve that yields these intersection numbers. Indeed this is true [**8**], and turned out to be the prototypical example of the theory.

Consider the plane curve

$$x=\frac{1}{2}y^2$$

known as the Airy curve in the theory, on account of its relationship to Kontsevich's matrix model. The underlying surface is a copy of $\mathbb{P}^1$. The function y is a suitable co-ordinate, but for clarity of exposition, we introduce a new co-ordinate, z, such that

$$x=\frac{1}{2}z^2$$

$$y=z$$

We have

$$B(z,z')=\frac{dzdz'}{(z-z')^2}$$

$dx = 0$ exactly when $z = 0$, so there is only one branch point. Conjugation is given by $\hat{z} = -z$, so the Eynard-Orantin recursion is

$$\omega_{g,k+1}(z_0, \ldots, z_k) = \operatorname*{Res}_{z \to 0} \frac{dz_0}{(z^2 - z_0^2) z dz} W_{g,n}(z, -z, z_{\mathbf{K}}).$$

For example

$$\begin{aligned}
\omega_{0,3} &= \frac{dz_1}{z_1^2 z_2^2 z_3^2} \\
\omega_{0,4} &= \frac{3 dz_1 dz_2 dz_3}{z_1^2 z_2^2 z_3^2 z_4^2} \left(\frac{1}{z_1^2} + \frac{1}{z_2^2} + \frac{1}{z_3^2} + \frac{1}{z_4^2} \right) \\
\omega_{1,1} &= \frac{dz_1}{8 z_1^4} \\
\omega_{1,2} &= \frac{dz_1 dz_2}{8 z_1^2 z_2^2} \left(\frac{5}{z_1^4} + \frac{3}{z_1^2 z_2^2} + \frac{5}{z_2^4} \right) \\
\omega_{2,1} &= \frac{105 dz_1}{128 z_1^{10}}
\end{aligned}$$

The result is that $\omega_{g,k+1}(z_0, \ldots, z_k)$ is a generating function for the intersection theory of certain canonical classes defined on the Deligne-Mumford compactification of the moduli space $\overline{\mathcal{M}}_{g,n}$ of genus g Riemann surfaces with n marked points and $2g - 2 + n > 0$.

Consider the line bundle $\mathcal{L}_i \to \bar{\mathcal{M}}_{g,n}$ where the fibre over a $(\Sigma, x_1, \ldots, x_n)$ is $T^*_{x_i}\Sigma$. Then let $\psi_i = c_1(\mathcal{L}_i)$ be the first Chern class of this line bundle. Then we can consider

$$\langle \tau_{a_1} \ldots \tau_{a_n} \rangle_{g,n} = \int_{[\overline{\mathcal{M}}_{g,n}]^{vir}} \psi_1^{a_1} \ldots \psi_n^{a_n}$$

which is zero if $\sum_j i_j \neq 3g - 3 + n$.

One can show that

$$\omega_{g,n}(z_1,\ldots,z_n) = \sum_{a_1,\ldots,a_n\geq 0} \langle \tau_{a_1}\ldots\tau_{a_n}\rangle_{g,n} \prod_{i=1}^{n} \frac{(2a_i+1)!!dz_i}{z_i^{2a_i+2}}$$

2.4. Other examples

Examples in enumerative geometry are abundant. The following are known to arise from the Eynard-Orantin framework. Among other problems, we have Eynard-Orantin data for, (or at least convincing evidence),

- Stationary Gromov-Witten invariants of $\mathbb{P}^1$[**4**, **22**]
- Hurwitz numbers [**9**].
- Gromov-Witten invariants of toric Calabi-Yau 3-folds [**12**].
- Gromov-Witten invariants of toric Calabi-Yau orbifolds [**14**]
- Weil-Petersson volumes [**7**]
- The ELSV formula [**10**, **11**]
- Semi-simple cohomological field theories

The last example is worth expanding upon further. In [**4**] it was shown that the one could do away with the global spectral curve and replace it with a *local* spectral curve, namely, local power series data for y and B about n "branch points". The function x is assumed to be of a standard form near these points. Using this, it was shown that through a complicated change of variables, the data of the partition function of a semi-simple cohomological field theory, could be identified with the correlation functions arising from a local spectral curve.

CHAPTER 3

Equivariant Gromov-Witten Theory of the Projective Line.

3.1. Moduli Spaces of Stable Maps and Non-equivariant Gromov-Witten Invariants of the Projective Line

We first recall the definition of Kontsevich's moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ (see [**21**] and [**2**]).

A stable map consists of a triple $(\Sigma, (x_1, \ldots, x_n), f)$, where Σ is a complex curve with only double singular points, $n \geq 0$ ordered, non-singular, distinct points $(x_1, \ldots, x_n)$ called the marked points, and a map $f : \Sigma \to \mathbb{P}^1$, such that f does not have any infinitesimal automorphisms, although we allow stable maps with a finite number of automorphisms. Define a special point to be either a marked point or singular point. A map is not stable if either: f is constant on a genus 0 irreducible component of Σ with strictly fewer than 3 special points or; if f is constant and Σ is a torus with no special points. The points of $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ consist of those those stable maps such that Σ is of arithmetic genus g with n marked points, and $f : \Sigma \to \mathbb{P}^1$ is of degree d.

We have the evaluation maps

$$\mathrm{ev}_j : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \to \mathbb{P}^1$$

$$\mathrm{ev}_j : (\Sigma, (x_1, \ldots, x_n), f) = f(x_j)$$

In non-equivariant Gromov-Witten theory relevant classes are defined as follows: given a class $\alpha \in H^*(\mathbb{P}^1, \mathbf{Q})$, we have the *primary classes*

$$\mathrm{ev}_j^*(\alpha) \in H^*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d), \mathbf{Q}).$$

For $j = 1, \dots, n$ let $\mathcal{L}_i$ denote the line bundle on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ with fibre at $(\Sigma, (x_1, \dots, x_n), f)$ defined as the cotangent line $T^*_{x_j}\Sigma$. Define $\psi_j = c_1(\mathcal{L}_j) \in H^*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d), \mathbf{Q})$. given an integer $a_j > 0$ the *descendent classes* are the classes $\psi_i^{a_j} \operatorname{ev}_j^*(\alpha)$.

The Gromov-Witten invariants for $\mathbb{P}^1$ are the intersection numbers

$$\int_{\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)\right]^{vir}} \prod_{1 \leq j \leq n} \psi_j^{a_j} \operatorname{ev}_j^*(\alpha_j)$$

Here $\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)\right]^{vir} \in H_*(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d), \mathbf{Q})$ is a homology class known as the virtual fundamental class, and is of the expected dimension, $2g + 2d - 2 + n$. Its construction is non-trivial [**18**, **21**]. Assuming the classes α_j are of pure degree, the above integral is non-zero only if

$$2g + 2d - 2 + n = \sum_{j=1}^{n} \left(\deg(\alpha_j) + a_j\right).$$

3.2. Equivariant Gromov Witten Invariants of the Projective Line

In equivariant Gromov-Witten theory [**18**], we must modify our definitions slightly. Consider $\mathbb{P}^1$ equipped with the action of $T = (\mathbf{C}^\times)^2$, coming from the usual action of $(\mathbf{C}^\times)^2$ on $\mathbf{C}^2$, $(t_1, t_2) \cdot (z_1, z_2) = (t_1 z_1, t_2 z_2)$. In the equivariant theory we replace $H^*(\mathbb{P}^1, \mathbf{Q})$ with $H^*_T(\mathbb{P}^1, \mathbf{Q})$, and so must modify our definitions of the primary classes and virtual fundamental cycle.

The action of T on $\mathbb{P}^1$ induces an action on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$. This action is equivariant with respect to the evaluation and forgetful maps. Let ET denote the classifying bundle for T, and let BT be its classifying space. Then we have the diagram

$$\begin{array}{ccc} ET \times_T \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) & \xrightarrow{\operatorname{ev}_j} & ET \times_T \mathbb{P}^1 \\ \big\downarrow{\scriptstyle \pi} & & \\ BT & & \end{array}$$

The evaluation map ev_j now gives map

$$\mathrm{ev}_j^* : H_*^T(\mathbb{P}^1, \mathbf{Q}) = H_*^T(ET \times_T \mathbb{P}^1, \mathbf{Q}) \to H(ET \times_T \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d), \mathbf{Q}),$$

while the ψ-classes may be defined as before.

Given $2g - 2 + 2d + n > 0$ so that $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ is non-empty, classes $\alpha_1, \dots \alpha_n \in H_T^*(\mathbb{P}^1, \mathbf{Q})$ and integers $a_1, \dots, a_n \geq 0$ we define the equivariant Gromov-Witten invariants

$$\langle \tau_{a_1}(\alpha_1) \cdots \tau_{a_n}(\alpha_n) \rangle_{g,n,d} = \int_{\left[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)\right]^{vir}} \prod_{1 \leq j \leq n} \psi_j^{a_j} \, \mathrm{ev}_j^*(\alpha_j)$$

by integration along the fibres of π. These invariants take values in

$$H^*(BT, \mathbf{Q}) = H_T^*(\{*\}, \mathbf{Q}) = \mathbf{Q}[s_1, s_2]$$

Using equivariant localization, [**23**], Okounkov and Pandharipande were able to find a Fock space expression for these invariants, and proved a conjecture in [**17**] that these invariants may be arranged in a certain generating function that satisfies the full 2-Toda integrable hierachy from [**24**].

For $2g - 2 + n + m > 0$, introduce the following generating functions of Gromov-Witten invariants

$$(3.1) \quad \left\langle\!\!\left\langle \frac{\alpha_1}{1 - z_1\psi_1}, \cdots, \frac{\alpha_n}{1 - z_n\psi_n}, \alpha_{n+1}, \dots \alpha_{n+m} \right\rangle\!\!\right\rangle_{g,n+m}$$
$$= \sum_{\substack{d,l \geq 0 \\ a_1 \dots a_n \geq 0}} \frac{Q^d}{l!} \langle \tau_{a_1}(\alpha_1) \dots \tau_{a_n}(\alpha_n) \tau_0(\alpha_{n+1}) \dots \tau_0(\alpha_{n+m}) \underbrace{\tau_0(\mathbf{t}) \dots \tau_0(\mathbf{t})}_{l \text{ times}} \rangle_{g,n+m+l,d} \prod_{i=1}^n z_i^{a_i}.$$

Here $\mathbf{t} = t_0 e + t_1 H$ where H is the negative of the equivariant first Chern class of the tautological bundle on $\mathbb{P}^1$ as in Section A.4. Q keeps track of degree and is the generator of the Novikov ring for $\mathbb{P}^1$.

It is known that if $n+m \geq 3$

$$\sum_{a_1,\dots,a_n \geq 0} \langle \tau_{a_1}(\alpha_1)\dots\tau_{a_n}(\alpha_n)\tau_0(\alpha_{n+1})\dots\tau_0(\alpha_{n+m})\rangle_{0,n+m,0} \prod_{i=1}^{n} z_i^{a_i}$$
$$= (z_1+\dots+z_n)^{n+m-3}\int_{\mathbb{P}^1}\alpha_1\dots\alpha_{n+m}$$

Using this as a definition when $m+n<3$, we can extend (3.1) to the unstable geometries. In particular, for $(g,n,m,d)=(0,2,0,0)$, we set

$$\left\langle\!\!\left\langle \frac{\alpha_1}{1-z_1\psi_1}, \frac{\alpha_2}{1-z_2\psi_2} \right\rangle\!\!\right\rangle_{0,2}$$
$$= \sum_{\substack{l,a_1,a_2\geq 0\\ d>0}} \frac{Q^d}{l!} \langle \tau_{a_1}(\alpha_1)\tau_{a_2}(\alpha_2)\underbrace{\tau_0(\mathbf{t})\dots\tau_0(\mathbf{t})}_{l \text{ times}}\rangle_{0,n+l,d} z_1^{a_1} z_2^{a_2} + \frac{1}{z_1+z_2}\int_{\mathbb{P}^1}\alpha_1\cup\alpha_2$$

In particular, setting $z_1 = 0$, we have

$$\left\langle\!\!\left\langle \alpha_1, \frac{\alpha_2}{1-\psi_2 z_2} \right\rangle\!\!\right\rangle_{0,2} = \sum_{\substack{l,a\geq 0\\ d>0}} \frac{Q^d}{l!} \langle \tau_0(\alpha_1)\tau_a(\alpha_2)\underbrace{\tau_0(\mathbf{t})\dots\tau_0(\mathbf{t})}_{l \text{ times}}\rangle_{0,n+l,d} z_2^a + \frac{1}{z_2}\int_{\mathbb{P}^1}\alpha_1\cup\alpha_2 \tag{3.2}$$

CHAPTER 4

The Spectral Curve for the Equivariant Gromov-Witten Theory of the Projective Line

4.1. The Spectral Curve Data

According to [**15**], the spectral curve data (Σ, x, y) for the equivariant Gromov-Witten theory of $\mathbb{P}^1$ is as follows

$$x(Y) = t_0 + Y + \frac{Q\,\mathrm{e}^{t_1}}{Y} + s_1 \log(Y) + s_2 \log\left(\frac{Q\,\mathrm{e}^{t_1}}{Y}\right), \qquad y(Y) = \log(Y). \tag{4.1}$$

This defines a $\mathbf{Z}$-cover $(x, y) \mapsto Y$ of $\mathbf{C}^\times$. The fundamental normalized differential form of the second kind [**16**] is taken to be

$$B(Y, Y') = \frac{dY \otimes dY'}{(Y - Y')^2}, \qquad Y, Y' \in \mathbf{C}^\times.$$

Note that this is not as expected. The function y is a single valued global co-ordinate on Σ, so we might expect to take $B(y, y') = \frac{dy dy'}{(y-y')^2}$. We will return to this in Section 4.5.

Let $\pi_i : \Sigma^n \to \Sigma$ be projection onto the i^{th} factor. For $g \geq 0$, $n > 0$, define forms $\omega_{g,n} \in \Gamma\left(\otimes_{i=1}^n \pi_i^*(T^*\Sigma)\right)$ on Σ^n as usual: $\omega_{0,1}(Y) = 0$, $\omega_{0,2}(Y, Y') = B(Y, Y')$, and

$$\omega_{g,n+1}(Y_1, \ldots, Y_{n+1}) = \sum_{\alpha=1}^{2} \operatorname*{Res}_{Y \to P_\alpha} \frac{\int_{\xi=Y}^{\hat{Y}} B(\xi, Y_{n+1})}{\Big(y(Y) - y(\hat{Y})\Big)\, dx(Y)} W_{g,n}(Y_1, \ldots, Y_n, Y, \hat{Y}). \tag{4.2}$$

where

$$W_{g,n+1}(Y_1,\dots,Y_n,Y,\hat{Y}) = \omega_{g-1,n+2}(Y_1,\dots,Y_n,Y,\hat{Y}) + \sum_{\substack{g_1+g_2=g \\ J\coprod K=\{1,\dots n\}}} \omega_{g_1,|J|+1}(Y_J,Y)\,\omega_{g_2,|K|+1}(Y_K,\hat{Y}).$$

The branch points we take are not the branch points of the map

$$x : \Sigma \to \mathbf{C}$$

$$(x,y) \to x$$

but rather the branch points of the map $(x,Y) \to x$. In Section 4.5 we discuss this discrepancy. There are two such branch points, namely P_1 and P_2. While y appears in (4.2), $\omega_{g,n}$ is independent of choice of branch of y.

These branch points are of order two. Recall that we define $\hat{Y}$ as the co-ordinate of the point on Σ near P_α satisfying $x(\hat{Y}) = x(Y)$ with $\hat{Y} \neq Y$. By definition, these branch points occur where $dx(Y) = 0$.

4.2. Idempotents in the Equivariant Cohomology

Let T be the two-dimensional torus $(\mathbf{C}^\times)^2$. We recall in Appendix A that the equivariant cohomology $H^*_T(\{*\},\mathbf{Q})$ of a point is the polynomial ring $\mathbf{Q}[s_1,s_2]$, where $s_1,s_2 \in H^2_T(\{*\},\mathbf{Q})$ are Chern classes of certain equivariant line bundles on BT. The equivariant cohomology $H^*_T(\mathbb{P}^1,\mathbf{Q})$ of $\mathbb{P}^1$ is the ring $\mathbf{Q}[s_1,s_2,H]/(H-s_1)(H-s_2)$ where H is the equivariant Chern class of the tautological bundle. We denote by e the unit in $H^*_T(\mathbb{P}^1,\mathbf{Q})$.

If we invert the element s_1-s_2, we may decompose the identity element $e \in H^0_T(\mathbb{P}^1,\mathbf{Q})$ into a pair of idempotent elements

$$\phi_1 = \frac{1}{s_1-s_2}c_1^T(\mathcal{O}(p_1)) = \frac{H-s_2}{s_1-s_2} \qquad \phi_2 = -\frac{1}{s_1-s_2}c_1^T(\mathcal{O}(p_2)) = \frac{H-s_1}{s_2-s_1} \tag{4.3}$$

which lie in the ring

$$H_T^*(\mathbb{P}^1, \mathbf{Q})[(s_1 - s_2)^{-1}] = \frac{\mathbf{Q}[s_1, s_2, H, (s_1 - s_2)^{-1}]}{(H - s_1)(H - s_2)}.$$

These classes generate $H_T^*(\mathbb{P}^1, \mathbf{Q})[(s_1 - s_2)^{-1}]$ as an $H_T^*(\{*\}, \mathbf{Q})[(s_1 - s_2)^{-1}]$-algebra and satisfy

$$\phi_1 + \phi_2 = e \qquad\qquad \phi_i\phi_j = \delta_{ij}\phi_i. \tag{4.4}$$

We will denote $s_1 - s_2$ by s.

The κ-class, which plays an important role in the main formula of the paper [**15**], is based on the $\hat{\Gamma}$-class of Iritani [**19**]. Given a T-equivariant line bundle $\mathcal{L}$ on $\mathbb{P}^1$, let $\kappa(\mathcal{L}, z)$ be defined by the formula

$$\kappa(\mathcal{L}, z) = (-z)^{c_1^T(T\mathbb{P}^1)z-1}\Gamma\left(1 - c_1^T(T\mathbb{P}^1)z\right)\exp\left(-2\pi\sqrt{-1}c_1^T(\mathcal{L})z\right). \tag{4.5}$$

Here, Γ is the gamma-function, whose Taylor expansion has the formula

$$\Gamma(1-t) = \exp\left(\gamma t + \sum_{k=2}^{\infty}\frac{\zeta(k)}{k}t^k\right).$$

The derivative $\Gamma'(1) = -\gamma$ is the Euler-Mascheroni constant.

Let $\mathcal{L} = \mathcal{O}(l_1p_1 + l_2p_2)$. In order to understand the formula for $\kappa(\mathcal{L}, z)$ better, we multiply by $e = \phi_1 + \phi_2$ and use the formulas $c_1^T(\mathcal{L})\phi_1 = l_1s\phi_1$ and $c_1^T(\mathcal{L})\phi_2 = -l_2s\phi_2$. In particular, $c_1^T(T\mathbb{P}^1)\phi_1 = s\phi_1$ and $c_1^T(T\mathbb{P}^1)\phi_2 = -s\phi_2$. We see that

$$\begin{aligned}\kappa(\mathcal{L}, z) = (-z)^{sz-1}\Gamma\left(1 - sz\right)\exp\left(-2\pi l_1\sqrt{-1}sz\right)\phi_1 \\ + (-z)^{-sz-1}\Gamma\left(1 + sz\right)\exp\left(2\pi l_2\sqrt{-1}sz\right)\phi_2\end{aligned}$$

This lies in the free $H_T^*(\mathbb{P}^1, \mathbf{C})((z))$-module $\hat{M} = H_T^*(\mathbb{P}^1, \mathbf{C})((z))[(-z)^{-sz}, (-z)^{sz}]$ which has commutative product

$$(-z)^{-sz}(-z)^{sz} = e.$$

The module $\hat{M}$ is introduced to handle a feature of the formula proved in [**15**], which we now recall.

Theorem 4.1 ([**15**, Theorems 2 and 3.8])**.** *Suppose $n > 0$ and $2g-2+n > 0$, and let $\mathcal{L}_1, \ldots, \mathcal{L}_n$ be equivariant line bundles on $\mathbb{P}^1$. Let $z_1, \ldots, z_n, s$ be positive real parameters such that s is not an integral multiple of z_i for any $1 \leq i \leq n$. Associate to a line bundle $\mathcal{L} = \mathcal{O}(l_1 p_1 + l_2 p_2)$ the path $\gamma(\mathcal{L})$ pictured in Figure 1. Then*

$$\begin{aligned}(4.6)\quad \int_{y_1 \in \gamma(\mathcal{L}_1)} \cdots \int_{y_n \in \gamma(\mathcal{L}_n)} \exp\left(z_1 x(\mathrm{e}^{y_1}) + \cdots + z_n x(\mathrm{e}^{y_n})\right) \omega_{g,n}(y_1, \ldots, y_n) \\ = (-1)^{g-1} \left\langle\!\!\left\langle \frac{z_1 \kappa(\mathcal{L}_1, z_1)}{1 - z_1 \psi_1}, \ldots, \frac{z_n \kappa(\mathcal{L}_n, z_n)}{1 - z_n \psi_n} \right\rangle\!\!\right\rangle_{g,n}\end{aligned}$$

and

$$(4.7)\qquad \int_{y_1 \in \gamma(\mathcal{L}_1)} \exp\left(z_1 x(\mathrm{e}^{y_1})\right) dy_1 = \left\langle\!\!\left\langle e, \frac{z_1 \kappa(\mathcal{L}_1, z_1)}{1 - z_1 \psi_1} \right\rangle\!\!\right\rangle_{0,2}$$

where we have used the extended definition (3.2)*.*

The contour $\gamma(\mathcal{L})$ is a path on the cover Σ on which integrand is single-valued. These two equations should be understood as an equality in $\hat{M} \otimes \cdots \otimes \hat{M}$ and $\hat{M}$ respectively.

The proof follows the general strategy of [**4**, Theorem 4.1] to find spectral curve data associated to a conformal Frobenius manifold. While the equivariant cohomology of $\mathbb{P}^1$ is not a conformal Frobenius manifold, the proof still works.

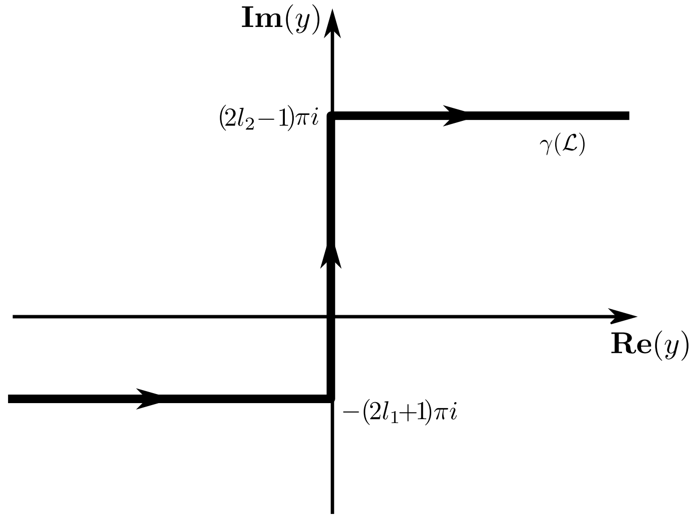

Figure 1. For a line bundle $\mathcal{L} = \mathcal{O}(l_1p_1 + l_2p_2)$, $\gamma(\mathcal{L})$ is as pictured.

We do some preliminary processing. First note that without loss of generality, we can set $Q = 1$. The Q dependence can be recovered by the shift $t_1 \to t_1 + \log Q$. We compute the location of the branch points. We have

$$dx(Y) = \left(1 - \frac{\mathrm{e}^{t_1}}{Y^2} + \frac{s}{Y}\right) dY$$

and so there are two branch points of $(x, Y) \mapsto Y$, namely $Y = P_1$ and, P_2. Thus

$$\frac{dx(Y)}{dY} = \frac{(Y - P_1)(Y - P_2)}{Y^2}.$$

The spectral curve is analytic rather than algebraic. This does not present a problem. If we assume $s, t_1 \in \mathbf{C}^\times$, there are no essential singularities in x or y near the branch points P_1, P_2. So, for each g, n, $\omega_{g,n}$ depends only on a finite number of terms of the series expansions of each of x, y, and Y near the branch points. We may compute $\omega_{g,n}$ as usual. As examples, we find

$$\begin{aligned}\omega_{0,3}(Y_1, Y_2, Y_3) = &-\frac{P_1^3 dY_1 dY_2 dY_3}{(P_1 - P_2)\,(Y_1 - P_1)^{\,2}\,(Y_2 - P_1)^{\,2}\,(P_1 - Y_3)^{\,2}} \\ &- \frac{P_2^3 dY_1 dY_2 dY_3}{(P_2 - P_1)\,(Y_1 - P_2)^{\,2}\,(Y_2 - P_2)^{\,2}\,(P_2 - Y_3)^{\,2}}\end{aligned}$$

$$\omega_{1,1}(Y_1) = \frac{dY_1}{72\,(P_1-P_2)^{\,3}}$$
$$\times\left(-\frac{9\,(P_1-P_2)^{\,2}P_1^3}{(P_1-Y_1)^{\,4}} + \frac{6\,(P_1-P_2)\,(P_1-2P_2)\,P_1^2}{(P_1-Y_1)^{\,3}}\right.$$
$$+\frac{(-5P_1^2+23P_2P_1-32P_2^2)\,P_1}{(P_1-Y_1)^{\,2}} + \frac{6\,(P_1-P_2)\,(2P_1-P_2)\,P_2^2}{(Y_1-P_2)^{\,3}}$$
$$\left.+\frac{P_2\,(32P_1^2-23P_2P_1+5P_2^2)}{(P_2-Y_1)^{\,2}} + \frac{9\,(P_1-P_2)^{\,2}P_2^3}{(P_2-Y_1)^{\,4}}\right)$$

4.3. Computing the Gromov-Witten Invariants from the Correlation Functions

For $i = 1, 2$, define

$$S_e^i(z) = \left\langle\!\!\left\langle e, \frac{\phi_i}{1-z\psi} \right\rangle\!\!\right\rangle_{0,2}$$

which, by the definition (4.3) of ϕ_i lie in $\mathbf{Q}[s_1, s_2, s^{-1}][[t_0, t_1]]((z))$. Let $(,)$ be the Poincaré pairing on $H_T^*(\mathbb{P}^1, \mathbf{Q})[s^{-1}]((z))$. Then $S_e^i(z)$ are components of the S-operator, which satisfies

$$(\alpha_1, S(\alpha_2)) = \left\langle\!\!\left\langle \alpha_1, \frac{\alpha_2}{1-z\psi} \right\rangle\!\!\right\rangle_{0,2}$$

for any classes $\alpha_1, \alpha_2 \in H_T^*(\mathbb{P}^1, \mathbf{Q})$.

The components

$$S_H^i(z) = \left\langle\!\!\left\langle H, \frac{\phi_i}{1-z\psi} \right\rangle\!\!\right\rangle_{0,2}$$

will make an appearance later.

Recall our definition of the κ classes (4.5) We compute $\left\langle\!\left\langle e, \frac{\phi_\alpha}{1-z\psi} \right\rangle\!\right\rangle_{0,2}$ via linear algebra. For line bundles $\mathcal{L}_1 = \mathcal{O}(l_1p_1 + l_2p_2)$ and $\mathcal{L}_2 = \mathcal{O}(l'_1p_1 + l'_2p_2)$, we define a matrix $C(l_1, l_2, l'_1, l'_2, z, s)$ such that

$$\begin{pmatrix} \kappa(\mathcal{L}_1, z) \\ \kappa(\mathcal{L}_2, z) \end{pmatrix} = C(l_1, l_2, l'_1, l'_2, z, s) \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Lemma 4.2. *Assume* $l_1 + l_2 - l_1' - l_2' \neq 0$. *Then* $\det C(l_1, l_2, l_1', l_2', z, s) = 0$ *only when* $z \in \frac{1}{l_1+l_2-l_1'-l_2'} s^{-1}\mathbf{Z} \setminus (s^{-1}\mathbf{Z})$. *Singularities at nonzero* $z \in s^{-1}\mathbf{Z}$ *are removable, and there is a simple pole at* $z = 0$.

This shows that if $l_1 + l_2 - l_1' - l_2' \neq 0$, then away from $z \in \frac{1}{l_1+l_2-l_1'-l_2'} s^{-1}\mathbf{Z} \setminus (s^{-1}\mathbf{Z})$, the equivariant Chern classes of $\mathcal{O}(l_1 p_1 + l_2 p_2)$ and $\mathcal{O}(l_1' p_1 + l_2' p_2)$ generate $H_T^*(\mathbb{P}^1, \mathbf{Q})$.

PROOF.

$$\det \begin{pmatrix} (-z)^{sz-1}\Gamma(1-sz)\,\mathrm{e}^{-2\pi l_1\sqrt{-1}sz} & (-z)^{-1-sz}\Gamma(1+sz)\,\mathrm{e}^{2\pi l_2\sqrt{-1}sz} \\ (-z)^{sz-1}\Gamma(1-sz)\,\mathrm{e}^{-2\pi l_1'\sqrt{-1}sz} & (-z)^{-1-sz}\Gamma(1+sz)\,\mathrm{e}^{2\pi l_2'\sqrt{-1}sz} \end{pmatrix} \tag{4.8}$$
$$= z^{-2}\Gamma(1-sz)\,\Gamma(1+sz)\left(\mathrm{e}^{2\pi(l_2'-l_1)\sqrt{-1}sz} - \mathrm{e}^{2\pi(l_2-l_1')\sqrt{-1}sz}\right)$$
$$= \frac{\pi s}{z\sin(\pi s z)}\left(\mathrm{e}^{2\pi(l_2'-l_1)\sqrt{-1}sz} - \mathrm{e}^{2\pi(l_2-l_1')\sqrt{-1}sz}\right)$$

The first factor has simple poles when $0 \neq z \in s^{-1}\mathbf{Z}$, and a double pole at $z = 0$. The second factor has simple zeros when $(l_1 + l_2 - l_1' - l_2')z \in s^{-1}\mathbf{Z}$. Therefore, singularities at finite, nonzero $z \in s^{-1}\mathbf{Z}$ are removable and the determinant is zero exactly when $\frac{1}{z} \in \frac{1}{l_1+l_2-l_1'-l_2'} s^{-1}\mathbf{Z} \setminus (s^{-1}\mathbf{Z})$, and has a simple pole at $z = 0$. □

This suggests that if we take $l_1 + l_2 - l_1' - l_2' = \pm 1$, we can find suitable chains γ_1, γ_2, in $\hat{M} \otimes_{\mathbf{Z}} C_*(\mathbb{P}^1)$ such that

$$\int_{y\in\gamma_1} \exp(zx(\mathrm{e}^y))\,dy = \left\langle\!\!\left\langle e, \frac{\phi_1}{1-z\psi} \right\rangle\!\!\right\rangle_{0,2} = S_e^1(z)$$
$$\int_{y\in\gamma_2} \exp(zx(\mathrm{e}^y))\,dy = \left\langle\!\!\left\langle e, \frac{\phi_2}{1-z\psi} \right\rangle\!\!\right\rangle_{0,2} = S_e^2(z).$$

See Section 4.4.

Our main result is that given arbitrary classes $\alpha_1, \ldots, \alpha_n \in H_T^*(\mathbb{P}^1, \mathbf{Q})$,

$$\left\langle\!\!\left\langle \frac{\alpha_1}{1 - z_1\psi_1}, \ldots, \frac{\alpha_n}{1 - z_n\psi_n} \right\rangle\!\!\right\rangle_{g,n}$$

may be expressed in terms of the functions S_e^1, S_H^1, S_e^2, S_H^2.

For convenience, define a coefficient ring

$$A = \mathbf{Q}\left[P_1, P_2, (P_1 - P_2)^{-1}, P_1^{-1}, P_2^{-1}\right] = \mathbf{Q}\left[s, \mathrm{e}^{-t_1}, \left(s^2 + 4\,\mathrm{e}^{t_1}\right)^{\pm\frac{1}{2}}\right].$$

Proposition 4.3. *For $n > 0$, $2g - 2 + n > 0$, there are polynomials $T_{g,n}^{\vec{\alpha},\vec{\beta}}(z_1, \ldots, z_n)$ with coefficients in A such that*

$$\left\langle\!\!\left\langle \frac{\phi_{i_1}}{1 - z_1\psi_1}, \ldots, \frac{\phi_{i_n}}{1 - z_n\psi_n} \right\rangle\!\!\right\rangle_{g,n} = \sum_{\beta_1, \ldots \beta_n \in \{e,H\}} T_{g,n}^{\vec{i},\vec{\beta}} \prod_{j=1}^{n} S_{\beta_j}^{i_j}(z_j) \tag{4.9}$$

The first observation we make is that the left hand side of (4.9) is equal to

$$\left\langle\!\!\left\langle \frac{\phi_{i_1}}{1 - z_1\psi_1}, \ldots, \frac{\phi_{i_n}}{1 - z_n\psi_n} \right\rangle\!\!\right\rangle_{g,n} = \int_{y_1 \in \gamma_{i_1}} \cdots \int_{y_n \in \gamma_{i_n}} \mathrm{e}^{\sum_i z_i x(Y_i)}\, \omega_{g,n}$$

The main idea is to find a differential form $\Omega_{g,n}$ such that for all line bundles $\mathcal{O}(l_1 p_1 + l_2 p_2)$

$$\int_{\gamma(\mathcal{L})} \mathrm{e}^{\sum_i z_i x(Y_i)}\, (\Omega_{g,n} - \omega_{g,n}) = 0$$

but where

$$\int_{\gamma(\mathcal{L})} \mathrm{e}^{\sum_i z_i x(Y_i)}\, \Omega_{g,n}$$

may be readily evaluated. The result is that (4.6) may be reduced to a product of integrals of the form (4.7). Finding $\Omega_{g,n}$ amounts to a finite procedure of alternately integrating by parts, and separating the result at each step into a sum of two parts: a desired term, and a term requiring further processing. The surprise is that (4.2) and the form of (4.6) together ensure that an $\Omega_{g,n}$ may be found in $R[Y_1^{-1},\dots,Y_n^{-1}]$ for some suitable ring of coefficients R not depending on the Y_i. In fact, no power greater than $\Omega_{g,n}$ turns out to be symmetric, and no power greater than Y_i^{-2} arises for $i=1,\dots,n$.

The proof is organized as follows. We have a series of lemmas. First we define a suitable alternative local coordinate to Y so that we may keep track of coefficients of the rational functions we obtain and show that A is correctly defined. Then we establish that $\omega_{g,n}$ is a rational n-linear form and finite at infinity, and that it has no residue at branch points $Y_i = P_\alpha$, and is thus the differential of a rational function. Next we show that this is the base case in an induction process that terminates with a form $\Omega_{g,n}$ suitable to prove the proposition.

In the discussion that follows, x and y always refer to the given functions (4.1) on the spectral curve, while Y, Y_i are the coordinates of points on $\mathbf{C}^\times$. We always assume that for $i=1,\dots n$, $Y_i \neq P_1, P_2$, and $z, z_i, \mathrm{e}^{t_1}, s > 0$, and that $P_1, P_2, (P_1-P_2), s \neq 0$. First, we derive a convenient coordinate in which we may deal with Y and $\hat{Y}$.

Lemma 4.4. *For $\alpha = 1,2$ there is a coordinate ξ_α defined in a neighbourhood of P_α and a power series $f_\alpha(\xi_\alpha) = \sum_{i=1}^\infty c_{\alpha,i}\xi_\alpha^{2i}$ with $c_{\alpha,i} \in A$ such that*

$$Y = P_\alpha + \xi_\alpha + f_\alpha(\xi_\alpha)$$

$$\hat{Y} = P_\alpha - \xi_\alpha + f_\alpha(\xi_\alpha)$$

PROOF. We compute $\xi = \xi_1$, $f(\xi) = f_1(\xi)$ and $c_k = c_{1,k}$ First

$$\begin{aligned} x(Y) - x(P_1) &= x(P_1 + \xi + f(\xi)) - x(P_1) \\ &= -P_2 \sum_{i=2}^\infty \frac{(-1)^i}{P_1^i}(\xi + f(\xi))^i + (P_1+P_2)\sum_{i=2}^\infty \frac{(-1)^i}{iP_1^i}(\xi + f(\xi))^i \end{aligned}$$

$$= \sum_{i=2}^{\infty} \frac{(-1)^i}{P_1^i} \left(\frac{P_1 + (1-i)P_2}{i} \right) (\xi + f(\xi))^i$$

In a similar way

$$x(\hat{Y}) - x(P_1) = \sum_{i=2}^{\infty} \frac{(-1)^i}{P_1^i} \left(\frac{P_1 + (1-i)P_2}{i} \right) (-\xi + f(\xi))^i$$

So

$$\begin{aligned}
0 &= x(Y) - x(\hat{Y}) \\
&= \sum_{i=2}^{\infty} \frac{(-1)^i}{P_1^i} \left(\frac{P_1 + (1-i)P_2}{i} \right) \left((\xi + f(\xi))^i - (-\xi + f(\xi))^i \right) \\
&= \sum_{i=2}^{\infty} \frac{(-1)^i}{P_1^i} \left(\frac{P_1 + (1-i)P_2}{i} \right) \sum_{j=0}^{i} \binom{i}{j} (1 - (-1)^{i-j}) \xi^{i-j} f(\xi))^j \\
&= \sum_{i=2}^{\infty} \frac{(-1)^i}{P_1^i} \left(\frac{P_1 + (1-i)P_2}{i} \right) \sum_{j=0}^{i} \binom{i}{j} (1 - (-1)^{i-j}) \xi^{i-j} \left(\sum_{k=2}^{\infty} c_k \xi^{2k} \right)^j
\end{aligned}$$

Collecting like powers of ξ, we find the first time c_m appears is in the ξ^{2m+1}, term, where $k = m$, $j = 1$, $i = 2$. Thus, the above may be written

$$0 = \sum_{m=2}^{\infty} \xi^{2m+1} \left(2 \frac{P_1 - P_2}{P_1^2} c_m + h_m(c_1, \ldots, c_{m-1}) \right)$$

where h_m is a polynomial with coefficients in $\mathbb{Q}[P_1, P_2, P_1^{-1}]$. Thus $h_m(c_1, \ldots, c_{m-1}) \in A$ and so $c_m \in A$ too. The same argument applies for ξ_2 *mutatis mutandis*. □

Note that by definition of $\hat{Y}$,

$$\left(\log(Y) - \log(\hat{Y}) \right)^{-1} = \frac{P_1 + P_2}{\hat{Y} + \frac{P_1 P_2}{\hat{Y}} - Y - \frac{P_1 P_2}{Y}} = \frac{(P_1 + P_2)}{(\hat{Y} - Y) \left(1 + \frac{P_1 P_2}{Y \hat{Y}} \right)} \tag{4.10}$$

We set

$$\omega(Y) = \left(y(Y) - y(\hat{Y})\right) dx(Y).$$

Using the definition of $\hat{Y}$, we have

$$\frac{1}{\omega(Y)} = \frac{(P_1 + P_2)Y^2}{(\hat{Y} - Y)(Y - P_1)(Y - P_2)\left(1 + \frac{P_1 P_2}{Y\hat{Y}}\right) dY}$$

Lastly

$$\int_{\xi=Y}^{\hat{Y}} B(\xi, Y_n) = \frac{dY_n}{Y - Y_n} - \frac{dY_n}{\hat{Y} - Y_n}$$

So the recursion can be written

$$\omega_{g,n}(Y_1, \ldots, Y_n) = \sum_\alpha \operatorname*{Res}_{Y \to P_\alpha} \left(\frac{dY_n}{Y - Y_n} - \frac{dY_n}{\hat{Y} - Y_n}\right) \frac{W_{g,n}(Y, \hat{Y})}{\omega(Y)}$$

Lemma 4.5. *Let $2g - 2 + n > 0$, $n > 0$ $\omega_{g,n}(Y_1, \ldots, Y_n)$ may be written*

$$\omega_{g,n}(Y_1, \ldots, Y_n) = \frac{p_{g,n}(Y_1, \ldots, Y_n) dY_1 \ldots dY_n}{\prod_{i=1}^n (Y_i - P_1)^{k_{g,n}} (Y_i - P_2)^{k_{g,n}}}$$

for some $p \in A\left[Y_1, \ldots, Y_n\right]$ symmetric in the Y_i and $k_{g,n} \in \mathbf{Z}_{\geq 0}$. It is finite at infinity.

This is just [**8**, Theorem 4.2] for our non-algebraic curve. The proof is the same—when Y_n is away from the branch points, the residue may be computed as a finite integral of an analytic function, and so there can be no pole. We establish the result in our setting to ensure that there is no logarithmic dependence on Y at the branch points of Y, and thus show that $\omega_{g,n}$ is rational.

We will perform an induction on $2g - 2 + n$ to establish the lemma. The result isn't true for the base case $\omega_{0,2}(Y_1, Y_2) = \frac{dY_1 dY_2}{(Y_1 - Y_2)^2}$, but this form possesses properties that are sufficient to allow it to serve as one in the inductive step.

PROOF. Note that the Y_n dependence comes about as a residue

$$\omega_{g,n}(Y_1,\ldots,Y_n)=\sum_{i=1,2}\operatorname*{Res}_{Y\to P_i}\left(\frac{dY_n}{Y-Y_n}-\frac{dY_n}{\hat{Y}-Y_n}\right)\frac{W_{g,n}(Y,\hat{Y})}{\omega(Y)}$$

We focus on the residue $Y = P_1$. As usual, to obtain this a residue, we compute a Laurent expansion of the integrand in terms of ξ_1 about zero and then take the coefficient of ξ_1^{-1}.

- First consider the factor

  $$\frac{dY_n}{Y-Y_n}-\frac{dY_n}{\hat{Y}-Y_n}$$

  This has a power series expansion in ξ_1 about $\xi_1 = 0$ with coefficients lying in the ring $\mathbb{Q}[(P_1-Y_n)^{-1}, c_{1,k}] \subset A[(P_1-Y_n)^{-1}]$ with $k = 2, 3, 4, \ldots$. Its constant term is zero. The coefficient of each term in this series has a pole at $Y_n = P_1$ of order at least two. Because of this it must be finite at $Y_n = \infty$, and this property passes to $\omega_{g,n}(Y_1, \ldots, Y_n)$.
- The factor $\frac{1}{\omega(Y)}$, has a double pole at $\xi_1 = 0$ and the coefficients of its Laurent series lie in A.
- Last, we have the factor $W_{g,n}(Y, \hat{Y})$. By induction, this has a Laurent series expansion in ξ_α with coefficients in $A[Y_i, (P_\alpha - Y_i)^{-1}]$ where $i = 1, \ldots, n-1$.

Suppose that the pole at $Y = P_\alpha$ has order m Therefore, after the residues at both P_1 and P_2, we can say

$$\sum_{\alpha=1}^{2}\sum_{i=2}^{m+1}\frac{dY_n}{(Y_n-P_\alpha)^i}r_{i,\alpha}(Y_1,\ldots,Y_{n-1})dY_1\ldots dY_{n-1}$$

with $r_{i,\alpha} \in A[Y_j, (Y_j - P_\beta)^{-1}]$ with $j = 1, \ldots n-1$, and $\beta = 1, 2$. Since $\omega_{g,n}(Y_1, \ldots, Y_n)$ is symmetric (Theorem 2.3) in the Y_i, we obtain the result. □

The following lemma establishes a base case for an induction step in the proof of the main result.

Lemma 4.6. *For* $1 \leq k \leq n$*,* $\alpha = 1, 2$*,*

$$\operatorname*{Res}_{Y_k \to P_\alpha} \mathrm{e}^{z_k x(Y_k)}\, \omega_{g,n}(Y_1, \ldots, Y_n) = 0$$

Proof. As $\omega_{g,n}$ is symmetric in its arguments, it suffices to check this for $k = n$ only. For the residue at P_1, we have

$$\begin{aligned}
\operatorname*{Res}_{Y_n \to P_1} \mathrm{e}^{z_n x(Y_n)}\, \omega_{g,n}(Y_1, \ldots, Y_n) &= \operatorname*{Res}_{Y_n \to P_1} \sum_{\beta=1}^{2} \operatorname*{Res}_{Y \to P_\beta} \mathrm{e}^{z_n x(Y_n)} \left(\frac{1}{Y - Y_n} - \frac{1}{\hat{Y} - Y_n} \right) \frac{W_{g,n}(Y, \hat{Y})}{\omega(Y)} \\
&= \operatorname*{Res}_{Y_n \to P_1} \operatorname*{Res}_{Y \to P_1} \mathrm{e}^{z_n x(Y_n)} \left(\frac{1}{Y - Y_n} - \frac{1}{\hat{Y} - Y_n} \right) \frac{W_{g,n}(Y, \hat{Y})}{\omega(Y)} \\
&= \operatorname*{Res}_{Y \to P_1} \operatorname*{Res}_{Y_n \to P_1, Y, \hat{Y}} \mathrm{e}^{z_n x(Y_n)} \left(\frac{1}{Y - Y_n} - \frac{1}{\hat{Y} - Y_n} \right) \frac{W_{g,n}(Y, \hat{Y})}{\omega(Y)} \\
&= \operatorname*{Res}_{Y \to P_1} 0 + \left(\mathrm{e}^{z_n x(Y)} \frac{W_{g,n}(Y, \hat{Y})}{\omega(Y)} \right) - \left(\mathrm{e}^{z_n x(\hat{Y})} \frac{W_{g,n}(Y, \hat{Y})}{\omega(Y)} \right) \\
&= \operatorname*{Res}_{Y \to P_1} 0 = 0.
\end{aligned}$$

A symmetrical argument applies to the residue $Y_n \to P_2$. We note that the above result works for any $B(Y, Y') = \frac{dY dY'}{(Y - Y')^2} + \mathrm{analytic}$ and arbitrary meromorphic function x with simple branch points away from $x = \infty$. □

The following two lemmas show that we can perform an iterative process to find $\Omega_{g,n}$. The first step in this process is to integrate by parts to find an intermediate rational function, and the second is write this intermediate function as a sum of two terms, one of the target form, and the second on which we reapply this process.

Lemma 4.7. *Suppose $\lambda(Y)dY$ is rational and finite at infinity with poles only at $Y = P_1$ and $Y = P_2$. If $\mathrm{Res}_{Y\to P_\alpha} \mathrm{e}^{zx(Y)}\,\lambda(Y)dY = 0$, then there exists a rational function $\eta(Y)$ that is zero at $Y = \infty$ satisfying $d\eta = \lambda(Y)dY$. Moreover*

$$\operatorname*{Res}_{Y\to P_\alpha} \mathrm{e}^{zx(Y)}\,\eta(Y)dx(Y) = 0$$

Proof. Since $\lambda(Y)dY$ is rational, it may be anti-differentiated. This does not introduce any logarithmic terms. To see this, note that in taking the limit $z \to 0$ in $\mathrm{Res}_{Y\to P_\alpha} \mathrm{e}^{zx(Y)}\,\lambda(Y)dY = 0$, we observe that $\lambda(Y)dY$ has no non-zero residues. Thus $\eta(Y)$ is rational. A constant of integration may be chosen so that $\lim_{Y\to\infty} \eta(Y) = 0$. The rest is just integration by parts.

$$\operatorname*{Res}_{Y\to P_\alpha} \mathrm{e}^{zx(Y)}\,\lambda(Y)dY = \operatorname*{Res}_{Y\to P_\alpha} \mathrm{e}^{zx(Y)}\,d\eta(Y) = -\operatorname*{Res}_{Y\to P_\alpha} z\,\mathrm{e}^{zx(Y)}\,\eta(Y)dx(Y) \qquad \square$$

We note that this holds for also holds for arbitrary rational x.

Lemma 4.8. *If $\lambda(Y)dY$ is a rational 1-form with coefficients in a ring F that includes z, P_1^{-1}, P_2^{-1}, with poles only at P_1 and P_2, and is finite at $Y = \infty$, and satisfying*

$$\operatorname*{Res}_{Y\to P_\alpha} \mathrm{e}^{zx(Y)}\,\lambda(Y)dY = 0$$

for $\alpha = 1, 2$; then there exists a rational form $\tilde{\lambda}(Y, z)dY = \frac{g_1(z)dY}{Y} + \frac{g_2(z)dY}{Y^2}$ such that

$$\mathrm{e}^{zx(Y)}\,\lambda(Y)dY - \mathrm{e}^{zx(Y)}\,\tilde{\lambda}(Y)dY = d\left(\mathrm{e}^{zx(Y)}\,\sigma(Y)\right)$$

for some rational $\sigma(Y)$.

Proof. We perform an induction on the sum of orders of poles of $\lambda(Y)dY$. For the base cases, note that if the $\lambda(Y)dY$ is regular Y, then $\lambda(Y)dY = 0$. The case where $\lambda(Y)dY$ has one simple pole at one or both $Y = P_1$ and $Y = P_2$ is excluded as $\operatorname{Res}_{Y\to P_\alpha} \mathrm{e}^{zx(Y)} \lambda(Y)dY = 0$. This is seen by taking the limit $z \to 0$.

For the inductive step, we integrate by parts. We assume that sum of the orders of all poles of the denominator of $\lambda(Y)dY$ is at least two. Consider the function $\eta(Y)$ as per Lemma 4.7.

$$\begin{aligned}
\mathrm{e}^{zx(Y)} \lambda(Y)dY &= \mathrm{e}^{zx(Y)} d\eta(Y) \\
&= -z\, \mathrm{e}^{zx(Y)} \eta(Y)dx(Y) + d\left(\mathrm{e}^{zx(Y)} \eta(Y)\right) \\
&= -\, \mathrm{e}^{zx(Y)} z\frac{(Y-P_1)(Y-P_2)}{Y^2}\eta(Y)dY + d\left(\mathrm{e}^{zx(Y)} \eta(Y)\right)
\end{aligned}$$

The order of poles at $Y = P_1$ and $Y = P_2$ is two less in $-z\frac{(Y-P_1)(Y-P_2)}{Y^2}\eta(Y)dY$ than $\lambda(Y)dY$. Perform a partial fraction decomposition to write

$$-z\frac{(Y-P_1)(Y-P_2)}{Y^2}\eta(Y) = \left(\frac{g_1(z)}{Y} + \frac{g_2(z)}{Y^2} + \tilde{\lambda}(Y)\right) dY$$

where $\tilde{\lambda}(Y)$ is regular at $Y = 0$. It is easy to see that we only require P_1^{-1}, P_2^{-1} be in F to perform the partial fraction decomposition.

$\tilde{\lambda}(Y)$ must necessarily have poles only at $Y = P_1$ and P_2, and must be finite at $Y = \infty$. Now $z\, \mathrm{e}^{zx(Y)} \frac{(Y-P_1)(Y-P_2)}{Y^2}\eta(Y)dY$ and $-\, \mathrm{e}^{\frac{x(Y)}{z}} \tilde{\lambda}(Y)dY$ differ by a form analytic at $Y = P_\alpha$, and so $\operatorname{Res}_{Y\to P_\alpha} \mathrm{e}^{\frac{x(Y)}{z}} \tilde{\lambda}(Y)dY = 0$ too.

Thus the induction step is complete. □

We note that for $\lambda(Y) = \frac{1}{(Y-P_1)^k(Y-P_2)^k}$, which arises in practice, we have a recursive formula for $g_1(z), g_2(z)$, and $\tilde{\lambda}(Y)$.

We now prove the main proposition.

PROOF. Recall that in Theorem 4.1, the integration path $\gamma(\mathcal{L})$ in the Y variable consists of $(-\infty,-1]$, some number of circles around the point $Y=0$ at radius 1, and the segment $[-1,0)$. So, for any rational function $\sigma(Y)$, we have $\int_{\gamma(\mathcal{L})} d\left(\mathrm{e}^{zx(Y)}\,\sigma(Y)\right)=0$.

For a line bundle $\mathcal{L}$, set $\gamma'(\mathcal{L})=\exp(\gamma(\mathcal{L}))$. We are interested in evaluating

$$\int_{Y_1\in\gamma'(\mathcal{L}_1)}\cdots\int_{Y_n\in\gamma'(\mathcal{L}_n)}\exp\left(z_1x(Y_1)+\cdots+z_nx(Y_n)\right)\omega_{g,n}.$$

Apply Lemma 4.8 to $\omega_{g,n}$, iteratively in each variable $Y_1,\ldots,Y_n$. The process we have described (that eliminates poles at P_1,P_2 and generates double poles at $Y_i=0$) is linear, and so can indeed be applied to each variable Y_i in turn. On each path of integration, $\exp(x(Y_i)$ decreases exponentially as $Y_i\to\infty$, so we do not need to keep track of the exact terms. So there exists a form

$$\Omega_{g,n}(Y_1,\ldots,Y_n,z_1,\ldots,z_n)=\frac{\prod_i dY_i}{\prod_i Y_i^2}f(Y_1,\ldots,Y_n,z_1,\ldots,z_n)$$

with $f\in A[z_1,\ldots,z_n,Y_1,\ldots,Y_n]$ that is affine in each Y_i when considered as a function of each variable Y_i independently. $\Omega_{g,n}$ has the property that for equivariant line bundles $\mathcal{L}_1,\ldots,\mathcal{L}_n$

$$\int_{Y_1\in\gamma'(\mathcal{L}_1)}\cdots\int_{Y_n\in\gamma'(\mathcal{L}_n)}\exp\left(z_1x(Y_1)+\cdots+z_nx(Y_n)\right)\omega_{g,n}$$
$$=\int_{Y_1\in\gamma'(\mathcal{L}_1)}\cdots\int_{Y_n\in\gamma'(\mathcal{L}_n)}\exp\left(z_1x(Y_1)+\cdots+z_nx(Y_n)\right)\Omega_{g,n}.$$

The right hand side of the above is readily evaluated. As we have said, $f(Y_1,\ldots,Y_n,z_1,\ldots,z_n)$ is at most linear in each of the Y_i. Because of this, we only need to know how to evaluate the integrals

$$\int_{Y\in\gamma'(\mathcal{L})}\exp\left(zx(Y)\right)\frac{dY}{Y}\quad\text{and}\quad\int_{Y\in\gamma'(\mathcal{L})}\exp\left(zx(Y)\right)\frac{dY}{Y^2}.$$

The former we have already know from (4.7) because $dy=d\log Y=dY/Y$, while the latter can be obtained by differentiating the former with respect to a parameter in x.

We have

$$\int_{y\in\gamma(\mathcal{L})} \exp\left(zx(Y)\right)\frac{dY}{Y^2} = \left(\frac{1}{z}\frac{\partial}{\partial q} - \frac{s_2}{q}\right)\int_{y\in\gamma(\mathcal{L})} \exp\left(zx(Y)\right)\frac{dY}{Y}$$
$$= \left(\frac{1}{z}\frac{\partial}{\partial q} - \frac{s_2}{q}\right)\left\langle\!\!\left\langle e, \frac{z\kappa(\mathcal{L},z)}{1-z\psi}\right\rangle\!\!\right\rangle_{0,2}$$

Now γ_i does not depend on q, and so we also have

$$\int_{\gamma_i} \exp\left(zx(Y)\right)\frac{dY}{Y^2} = \left(\frac{1}{z}\frac{\partial}{\partial q} - \frac{s_2}{q}\right)\left\langle\!\!\left\langle e, \frac{\phi_i}{1-z\psi}\right\rangle\!\!\right\rangle_{0,2}.$$

In fact, by [**15**, sec. 2.5] and in turn [**18**], we have

$$\frac{1}{z}\frac{\partial}{\partial q}\left\langle\!\!\left\langle e, \frac{\phi_i}{1-z\psi}\right\rangle\!\!\right\rangle_{0,2} = \frac{1}{q}\left\langle\!\!\left\langle H, \frac{\phi_i}{1-z\psi}\right\rangle\!\!\right\rangle_{0,2} = \frac{1}{q}S^i_H,$$

thus

$$\int_{\gamma_i} \exp\left(zx(Y)\right)\frac{dY}{Y^2} = \frac{1}{q}S^i_H - \frac{s_2}{q}S^i_e.$$

And so we have proved the result □

The result here extends to a broader class of spectral curves than that studied here. Given correlation functions $\omega_{g,n}$ satisfying Eynard-Orantin recursion for a rational x, we can make a statement about the integral

$$\int_{x_i\in\gamma_1}\cdots\int_{x_n\in\gamma_n} \mathrm{e}^{\sum_i z_i x_i}\,\omega_{g,n} \tag{4.11}$$

for contours γ_i on which $\mathrm{e}^{z_i x_i}$ agrees at its end-points. The form of $\Omega_{g,n}$ changes, but the statement will be that (4.11)will be be reduced to integrals

$$\int_\gamma \mathrm{e}^{zx(Y)}(Y-Q)^{-k}dY \tag{4.12}$$

for Q a pole of dx of degree m and $0 < k < m$.

To relate this to a counting problem, however, one needs results similar to Theorem 4.1, (4.6). Some method of identifying quantities of the form (4.12) with generating functions related to the counting problem would also be desirable.

We finish the section by listing some $\Omega_{g,n}$ obtained from this process.

$$\Omega_{0,3} = \frac{z_1 z_2 z_3 dY_1 dY_2 dY_3}{Y_1 Y_2 Y_3}\left(-q - s^2 + qs\left(\frac{1}{Y_1} + \frac{1}{Y_2} + \frac{1}{Y_3}\right) - q^2\left(\frac{1}{Y_1Y_2} + \frac{1}{Y_1Y_3} + \frac{1}{Y_2Y_3}\right)\right)$$
$$\Omega_{1,1} = \frac{z_1 dY_1}{Y_1}\left(\frac{-\frac{qs}{3(4q+s^2)} - \frac{qz_1}{12}}{Y_1} - \frac{5\left(12q+s^2\right)}{72\left(4q+s^2\right)} + \frac{sz_1}{24}\right)$$

4.4. Computation of the *S* operator

The determinant computed earlier suggests that we take line bundles $\mathcal{L} = \mathcal{O}(l_1p_1 + l_2p_2)$ and $\mathcal{L}' = \mathcal{O}(l_1'p_1 + l_2'p_2)$ with $l_1 + l_2 - l_1' - l_2' = 1$. Actually, if we set $l_2 = l_2'$, and $l_1 = l_1' + 1$, our expression for $S_e^1(z)$ simplifies.

To this end, set $\mathcal{L}_1 = \mathcal{O}\left(l_1p_1 + l_2p_2\right)$ and $\mathcal{L}_2 = \mathcal{O}\left((l_1+1)p_1 + l_2p_2\right)$. Then we have

So, taking advantage of (4.8)

$$\begin{aligned}\phi_1 &= \frac{z\sin(\pi sz)}{\pi s}\frac{(-z)^{-1-sz}\Gamma(1+sz)\,\mathrm{e}^{2l_2\pi\sqrt{-1}sz}}{\left(\mathrm{e}^{2(l_2-l_1)\pi\sqrt{-1}sz} - \mathrm{e}^{2(l_2-l_1-1)\pi\sqrt{-1}sz}\right)}\left(\kappa\left(\mathcal{L}_1,z\right) - \kappa\left(\mathcal{L}_2,z\right)\right)\\ &= \frac{1}{2\pi\sqrt{-1}s}(-z)^{-1-sz}\Gamma(1+sz)\,\mathrm{e}^{(2l_1+1)\pi\sqrt{-1}sz}\left(z\kappa\left(\mathcal{L}_1,z\right) - z\kappa\left(\mathcal{L}_2,z\right)\right)\end{aligned}$$

If we define the contour $\gamma_{1,l_1,l_2} = \gamma\left(\mathcal{O}(l_1p_1 + l_2p_2)\right) - \gamma\left(\mathcal{O}((l_1+1)p_1 + l_2p_2)\right)$, then

$$S_e^1(z) = \left\langle\!\!\left\langle e, \frac{\phi_1}{1 - z\psi}\right\rangle\!\!\right\rangle_{0,2} = -\frac{1}{2\pi\sqrt{-1}s}(-z)^{-1-sz}\Gamma(1+sz)\,\mathrm{e}^{(2l_1+1)\pi\sqrt{-1}sz}\int_{\gamma_{1,l_1,l_2}} \mathrm{e}^{zx(\mathrm{e}^y)}\,dy$$

Note that γ_{1,l_1,l_2} is the contour as pictured in figure 2.

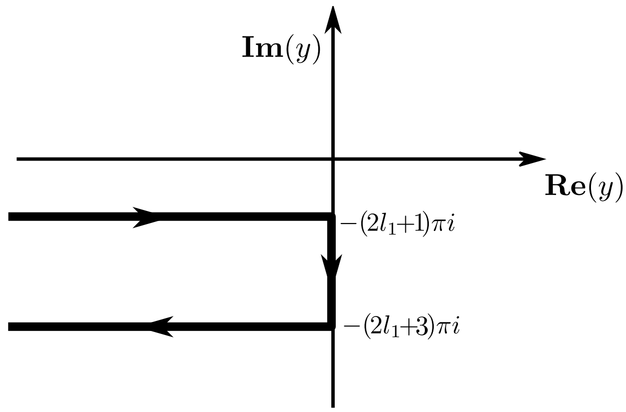

Figure 2. The contour γ_{1,l_1,l_2} is as pictured. It is independent of l_2.

We follow along the calculation given in [**15**].

$$
\begin{aligned}
&\int_{\gamma_{1,l_1,l_2}} \mathrm{e}^{zx(\mathrm{e}^y)}\,dy = \mathrm{e}^{zt_0+zt_1(s_1+s_2)/2}\int_{\gamma_{1,l_1,l_2}} \mathrm{e}^{z(2\exp(t_1/2)\cosh(y)+sy)}\,dy\\
&= \mathrm{e}^{zt_0+zt_1(s_1+s_2)/2}\int_{-\infty}^{0} \mathrm{e}^{z\left(-2\exp(t_1/2)\cosh(y)+s\left(y-(2l_1+1)\pi\sqrt{-1}\right)\right)}\,dy\\
&\qquad + \mathrm{e}^{zt_0+zt_1(s_1+s_2)/2}\int_{\pi\sqrt{-1}}^{-\pi\sqrt{-1}} \mathrm{e}^{z\left(2\exp(t_1/2)\cosh(y)+s\left(y-(2l_1+2)\pi\sqrt{-1}\right)\right)}\,dy\\
&\qquad + \mathrm{e}^{zt_0+zt_1(s_1+s_2)/2}\int_{0}^{-\infty} \mathrm{e}^{z\left(-2\exp(t_1/2)\cosh(y)+s\left(y-(2l_1+3)\pi\sqrt{-1}\right)\right)}\,dy\\
&= \mathrm{e}^{zt_0+zt_1(s_1+s_2)/2-sz(2l_1+2)\pi\sqrt{-1}}\\
&\quad 2\pi\sqrt{-1}\left(\frac{\sin(sz\pi)}{\pi}\int_0^{\infty} \mathrm{e}^{-2z\exp(t_1/2)\cosh(y)-szy}\,dy - \frac{1}{\pi}\int_0^{\pi} \mathrm{e}^{2z\exp(t_1/2)\cos(u)}\cos(szu)\,du\right)\\
&= -2\pi\sqrt{-1}\,\mathrm{e}^{zt_0+zt_1(s_1+s_2)/2-sz(2l_1+2)\pi\sqrt{-1}}\,I_{sz}\left(2z\exp(t_1/2)\right)
\end{aligned}
$$

The integral representation for the modified Bessel function $I_\beta(\zeta)$ is given in [**25**, p.181], and is valid for $Re\,(2z\exp(t_1/2)) > 0$. The series for the modified Bessel function about $z = 0$ has monodromy, but we find

$$
S_e^1(z) = \left\langle\!\!\left\langle e, \frac{\phi_1}{1-z\psi}\right\rangle\!\!\right\rangle_{0,2} = \frac{1}{s}z^{-1-sz}\Gamma(1+sz)\,\mathrm{e}^{zt_0+zt_1(s_1+s_2)/2}\,I_{sz}\left(2z\exp(t_1/2)\right)
$$

$$= \frac{1}{s} z^{-1-sz}\, \mathrm{e}^{zt_0+zt_1(s_1+s_2)/2} \sum_{m=0}^{\infty} \frac{1}{m!\,(1+sz)^{(m)}} \,(z\exp(t_1/2))^{2m+sz}$$

$$= \frac{1}{s}\, \mathrm{e}^{zt_0+s_1zt_1} \sum_{m=0}^{\infty} \frac{(\mathrm{e}^{t_1})^m\, z^{2m-1}}{m!\,(1+sz)^{(m)}}$$

where $(1+\zeta)^{(m)} = \prod_{j=1}^{m}(m+\zeta)$.

In an analogous way, we can compute $S_e^2(z)$. Set $\mathcal{L}_3 = \mathcal{O}(l_1p_1+(l_2+1)p_2)$, $\mathcal{L}_4 = \mathcal{O}(l_1p_1+l_2p_2)$. We find

$$\phi_2 = \frac{\sqrt{-1}}{2\pi s} z^{sz}\Gamma(1-sz)\,\mathrm{e}^{-(2l_2+2)\pi\sqrt{-1}sz}\left(-\kappa(\mathcal{L}_3,z)+\kappa(\mathcal{L}_4,z)\right)$$

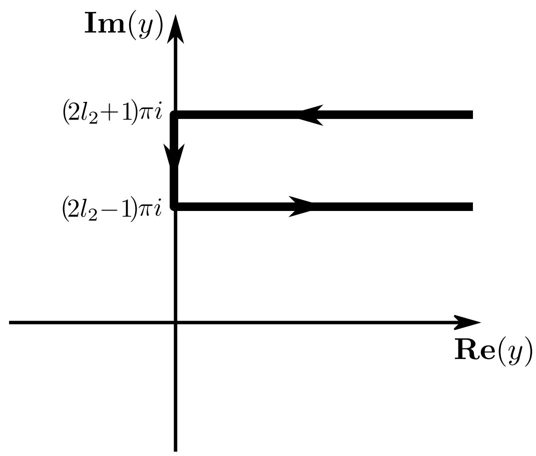

Figure 3. The contour γ_{2,l_1,l_2}. This contour is independent of l_1.

So if we define the contour $\gamma_{2,l_1,l_2} = \gamma(\mathcal{O}(l_1p_1+l_2p_2)) - \gamma(\mathcal{O}(l_1p_1+(l_2+1)p_2))$ as pictured in figure 3, we can perform a similar calculation as before, we find

$$S_e^2(z) = \left\langle\!\!\left\langle e, \frac{\phi_2}{1-z\psi}\right\rangle\!\!\right\rangle_{0,2} = \frac{\sqrt{-1}}{2\pi s} z^{sz}\Gamma(1-sz)\,\mathrm{e}^{-(2l_2+2)\pi\sqrt{-1}sz}\int_{\gamma_{2,l_1,l_2}} \mathrm{e}^{zx(\mathrm{e}^y)z}\,dy$$

$$= -\frac{1}{s} z^{sz}\Gamma(1-sz)\,\mathrm{e}^{zt_0+zt_1(s_1+s_2)/2}\, I_{-sz}(2z\exp(t_1/2))$$

$$= -\frac{1}{s} z^{sz} \, \mathrm{e}^{zt_0 + zt_1(s_1+s_2)/2} \sum_{m=0}^{\infty} \frac{1}{m! \, (1-sz)^{(m)}} \left(z\sqrt{\mathrm{e}^{t_1}} \right)^{2m-sz}$$

$$= -\frac{1}{s} \, \mathrm{e}^{zt_0 + zs_2 t_1} \sum_{m=0}^{\infty} \frac{(\mathrm{e}^{t_1})^m \, z^{2m+1}}{m! \, (1-sz)^{(m)}}.$$

4.5. The Finite Approximation to the Spectral Curve

We return to the question of $B(Y, Y')$ raised earlier about the covering $\Sigma \to \mathbf{C}^\times$. There are infinitely many branch points over P_1 and P_2 coming from the branches of $y = \log(Y)$. From the point of view of [**4**], we should not worry. We can define the spectral curve as a *local* spectral curve as mentioned in Section 2.4 and be done with it. However, Theorem 4.1 is non-local in nature, and so we are tempted to pursue this puzzle further.

Actually, we can view $B(Y, Y')$ as the push-forward of the bilinear form $\frac{dydy'}{(y-y')^2}$ on Σ. The difficulty lies is the fact that the fibres of our covering map contain $\mathbf{Z}$-many points, and we are left with a silly scalar factor of $\sum_{i \in \mathbf{Z}} 1$. Motivated by the following lemma, we will approximate the infinite cover of $\mathbf{C}^\times$ by a finite one.

Lemma 4.9. *Let $Z = p(z) = z^N$, $Z' = p(z') = z'^N$. Then the push-forward of the bilinear form $\frac{dzdz'}{(z-z')^2}$ under p (twice) is*

$$p_* p_* \frac{dzdz'}{(z-z')^2} = N \frac{dZdZ'}{(Z-Z')^2}.$$

PROOF. If z_0 is some fixed pre-image of Z, we have

$$p_* p_* \frac{dzdz'}{(z-z')^2} = p_* \frac{dz' dZ z_0}{NZ} \sum_{k=1}^{N} \frac{\mathrm{e}^{2\pi\sqrt{-1}k/N}}{(z_0 \, \mathrm{e}^{2\pi\sqrt{-1}k/N} - z')^2}$$

$$= p_* \frac{dz' dZ N z'^{N-1}}{(Z - z'^N)^2}$$

$$= \sum_{k'=1}^{N} \frac{dZdZ'}{(Z-Z')^2}$$

$$= N\frac{dZdZ'}{(Z-Z')^2}$$

Here we have used the result

$$\sum_{k=1}^{N}\frac{e^{2\pi\sqrt{-1}k/N}}{(e^{2\pi\sqrt{-1}k/N}-\lambda)^2} = \frac{d}{d\lambda}\sum_{k=1}^{N}\frac{e^{2\pi\sqrt{-1}k/N}}{(e^{2\pi\sqrt{-1}k/N}-\lambda)}$$
$$= \frac{d}{d\lambda}\frac{N}{1-\lambda^N}$$
$$= -\frac{N^2\lambda^{N-1}}{(1-\lambda^N)^2} \qquad \square$$

Our goal is to show that we can define a curve Σ_N and correlation functions $\omega_{g,k}^N$ as per Eynard and Orantin (with a slight modification), so that $\lim_{N\to\infty}\omega_{g,k}^N = \omega_{g,n}$, with $\omega_{g,n}$ the correlation functions for the curve Σ.

Define a curve $\Sigma_N \subset \mathbf{C}^3$ as the locus

(4.13) $$x = N\left(t_0 + s_2 t_1 + Y + e^{t_1}Y^{-1} + s\log(Y)\right), \qquad y = Y^{\frac{1}{N}}.$$

Now y is no longer a global coordinate. As a fix, we take $\log(Y) = N\mathrm{Log}\,(y)$, where Log is a branch of the logarithm with cut avoiding the $2N$ branch points of the map $x : \Sigma_N \to \mathbf{C}^\times$. The parameter x is restricted accordingly. Notice that, where $Y^{\frac{1}{N}} = \exp\left(\frac{1}{N}\mathrm{Log}(Y)\right)$,

$$\lim_{N\to\infty} NY^{\frac{1}{N}} - N = \mathrm{Log}(Y)$$

This suggests that we should have taken $y = NY^{\frac{1}{N}} - N$. This is possible, but unnecessary. It may be absorbed into x as the factor N as we have done.

Let $\xi_i = y\circ\pi_i : \Sigma_N^m \to \mathbf{C}$, $Z_i = Y\circ\pi_i : \Sigma_N^m \to \mathbf{C}$ be the y and Y coordinates of the i^{th} factor respectively. Consider the following quantities defined on Σ_N as follows:

$$\omega^N_{0,1}(\xi_1) = 0$$

$$B^N(y,y') = \frac{dydy}{(y-y')^2}$$

$$\omega^N_{0,2}(\xi_1,\xi_2) = \frac{N^2\xi_1^{N-1}\xi_2^{N-1}d\xi_1 d\xi_2}{(\xi_1^N-\xi_2^N)^2} = \frac{dZ_1dZ_2}{(Z_1-Z_2)}$$

$$\omega^N_{g,k+1}(\xi_1,\dots,\xi_{k+1}) = \sum_{p\in\Sigma_N, dx(p)=0} \operatorname*{Res}_{y\to p} \frac{\int_{\xi=y}^{\hat{y}} B^N(\xi_{k+1},\xi)}{(y-\hat{y})dx} W^N_{g,n+1}(y,\hat{y},\xi_K) \tag{4.14}$$

$$W^N_{g,n+1}(y,\hat{y},\xi_K) = \left(\omega^N_{g-1,k+2}(y,\hat{y},\xi_K) + \sum_{\substack{g_1+g_2=g\\ J\coprod J'=K}} \omega^N_{g_1,|J|+1}(y,\xi_J)\omega^N_{g_2,J'}(\hat{y},\xi_J)\right)$$

We have made the expected choice for $B^N(y,y')$, but for $\omega^N_{0,2}(\xi_1,\xi_2)$ we have taken the push-forward of B^N to $\{Y\in \mathbf{C}\}$ and scaled it by N^{-1}. Now $dx=0$ at $Y=P_1,P_2$ as before, and we have $2N$ branch points satisfying either $y^N=P_1$ or $y^N=P_2$.

Proposition 4.10. *If $2g-2+k>0$, as $N\to\infty$, in terms of $Z_i=Y(\xi_i)$, we have*

$$\omega^N_{g,k}(\xi_1,\dots,\xi_k) = \omega_{g,k}(Z_1,\dots,Z_k) + O\left(\frac{1}{N}\right).$$

The proof of this is the goal of the rest of this section.

Lemma 4.11. *The correlation functions $\omega^N_{g,k}(\xi_1,\dots,\xi_k)$ are single-valued in the coordinate $Z_j=\xi_j^N$, $j=1,\dots k$. Moreover, the recursion as in (4.14) reduced to a sum of residues at exactly the two principal roots: $p_1=P_1^{\frac{1}{N}}$ and $p_2=P_2^{\frac{1}{N}}$. We have*

$$\omega^N_{g,k+1}(Z_1,\dots,Z_{k+1}) = \sum_{i=1}^{2} \operatorname*{Res}_{y\to p_i} \frac{1}{(y-\hat{y})dx} \int_{\zeta=Y}^{\hat{Y}} \frac{d\zeta dZ_{k+1}}{(\zeta-Z_{k+1})^2} W^N_{g,n+1}(y,\hat{y},\xi_K). \tag{4.15}$$

Had we taken $\omega^N_{0,2}(\xi_1,\xi_2)=B^N(\xi,\xi')$, this lemma would not be true. This is what motivated our choice for $\omega^N_{0,2}$.

PROOF. The proof is by induction on $2g-2+k$. It is true in the base case, $\omega_{0,2}$. For the induction step, we need to show $\omega_{g,k+1}(\xi_1,\ldots,\xi_{k+1})$ may be written in terms of Z_{k+1} and dZ_{k+1}. Now by induction, the factor $W^N_{g,k+1}$ is invariant under the transformation $p \to \mathrm{e}^{2\pi\sqrt{-1}/N}\,p$. So we only need to check that for $i=1,2$, the quantity

$$\sum_{p\in\Sigma_N, y(p)^N=P_i} \operatorname*{Res}_{y\to p} \frac{\int_{\xi=y}^{\hat{y}} B^N(\xi_{k+1},\xi)}{(y-\hat{y})dx}$$

is single valued in Z_{k+1}.

Consider $\hat{y}$ near a branch point p. Then

$$\begin{aligned} x\left(\left(\mathrm{e}^{2\pi\sqrt{-1}/N}\,\hat{y}\right)^N\right) &= t_0 + s_2 t_1 + \hat{y}^N + \frac{q}{\hat{y}^N} + s\log(\hat{y}^N) \\ &= t_0 + s_2 t_1 + y^N + \frac{q}{y^N} + s\log(y^N) \\ &= x\left(\left(\mathrm{e}^{2\pi\sqrt{-1}/N}\,y\right)^N\right). \end{aligned}$$

So if we have $y,\hat{y}$ near $y=p$, we have $\mathrm{e}^{2\pi\sqrt{-1}/N}\,y, \mathrm{e}^{2\pi\sqrt{-1}/N}\,\hat{y}$ near $\mathrm{e}^{2\pi\sqrt{-1}/N}\,p$. Let p_1 be the principal root of $P_1^{\frac{1}{N}}$ and $\hat{y}$ is the conjugation of y near this point,

$$\begin{aligned} &\sum_{p\in\Sigma_N, y(p)^N=P_1} \operatorname*{Res}_{y\to p} \frac{1}{(y-\hat{y})dx} \int_{\xi=y}^{\hat{y}} B^N(\xi_{k+1},\xi) \\ &= \sum_{k=1}^{N} \operatorname*{Res}_{y\to \mathrm{e}^{2\pi\sqrt{-1}k/N}\,p_1} \frac{1}{(y-\hat{y})dx} \int_{\xi=\mathrm{e}^{2\pi\sqrt{-1}k/N}\,y}^{\mathrm{e}^{2\pi\sqrt{-1}k/N}\,\hat{y}} \frac{d\xi_{k+1}d\xi}{(\xi_{k+1}^2-\xi^2)} \\ &= \operatorname*{Res}_{y\to p_1} \frac{1}{(y-\hat{y})dx} \int_{\xi'=y}^{\hat{y}} \sum_{k=1}^{N} \frac{\mathrm{e}^{-2\pi\sqrt{-1}k/N}\,d\xi_{k+1}d\xi'}{(\xi_{k+1}-\mathrm{e}^{-2\pi\sqrt{-1}k/N}\,\xi')^2} \\ &= \operatorname*{Res}_{y\to p_1} \frac{1}{(y-\hat{y})dx} \int_{\zeta=Y}^{\hat{Y}} \frac{N\xi_{k+1}^{N-1}d\xi_{k+1}d\zeta}{(\zeta-\xi_{k+1}^N)^2} \\ &= \operatorname*{Res}_{y\to p_1} \frac{1}{(y-\hat{y})dx} \int_{\zeta=Y}^{\hat{Y}} \frac{d\zeta dZ_{k+1}}{(\zeta-Z_{k+1})^2} \end{aligned}$$

which is what we need to show[1]. □

Observe that the explicit dependence of the correlation functions on x and y lies entirely on the factor $\frac{1}{(y-\hat{y})dx}$ in the recursion formula.

Lemma 4.12. *For $i = 1, 2$, let $y = Y^{\frac{1}{N}}$ denote the primary branch near $Y = P_i$. In the limit $N \to \infty$,*

$$(y - \hat{y})dx = \frac{1}{N}\left(\log(Y) - \log(\hat{Y})\right) dx + O\left(\frac{1}{N}\right).$$

PROOF. For $i = 1, 2$, let $y = Y^{\frac{1}{N}}$ denote the primary branch near $Y = P_i$. Then

$$\begin{aligned}\frac{1}{N}(Ny - N\hat{y})dx &= \frac{1}{N}\left(N\, \mathrm{e}^{\log(Y)/N} - N\, \mathrm{e}^{\log(\hat{Y})/N}\right) dx \\ &= \frac{1}{N}\left(\log(Y) - \log(\hat{Y})\right) dx + O\left(\frac{1}{N}\right)\end{aligned}$$

□

PROOF. (Proof of Proposition 4.10) The proof is by induction on $2g - 2 + k$. For the base case $\omega_{0,2}$ it holds by definition.

By Lemma 4.11, we see that (4.14) reduces to the sum of residues at the two principal roots p_1 and p_2, as in (4.15). However, the explicit dependence on the branch of y is in the factor

$$\frac{1}{(y - \hat{y})\, dx} = \frac{N}{\left(\log(Y) - \log(\hat{Y})\right) dx} + O\left(\frac{1}{N}\right).$$

□

The factor of N in the denominator is cancelled by the factor of N in (4.13). So

$$\omega^N_{g,n} - \omega_{g,n} = O\left(\frac{1}{N}\right).$$

[1]We note that the above calculation proceeds exactly as above in the case of the infinite cover $\Sigma \to \mathbf{C}^\times$ with $B(y, y') = \frac{dy dy'}{(y-y')^2}$. The argument fails, however, because we cannot show that $W^N_{g,k+1}$ is invariant of the choice of branch of y.

APPENDIX A

The Equivariant Cohomology of the Projective Line

Let $\mathcal{O}(1)$ be the dual of the tautological bundle on $\mathbb{P}^1$, equipped with the natural action of the torus T, and let $H = c_1^T(\mathcal{O}(1))$ be its equivariant first Chern class. In this appendix, we show that the equivariant cohomology of $\mathbb{P}^1$ is gven by

$$H_T^*(\mathbb{P}^1, \mathbf{Q}) = \frac{\mathbf{Q}[H, s_1, s_2]}{(H - s_1)(H - s_2)}$$

where $H_T^*(\{*\}, \mathbf{Q}) = \mathbf{Q}[s_1, s_2]$. The multiplicative identity in $H_T^*(\mathbb{P}^1, \mathbf{Q})$ is denoted by e.

A.1. Equivariant Cohomology

Given a manifold X with smooth G action, the naïve definition of equivariant cohomology is to compute the cohomology of X/G. When the action of G is free, this gives the correct result, but for non-free G actions this gives a poorly behaved cohomology theory. The idea of equivariant cohomology is to replace X with a homotopy-equivalent space $\tilde{X}$ that carries a free G action, and then compute the cohomology of $\tilde{X}/G$.

Let EG be the total space of a weakly contractible principal right G-bundle with base BG; EG is referred to as the classifying bundle for G. The resolution $\tilde{X}$ may be taken to be $EG \times X$.

Given spaces Y and X with right and left actions by a topological group G, denote by $Y \times_G X$ the quotient of $Y \times X$ by the equivalence relation $(y \cdot g, x) \sim (y, g \cdot x)$, where $x \in X$, $y \in Y$, $g \in G$.

Given a group G and a topological space X on which G acts continuously on the left, and a coefficient ring R, the equivariant cohomology $H_G^*(X, R)$ is defined to be the ordinary (singular) cohomology ring $H^*(EG \times_G X, R) = H^*(\tilde{X}/G, R)$.

The classifying bundle EG is usually constructed as an inductive limit $\lim_{m\to\infty} EG_m$ of finite dimensional approximations EG_m. Standard theorems [**1**] state that for a given G, i, and X, for all m sufficiently large,

$$H^i(EG \times_G X) = H^i(EG_m \times_G X).$$

There are also equivariant analogues of the Chern classes. Given an equivariant vector bundle V over a G-space X, we can define a vector bundle $\mathbb{V} = EG \times_G V$ over $EG \times_G X$. By definition, the equivariant total Chern class $c^G(V) \in H_G^{2*}(X)$ of V is the total Chern class $c(\mathbb{V})$ of $\mathbb{V}$.

In this thesis, we are interested in the case $G = T = (\mathbf{C}^*)^2$, and $X = \mathbb{P}^1$, with action $(\theta_1, \theta_2) \cdot [z_1, z_2] = [\theta_1 z_1, \theta_2 z_2]$. We take

$$ET = \lim_{m\to\infty} ET_m = \lim_{m\to\infty} \big(\mathbf{C}^m \setminus \{0\}\big)^2.$$

Note that BT is may be identified as $\cup_{m=1}^{\infty} BT_m$, where $BT_m = \mathbb{P}^{m-1}$. Denote the pullbacks of the tautological line bundles $\mathcal{O}(-1)$ in the first and second factors of BT by $\mathcal{L}_1$ and $\mathcal{L}_2$.

A.2. The Equivariant Cohomology of a Point

First we compute the equivariant cohomology of a point. In general, $EG \times_G \{*\} = BG$, so we need to compute $H^*(BT_m, \mathbf{Q})$. Let $\mathcal{L}_1^*$ and $\mathcal{L}_2^*$ be the duals of the tautological line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$. It can be shown that the first Chern classes $s_1 = c_1(\mathcal{L}_1^*)$ and $s_2 = c_1(\mathcal{L}_2^*)$ generate the cohomology of BT_m with relations in degree $2m+2$. In the limit $m \to \infty$, these relations disappear, and so $H^*(BT, \mathbf{Q}) = \mathbf{Q}[s_1, s_2]$. In fact, s_i, $i = 1, 2$, is none other than the equivariant Chern class

$c_1^T(\mathcal{L}_i^*)$ for the line bundle $\mathcal{L}_i^*$ over a point with T action given by the character $\lambda_i(\theta_1, \theta_2) = \theta_i^{-1}$. Together, λ_1 and λ_2 generate the lattice $\mathrm{Hom}(T, \mathbf{C}^*)$.

A.3. Equivariant Line Bundles on the Projective Line

Consider the open cover of $\mathbb{P}^1$ by two open sets U_1 and U_2, which are the complement respectively of the fixed points $p_2 = [0; 1]$ and $p_1 = [1; 0]$ of the torus action. Thus, U_i is a neighborhood of p_i. An equivariant line bundle $\mathcal{L}$ on $\mathbb{P}^1$ is determined by equivariant line bundles $\mathcal{L}|_{U_i}$ on U_1 and U_2, and an equivariant isomorphism between the line bundles $\big(\mathcal{L}|_{U_1}\big)|_{U_1 \cap U_2}$ and $\big(\mathcal{L}|_{U_1}\big)|_{U_1 \cap U_2}$ on U_1 and U_2. The restrictions are in turn determined by characters $\lambda_1^{m_1}\lambda_2^{m_2}$ and $\lambda_1^{n_1}\lambda_2^{n_2}$ of T, which represent the stalks of these line bundles at p_1 and p_2 respectively. In order for the restrictions of these line bundles to $U_1 \cap U_2$ to be isomorphic, it is necessary and sufficient that $m_1 + m_2 = n_1 + n_2$, so that the restrictions of the characters to the diagonal $S^1 = \{\lambda_1 = \lambda_2\} \subset T$ are equal. The underlying non-equivariant line bundle of the resulting equivariant line bundle is $\mathcal{O}(m_1 - n_1)$.

The tautological bundle $\mathcal{O}(-1)$ equipped with the natural T-action has $m_1 = n_2 = -1$ and $m_2 = n_1 = 0$. Denote by H the equivariant Chern class $c_1^T(\mathcal{O}(1)) = -c_1^T(\mathcal{O}(-1))$.

Define $\mathcal{O}(l_1p_1 + l_2p_2)$ to be the line bundle with $m_1 = -m_2 = l_1$ and $n_1 = -n_2 = -l_2$. As a non-equivariant bundle it is isomorphic to $\mathcal{O}(l_1 + l_2)$. For any equivariant line bundle $\mathcal{L}$, there are unique $l_1, l_2, k \in \mathbf{Z}$ such that $\mathcal{L} = \mathcal{O}(l_1p_1 + l_2p_2) \otimes \mathcal{O}(k)$. We have

$$c_1^T(\mathcal{O}(l_1p_1 + l_2p_2)) = (l_1 + l_2)H - l_2s_1 - l_1s_2.$$

Lemma A.1. *The tangent bundle $T\mathbb{P}^1$, with the T-action induced by the action of T on $\mathbb{P}^1$, is isomorphic to $\mathcal{O}(p_1 + p_2)$.*

In particular, $c_1^T(T\mathbb{P}^1) = 2H - s_1 - s_2$.

A.4. The Equivariant cohomology of the Projective Line

We may compute the equivariant cohomology

$$H_T^*(\mathbb{P}^1, \mathbf{Q}) = H^*(ET \times_T \mathbb{P}^1, \mathbf{Q}).$$

by identifying the space $ET \times_T \mathbb{P}^1$ with the projective bundle $\mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)$ associated to the plane bundle $\mathcal{L}_1 \oplus \mathcal{L}_2$.

By definition, $H = c_1^T(\mathcal{O}(1))$ is the first non-equivariant Chern class of the line bundle

$$ET \times_T \mathcal{O}(1) = \mathcal{O}(1)_{\mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)} \to \mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2).$$

This is a cohomology class of $\mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)$ whose restriction to each fiber of the projection map $\mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2) \to BT$ is the hyperplane class of that fiber. By the Leray-Hirsch theorem, the classes 1 and H generate $H^*(ET \times_T \mathbb{P}^1, \mathbf{Q})$ as an $H^*(BT, \mathbf{Q})$-module.

Denote the unit of $H_T^*(\mathbb{P}^1, \mathbf{Q})$ by e. It remains to compute the relations satisfied by H. To this end, consider the plane bundle $S = (\mathcal{L}_1 \oplus \mathcal{L}_2) \otimes \mathcal{O}(1)_{\mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)}$. Now

$$\begin{aligned} c(S) &= c(\mathcal{L}_1 \otimes \mathcal{O}(1)_{\mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)}) c(\mathcal{L}_2 \otimes \mathcal{O}(1)_{\mathbb{P}(\mathcal{L}_1 \oplus \mathcal{L}_2)}) \\ &= (e - s_1 + H)(e - s_2 + H). \end{aligned}$$

This shoes that $c_1(S) = 2H - s_1 - s_2$ and $c_2(S) = (H - s_1)(H - s_2)$. But S contains the trivial bundle as a sub-bundle and so $c_2(S) = 0$. Thus

$$H_T^*(\mathbb{P}^1, \mathbf{Q}) = \frac{\mathbf{Q}[s_1, s_2, H]}{(H - s_1)(H - s_2)}$$

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